A short lecture series on Contour Integration in the Complex Plane
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Lecture - 16
Contour integration: rectangular contour

Good morning, welcome to this next lecture on complex variables. Without, before going full blown into another example using a branch cut, I wanted to do one more standard example, ok; you will find these in many textbooks. So, that example is these, consider. So, this is the next example.

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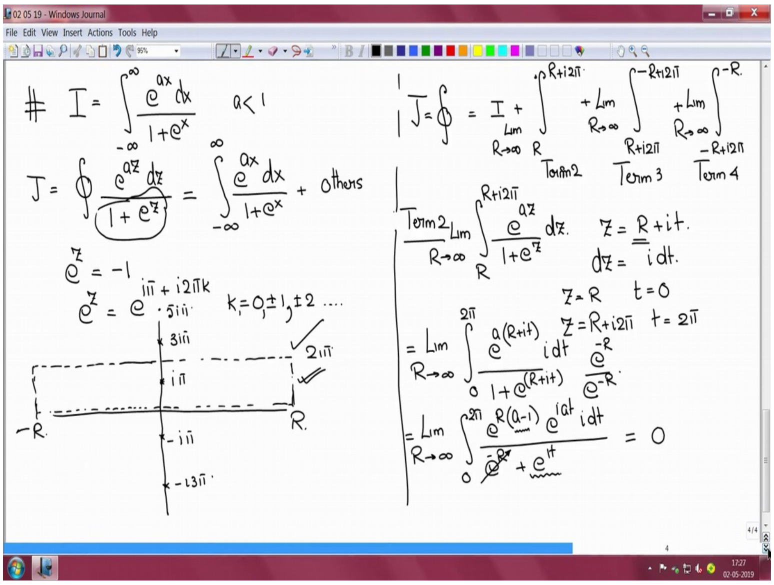

Consider I is equal to integral minus infinity to infinity, e to the power a x dx, by 1 plus e to the power x, with a being less than 1, a is a real number less than 1, ok. Now, as usual, we will replace this with a contour integral. So, this is a contour integral and we replace x with z. So, we have, e to the power a z dz, by 1 plus e to the power z, ok. Now, before proceeding any further, let us look at the roots of the denominator, ok.

What are the roots of the denominator? I have e to the power z is equal to minus 1, ok, which means I have e to the power z is equal to e to the power i pi plus i twice pi k. And k can go from 0 plus minus 1, plus minus 2, etcetera ok. So now, what do we have? We have a complex plane like this and the poles are at let us say i pi, thrice i pi, 5 times i pi and again at minus i pi, minus i 3 pi, etcetera.

So, these are the poles. Now, as usual I will choose so that I have this portion, which is minus infinity to infinity, e to the power a x d x by 1 plus e x plus other terms. So, we will see what those are; so, let us see how the contour goes. So, I come from minus infinity, there is no problem on the real axis, I go straight to plus infinity, then I close here, go up and I go through this, where it is twice i pi and I come back all the way to minus infinity and then I close the contour.

So, if I initially set this to be minus R and this to be plus R. So, I have the following portions, I have J which is the closed contour integral, which is equal to the portion I want, I plus I have integral, ok, R to R plus i twice pi. On all of them limit R tending to infinity, ok, plus limit R tending to infinity, integral R plus i twice pi here, to minus R plus i twice pi, plus limit R tending to infinity, integral minus R plus i twice pi, minus R. So, this is term 2, this is term 3, this is term 4.

So, let us look at term 2, let us look at term 2. Term 2 is integral, ok, R to R plus i twice pi, limit R tending to infinity, e to the power a z, by 1 plus e to the power z dz, ok. Now, on this curve z can be written as R plus i t, then d z here, R is a constant. So, d z is equal to i dt, when z is equal to R, then t is equal to 0, when z is equal to R plus i twice pi, t is equal to twice pi, ok. So now, I have this is equal to limit R tending to infinity, integral 0 to twice pi, e to the power a, R plus i t, by 1 plus e to the power R plus i t, into i dt ,ok.

Now, let me multiply the numerator by e to the power of minus R and denominator by e to the power of minus R. So, I get, limit R tending to infinity, integral 0 to twice pi, e to the power R, a minus 1, e to the power of i at, i dt by, e to the power of minus R, plus e to the power i t. Now, as R tends to infinity, you can see that this goes to 0. So, the denominator only e to the power i t survives. In a numerator a is less than 1. So, as R tends to infinity, this term goes to 0, so the integrand goes to 0, so this goes to 0, ok. Next, that was term 2, next is term 3, ok. What happens to term 3?

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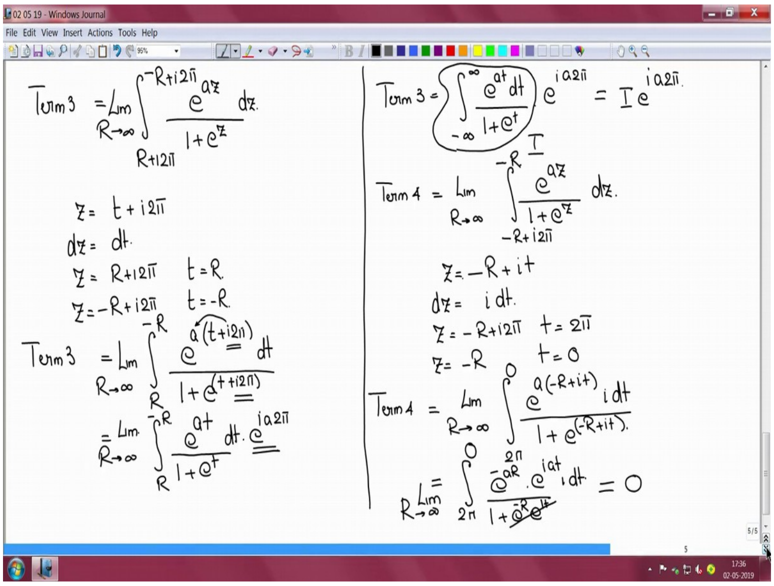

Term 3 is equal to integral R plus i twice pi into to, minus R plus i twice pi ok, e to the power a z, by 1 plus e to the power z dz. On this part of the curve, we can write z as equal to t plus i twice pi. So, that d is z equal to dt, when z is equal to R plus i twice pi, t is equal to R and when z is equal to minus R plus i twice pi, t is equal to minus R and so term 3, term 3 is equal to integral R to minus R, ok. I forgot limit R tending to infinity, e to the power a z is R plus i twice pi, dz is dt, by 1 plus e to the power z, e to the power of R plus i twice pi.

Now, I have limit R tending to infinity, integral R to minus R, e to the power a R, by 1 plus e to the power R. I am sorry; I am sorry, there is a mistake. There is as an error here, my mistake. This should be t and this should be t. So, this becomes e to the power a t and this becomes e to the power t, into dt, into e to the power of i a twice pi. The denominator, we have e to the power i twice pi so that is 1, in the numerator we have e to the power i twice pi into a.

So, that comes off comes here, ok. So, we now have term 3, 3 is equal to, let us straight away put it, minus infinity to infinity, e to the power a t d t ,by 1 plus e to the power t, into e to the power i a twice pi ,ok, but this part is my original integral that I wanted, this is I itself, ok. So, this is equal to I times e to the power of i a twice pi, ok; now, for term 4. Term 4 is equal to, limit R tending to infinity, integral minus R plus i twice pi to minus R, e to the power a z, by 1 plus e to the power z dz.

Now, on this part of the curve, we write z is equal to t, z is equal to minus R plus i t. So, that d z is equal to i dt and when z is equal to minus R plus i twice pi, t is equal to twice pi. And, when z is equal to minus R, t is equal to 0. Then my term 4 is equal to, limit R tending to infinity, integral twice pi to 0, e to the power a z, a minus R plus i t, dz is i dt, divided by 1 plus e to the power z, e to the power of minus R plus i t. Now, what we have here is integral twice pi to 0, e to the power of minus a R, e to the power of i a t, by 1 plus e to the power of minus R, e to the power i t, limit R tending to infinity, i dt.

Now, this goes to 0, because R is tending to infinity and this numerator also goes to 0, a numerator multiplies everything. So, this goes to 0, this goes to 0. So, now we have, there is one problem that I had. The limits on term 3 are, R in the denominator minus R in the numerator, ok. So, the limits here, would be plus infinity here and minus infinity here, plus infinity to minus infinity, ok. So, this is, minus i e to the power i a twice pi. So, now, what we have?

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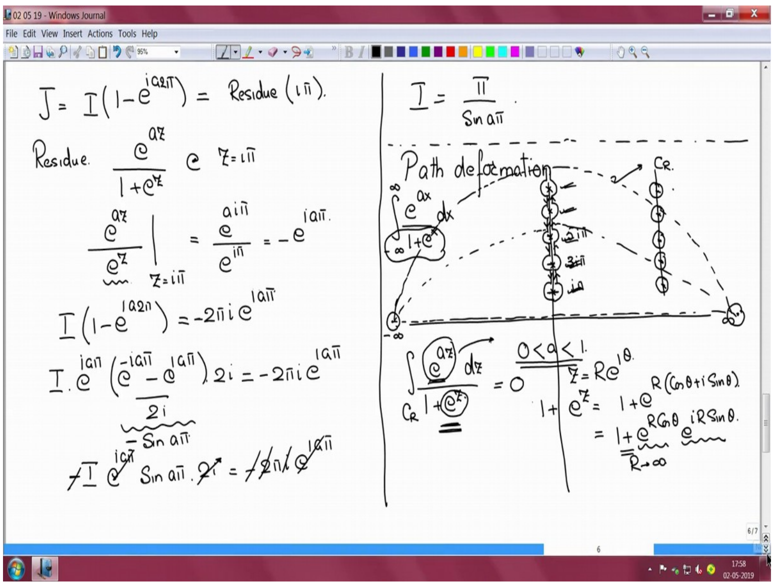

For J is I, into 1 minus e to the power of i a twice pi and that is equal to the residue, we are going to include just one pole and that residue is at i pi, ok. So, we have to find the residue at i pi, residue of e to the power a z, by 1 plus e to the power z, at z equal to i pi. Now, we have seen before, that if you have two functions p and q that are analytic, then we can find the residue by the numerator e to the power a z, divided by the derivative of the denominator, which is e to the power z, evaluated at z equal to i pi, provided the denominator does not go to 0 there. So, it does not.

So, this is equal to, e to the power a i pi ,by e to the power i pi, which is minus e to the power i a pi, ok. So now, what do I have? I have I, into 1 minus e to the power of i a twice pi, is equal to twice pi i ,times e to the power i a pi ,with a minus, ok. So, here what do we do? I take I ,i a pi outside, I get e to the power minus i a pi, plus e to the power i a pi, ok, divide by twice i ,multiply by twice i ,that is equal to minus twice pi i ,e to the power of i a pi. This part is sine of a pi; sine of a pi. So, what do I have? I have I ,e to the power of i a pi ,sine of a pi, into twice i ,is equal to minus twice pi i ,e to the power of i a pi. Sorry, this is a minus over here, ok. So, this gives me minus sine a pi.

So, I have a minus over here. So now, if I cancel twice I, twice I, cancel the minus ,cancel the e to the power i a pi, then I have, I given by pi divided by sine a pi, that is the answer. Now, we will use; do the same problem using a second method called the method of path deformation. So, let me draw the complex plane, I have this integral, which goes from minus infinity to infinity, e to the power a x, over 1 plus e to the power of x dx. So, we start at minus infinity, we move through the origin to plus infinity. So, that is the entire integral and my poles in the upper half are at i pi, thrice i pi, five i pi and so on.

So, now I will deform the path, ok. Let us say I have deformed path till here, I come down, I go around the pole, I come down, I go around the pole, I come down, I go around the pole, join up, go up the pole, go around, go up, go around and then come back down here. All the while the end points are anchored, minus infinity and plus infinity they are anchored.

So, in the limit what happens is I can take deform it along at contour that goes on a circular arc at radius infinity, I come down, I go around the pole, I come down, go around the pole, I come down, I go come to the bottom, i pi pole and go up go off to infinity and come back.

Now, what happens is that this function is poorly behaved only at these isolated singularities and in between it is well behaved. So, the integral that goes down between these poles and that which goes up again between the poles, they cancel out, ok. So, as a result I will draw it here, as a result I am left with merely counterclockwise integrals around isolated poles, ok.

Now, let me just give an approximate proof of what happens on this portion of the deformed contour, ok. I have integral over the CR e to the power a z d z, by 1 plus e to the power z. In here as z tends to infinity, z is moving on a circular arc R e to the power of i theta. So, that 1 plus e to the power z is equal to 1 plus e to the power R cos theta plus i sin theta and this is equal to 1 plus e to the power R cos theta e to the power i R sin theta.

Now, this is an oscillatory part it does not bother us, but this here as R tends to infinity ok. As R tends to infinity, this denominator blows up and the numerator has a in it, which lies between 0 and 1, a lies between 0 and 1 and so what happens is that, this e to the power z as R tends to infinity will overwhelm the numerator and so, this integrand will go to 0. Exponent is a very rapidly increasing function.

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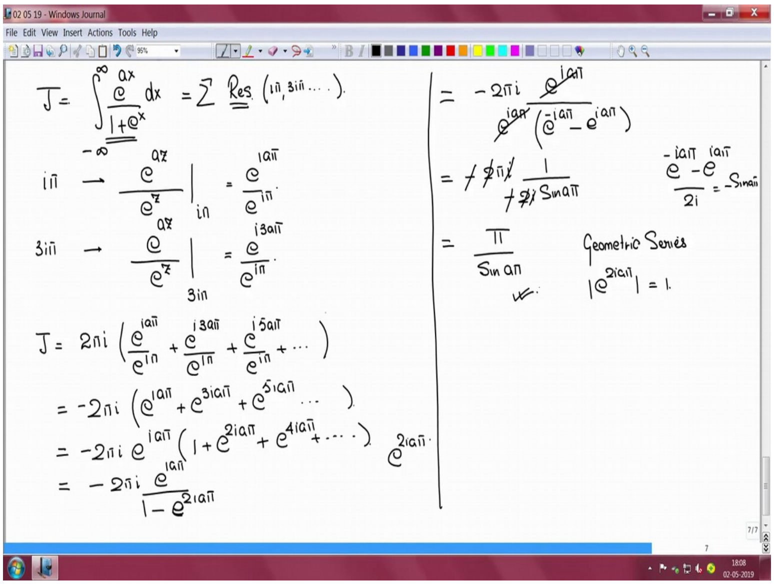

So, this e to the power z will overwhelm e to the power a z, because of the limits on a. So, this integral goes to 0. So, now what are we left with? We are left with e to the power a x, which is originally from minus infinity to infinity, ok. We are left with a series of infinite sums ok, of residues at all the poles; i pi, thrice i pi and so forth. So, how do we write this? We write this as, what is the residue at let us say i pi? The residue at i pi is e to the power a z, divided by the derivative of 1 plus e to the power z, which is e to the power z evaluated at i pi, which is equal to e to the power i a pi by e to the power of i pi.

Similarly, the residue at thrice i pi is e a to the power of a z by derivative of 1 plus e to the power z, which is e to the power z evaluated at thrice i pi, which is equal to e to the power i thrice a pi by e to the power i thrice pi, which I write as e to the power i pi ok, the same thing. So now, what do I have? My J is equal to twice pi I, into e to the power i a pi by e to the power i pi, plus e to the power i thrice a pi by e to the power i pi, plus e to the power i 5 a pi by e the power i pi and so forth.

So, this is equal to minus twice pi I, into e to the power i a pi, plus e to the power thrice i a pi, plus e to the power 5 i a pi and so on. This is a geometric series. This is equal to minus twice pi I, e to the power of i a pi, 1 plus e to the power twice i a pi, plus e to the power 4 i a pi and so forth. So, this is a geometric series with the ratio e to the power twice i a pi.

So, what does this do? This gives me minus twice pi I, e to the power i a pi, divided by 1 minus e to the power twice i a pi. So, this becomes equal to minus twice pi i, e to the power i a pi, by e to the power i a pi, into e to the power minus i a pi, minus e to the power i a pi, ok. If we simplify this ok, this is equal to minus twice pi i. Now, e to the power of minus i a pi, minus e to the power i a pi, by twice i is equal to minus sine a pi.

So, we have 1 by, minus twice i sine a pi minus 2 i, this is equal to pi by sine a pi. Now, we do know that when we do geometric series, the ratio must be strictly less than 1, ok, e to the power twice i a pi, magnitude is just 1, ok, but in this case, this series works, ok. So, with this, I close this lecture. We will continue with branch cut examples, again from the next class.

Thank you.