A short lecture series on Contour Integration in the Complex Plane

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Lecture - 15
Infinite branch cut example

Good morning, welcome to this next lecture on complex variables.

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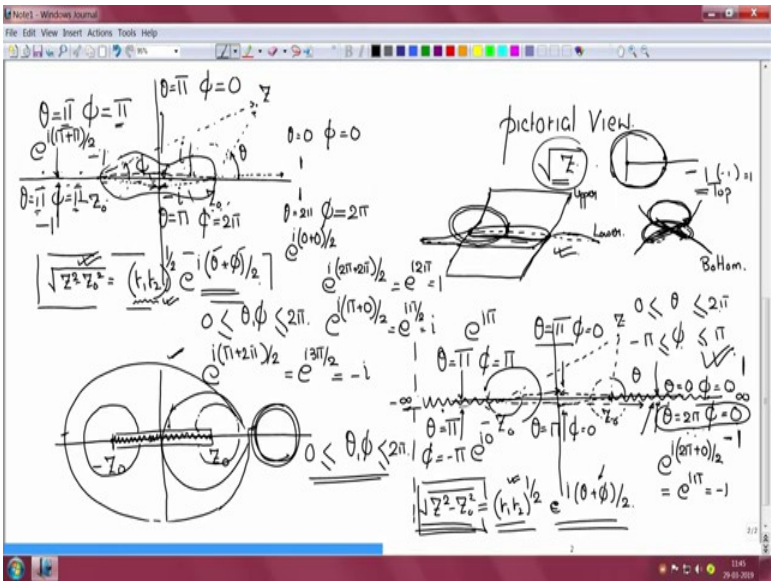

We were looking at this function which is square root of z square minus z 0 square, which is written as r 1 r 2 to the power half e to the power i theta plus phi by 2. We are looking at the second case where theta lies between 0 to 2 pi, but phi lies between minus pi and pi. So, let us examine how the function behaves and again I mentioned r 1 r 2 to the power half is just a magnitude. So, it does not bother us. So, let us see what is theta over here, theta over here just above the x axis is 0, theta is 0. And what is phi? Phi is also 0; phi is also 0, just above the x axis. What is below? Just below here, what is theta, just below here theta now has to go full round so, theta is 2 pi.

But, what is phi? Phi is still 0 ok. If I take a phasor from here and just below the x axis, this angle is 0, I am begin from minus pi come around phi is 0 here. So, phi is still 0. Therefore, you see the difference now. I get e to the power 0 plus 0 by 2 which is 1, I get e to the power i pi ply twice pi plus 0 by 2 which is i pi so, minus 1. So, I have a change in sign now. I have 1 over here minus 1 over here ok.

For this case e to the power i twice pi plus 0 by 2 is equal to e to the power of i pi which is minus 1 for this lower case, ok. Now, I have a jump in value. If I approach from top I get 1, if I approached from bottom I get minus 1 for this function. Let us examine in between ok, in between, let us say so now, my z minus z 0 looks like this and z plus z 0 looks like this. So, we will have see what are the theta phi values. So, theta for this case is pi and phi for this case here is 0, just as before.

If we approach from below, if we approach from below theta for this case is still pi and phi for this case is still 0, phi is 0. So, theta pi have same values and theta phi have same values approaching from above and below. So, this function is continuous here, let us examine what happens here. Here we have theta is equal to; theta is equal to pi, theta is equal to pi and here phi is equal to pi also, ok. Because why? Phi is going from minus pi goes around once and comes back here as pi. What about below? Theta is what is theta here? Theta here is pi, theta here is pi, but phi here is minus pi, phi here is minus pi.

So, above you get, I get e to the power of i pi, here i get e to the power i 0 so, the function jumps here also. So, interestingly this region is forbidden; this region is forbidden and this region is allowed you can cross here, ok. So, these become the branch cuts, they go off to infinity, they go off to infinity, ok. So, with this knowledge let us look at a problem that involves a branch cut ok, the first problem involving the branch cut.

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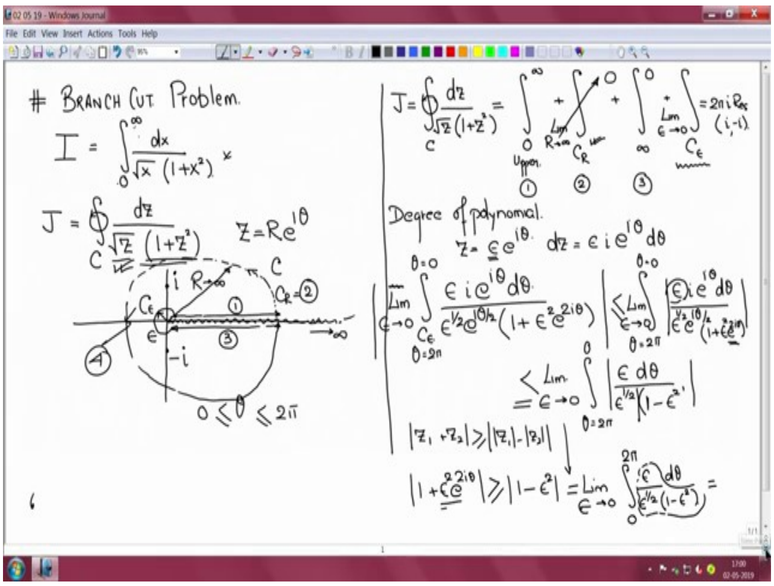

So, we have I, given by again a real line integral, 0 to infinity dx by square root of x and 1 plus x square ok; you can see that the integral has to go from 0 to infinity the positive x axis. So, as before we construct another function J, which will be going around a closed contour C and we replace x with z. So, we have z by square root of z into 1 plus z square ok. Now, fairly obvious, the cut we are going to take is here ok, we are going to parameterize z in terms of R e to the power of i theta and theta will be between limits 0 and twice pi.

So, that now for square root of z, for square root of z, this becomes the cut, as seen before this becomes the cut and as I said we need the portion to be integrated must be part of my contour. So, 0 to infinity must be part of my contour. So, how does that work out ok? So, watch this. We start here, just away from 0, then we move above the cut; we move above the cut to infinity, ok. So, let this be infinity, then I take a semicircular I mean I said take a circular arc, take a circular arc, the circular, ok.

I go around, I come back here and this is a cut so, I cannot cross. So, now I go along the cut till 0 and I take an epsilon circular contour a radius with epsilon circular contour, means a circular contour with radius epsilon and this here is R going off to infinity and I will join up, ok. So, this is a closed contour now and my closed contour because of square root of z has a branch cut that goes from 0 to infinity, 0 to infinity. And I have isolated poles at z equal to plus i and z equal to minus I, z equal to plus sign, z equal to minus sign. So now, how to solve this problem? So, my J is equal to integral over this closed contour C, ok.

This is my C, it is taken in the positive sense counterclockwise sense and I have dz over square root of z into 1 plus z square; now I write down the individual portions ok. So, let me call this 1, let me call this circular arc CR as 2, ok, then I have this portion coming from infinity back I will call it 3, then the C epsilon you call it 4. So, I have 4 portions, 4 legs. So, I have the integral going from 0 to infinity upper, above the cut, ok. Then I have this integral over this circular contour with radius going off to infinity, ok. Then I have the portion that comes from infinity to 0, this portion 3, this portion 1, this portion 2, this portion 3.

And I have lastly this integral over C epsilon limit epsilon tending to 0; now there are 2 isolated poles inside the contour; plus minus i. So, I will get residues from both twice pi i residue, from i and minus I, ok. So, we immediately if you look at this CR integral, using the degree of polynomial theorem; degree of polynomial theorem, the degree of the denominator is at least 2 greater than the numerator; there is nothing in the numerator, ok. So, CR can be sent to 0 immediately, CR is 0. Now I am left with C epsilon, C epsilon. Let us see how C epsilon integral looks like, ok, on C epsilon; on C epsilon my z is equal to epsilon e to the power of i theta, ok.

Epsilon being constant I will move along theta. So, dz is equal to epsilon i e to the power of i theta d theta, ok. Then, if we substitute this back into J, ok, I have J, I am sorry into C epsilon. So, I have integral along C epsilon, ok, we will put the theta limits on that in a minute. So, numerator is dz, which is epsilon i e to the power of i theta d theta, dz by square root of z is epsilon to the power half e to the power of i theta by 2. And then I have, 1 plus epsilon square e to the power of twice i theta and whole thing has limit epsilon tending to 0 and because of the theta definition, we have 0 here, twice pi here.

The theta limits on this are theta is 2 pi and theta is 0 here, ok. Now, the magnitude of this integral, the magnitude of this particular integral is less than equal to the integral of the magnitude and limit epsilon tending to 0. We can informally see that limit epsilon tending to 0, epsilon sits separately on top. So, this will go to 0 or if we still want to do it formally what we do is, we say this is less than, strictly less than limit epsilon tending to 0, integral theta equal to twice pi to theta equal to 0, ok.

Then we replace so, we have epsilon now i e to the power i theta go away, epsilon d theta by epsilon to the power of half, e to the power i theta magnitude is 1. Then we replace this with a smaller quantity, ok, we replace this with a smaller quantity, so, that less than strictly holds, ok. So, we know those particular inequalities, which is magnitude of z 1 plus z 2 is greater than or equal to the magnitude of the magnitude of z 1 minus the magnitude of z 2, ok.

So, here I have e to the power of twice i theta which I want to get rid of. So, what I write as 1 is a magnitude of 1 minus epsilon square. What I am saying is that magnitude of 1 plus epsilon square e to the power twice i theta is greater than or equal to the magnitude of magnitude of 1 which is 1 minus the magnitude of this which is epsilon square, ok. So now, what happens is, this is equal to, the integral is equal to limit epsilon tending to 0 integral, here I will switch the integral limits because we need a positive value this modulus we have up here is a positive value. So, we have to switch the integral limits.

So, I get epsilon by epsilon to the power half into 1 minus epsilon square into d theta. This quantity can be pulled out because it is not a function of theta, ok.

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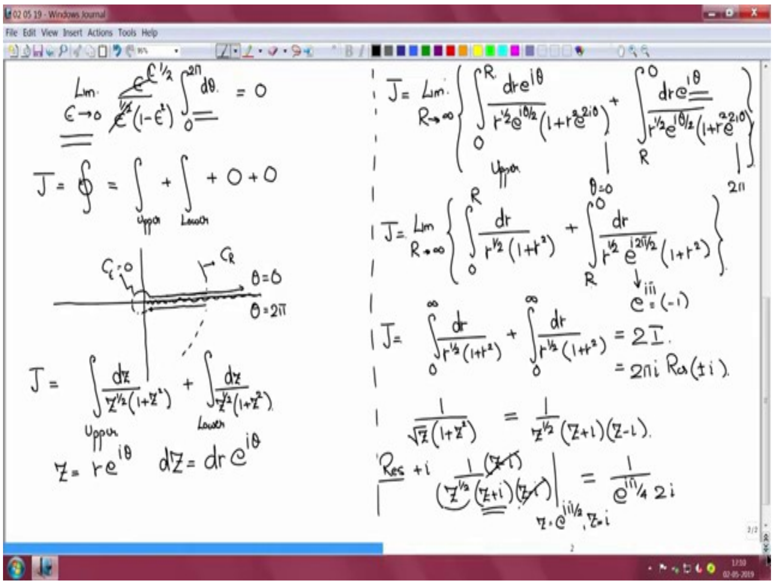

And so on the next page, I have, limit epsilon tending to 0 epsilon over epsilon to the power half, 1 minus epsilon square integral with the switched limits d theta. Because, this integral gives me twice pi, but here this epsilon to the power half cancels with epsilon and I get a epsilon to the power half and limit epsilon tending to 0 this goes to 0. Now, what have we achieved? We have J given by this integral over the closed contour and I have the integral on the upper portion plus the integral on the lower portion plus the C epsilon contour which gave 0, plus the CR which gave 0.

Now, let us look at the branch cut once again. We have this upper portion. So, we have a branch cut here, we have the upper portion that goes to infinity. We get into CR and CR 0, we come back over here, then we go to the lower portion, then we close with C epsilon and C epsilon is also 0. So, we now have, J is equal to integral dz over z to the power half 1 plus z square, in the upper part plus integral lower dz over z to the power half 1 plus z square. Now, let in both these cases, z be equal to r e to the power of i theta and dz is equal to dr e to the power of i theta because, theta is a constant on each of these portions.

So, now J becomes equal to limit R tending to infinity, integral 0 to R, dz which is d r e to the power of i theta by z to the power half which is r to the power half e to the power of i theta by 2 into 1 plus z square, 1 plus r square e to the power twice i theta; this is the upper portion, plus integral R to 0, d r e to the power of i theta by the same expression, r to the power half e to the power of i theta by 2, 1 plus r square e to the power of twice i theta. Here, theta is equal to 0, this is theta 0, theta is 0 here, and here theta is equal to twice pi.

So, here theta is equal to twice pi. So, we get J is equal to, limit R tending to infinity, 0 to R, theta is 0. So, we get d r over r to the power half, e to the power i 0, 1 plus r square, plus integral R to 0, here theta is 2 pi. So, I get again d r divided by R to the power half, e to the power of i twice pi by 2, into 1 plus r square. This gives me e to the power of i pi, which is a minus 1 and I can use this minus 1 and switch the limits on the integral, ok. So, if I do that, now integral 0 to infinity d r by r to the power half, 1 plus r square, plus again integral 0 to infinity, d r by r to the power half, into 1 plus r square.

So, this is essentially twice the integral I want, this is equal to 2 times I which should be equal to twice pi i times the residue at plus and minus i. Now, for the residue ok, the function we have is given by, 1 over, function we have is given by 1 over root z, into 1 plus z square. And so, we write this as 1 over z to the power half, z plus i into z minus i.

So, the residue at plus i; residue at plus i is given by 1 over z to the power half z plus i into z minus I, multiplied by z minus I, evaluated at either z equal to e to the power of i pi by 2 or z equal to i. So, we get a cancellation and for z to the power half, I use z equal to e to the power i pi by 2. So, 1 over e to the power of i pi by 4 and here I use z equal to i so, I get twice I, ok.

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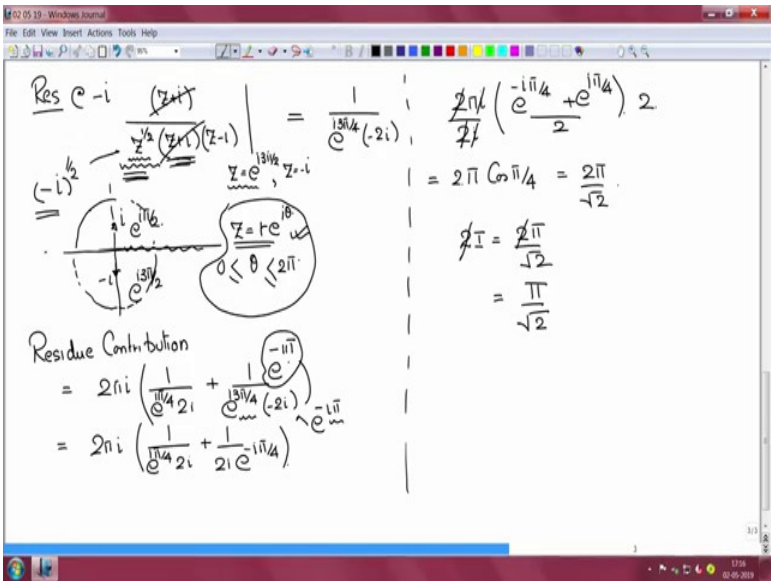

For the residue at minus i, I have z plus i divided by z to the power half, into z plus I, into z minus I, evaluated at z is equal to e to the power of i 3 pi by 2 or z equal to minus i. Let me explain why I have this particular expression ok. I have a branch cut; I have a branch cut and on the branch cut z is written as r e to the power of i theta, theta varying from 0 to twice pi.

So, theta varies it is 0 over here, this pi by 2 over here, it is pi over here, it is 3 pi by 2 over here and so forth. Now, when evaluating the part of the function which has given us the branch cut, we have to respect this definition so, that we do not go incorrect. I have poles at plus i and minus i. So, if I use minus i for this, as I am doing for this part of the function, if I say minus i to the power half then the square root has 2 values; I do not know which value to choose. So, it is better to use the branch cut definition, z here is e to the power i 3 pi by 2 and z here is e to the power of i pi by 2.

For the other part it does not matter, you can use z equal to i here z equal to minus i here, but for the part that is causing the branch cut, you have respect this definition, ok. So now, this gives me 1 over e to the power i 3 pi by 4, into minus twice i. So, the residue contribution; residue contribution is equal to twice pi i, into 1 over e to the power of i pi by 4 twice i, plus 1 by e to the power of i 3 pi by 4, into minus twice i, ok.

Now, here I will multiply the numerator and denominator by e to the power minus i pi, denominator by e to the power of minus i pi. So, that I get twice pi i, into 1 over e to the power of i pi by 4 twice i, now e to the power i pi gives me a negative. So, I get a negative and that negative cancels this negative so, the negative is gone. So, I get a plus 1 by twice i, but this e to the power i 3 by 4 and minus i pi gives me e to the power minus i pi by 4.

And so, I have twice pi I, take twice i outside, e to the power of minus i pi by 4, plus e to the power i pi by 4, ok. If I divide this by 2, multiply by 2, cancel twice i twice i, I get pi into 2 cos pi by 4; this is equal to twice pi by root 2 and on the left I had twice I 2 times I is now is equal to twice pi by root 2. And so, the answer is pi by root 2 ok. So, with this I close today's lecture.

Thank you.