A short lecture series on Contour Integration in the Complex Plane
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Lecture - 14
Finite Branch Cut

Good morning and welcome to this next lecture on Complex variables.

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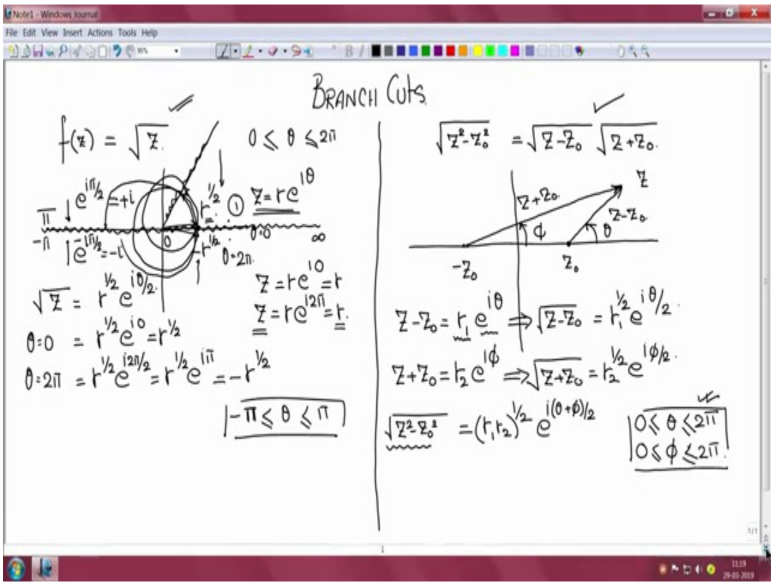

If you recall last lecture, we had started looking at Branch Cuts and we started by looking at a function of the form f of z equal to square root of z and we said that if we were doing closed contour integrals using this function, then let us say we have a closed curve of this form.

And we start over here; we start over here and we parameterize this circle as z is equal to r e to the power of i theta, then if we start measuring theta over here, theta equal to 0 above and theta equal to 2 pi below, then at this point z is equal to r e to the power of i 0 which is r and once you go around full circle and come back here, theta is 2 pi and z is equal to r e to the power of i 2 pi, which is again equal to r.

So, the same point on the plane the z value acquires this acquires the same value r, but look at the function square root of z. So, square root of z now on this curve which is parameterized as r e to the power i theta is r the magnitude to the power half, e to the power i theta by 2. So, at z equal to 0, I am sorry at theta equal to 0; theta equal to 0 we will get r to the power half, e to the power i 0, which is equal to r to the power half.

So, here we get r to the power half for the function. At theta equal to twice pi, we get r to the power half, e to the power of i twice pi i theta by 2 which is r to the power half, e to the power of i pi, e to the i pi is minus 1, this gives me minus r to the power half. So, here I get minus r to the power half. So, as we approach from above the x axis, we get r to the power half, which as we approach from below we get minus r to the power half, so this line is a line of discontinuity.

Now, based on the values I choose for theta limits, here theta is going between 0 and twice pi, theta can go from minus pi to pi also. theta can go from minus pi, we have minus pi over here and theta can go full circle and come back above which is pi, ok. If we choose these limits, then this line becomes the line of discontinuity because I will get e to the power i pi by 2 here which is equal to plus i and I get e to the power of minus i pi by 2 here the minus i. So, this line will become the line of discontinuity for square root of z.

So, based on this choice of theta, I will get a line of discontinuity. If I start at some particular angle for theta then I think that can become the line of discontinuity. I go around and come back and this could be the line of, so the choice is on theta, ok. So, that is as far as this branch cut going from 0 to infinity. Its an infinite branch cut.

Now, we started looking at the next function which causes branch cuts. It is also a square root function, but that looks like square root of z square minus z 0 square, ok. We write this as square root of z minus z 0, into square root of z plus z 0, for now let z 0 be on the real axis. So, that is my axis. Let z 0 be here. Let minus z 0 be here, then if z happens to be somewhere here, let us say this is my z then z minus z 0 is this is my z minus z 0 phasor and this is my z plus z 0 of phasor.

So, I will represent these in terms of magnitudes and arguments. So, this I will call it theta. This will be denoted by phi. So we say z minus z 0 is equal to r 1 e to the power of i theta which implies. So, this is the magnitude positive number and this is the phase. So, z minus z 0 square root is going to be equal to r 1 to the power half, e to the power of i theta by 2. Similarly, z plus z 0, we denote it by r 2 e to the power of i phi, which implies square root of z plus z naught is equal to r 2 to the power of half e to the power i phi by 2.

Now the product. We have the product over here, the product over here. So, square root of z square minus z 0 square is equal to r 1, r 2 to the power half and e to the power i theta plus phi by 2, ok. Now, just as it happened in this case, square root of z case, here also we have to choose limits on theta and phi. So, as a beginning let us choose theta twice pi and so is phi, ok. Now, let us see what happens in the complex plane with regard to this function and with regard to this definition of theta and phi, how the well behaved ness comes about using this definition of theta and phi.

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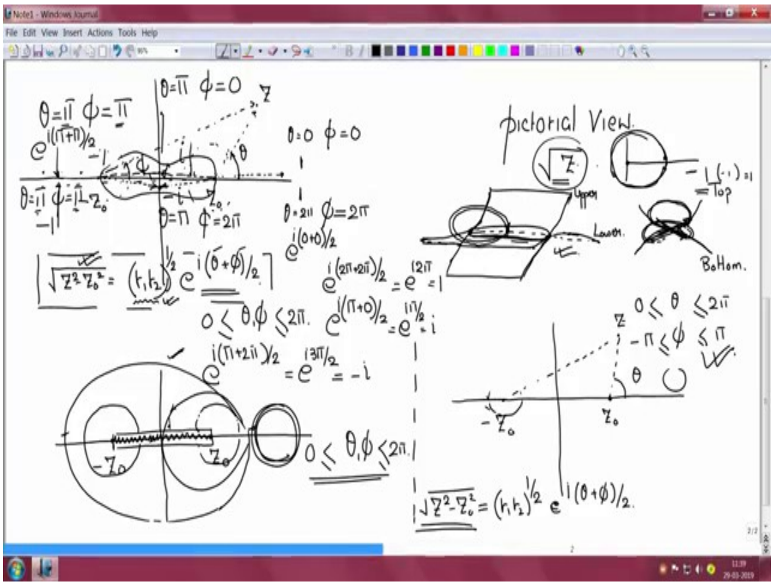

z 0 is here, minus z 0 is here let us say z is here somewhere I keep it light because we will be writing many things, ok. So, what do I have? I have square root of z square minus z 0 square is equal to r 1, r 2 to the power half e to the power i theta plus phi by 2 and theta, phi, theta comma phi are defined this way, ok. So, now, let us see if we begin here; if we begin here, if my z is very close to the real axis from above, ok, then theta is measured this way and phi is also measured this way.

So, here theta is equal to 0, here phi is also equal to 0. So, the argument is going to be e to the power i 0 plus 0 by 2, 0 plus 0 by 2, so e to the power i 0, so it will be 1 here ok; that means, I will get r 1, r 2 to the power half, ok, so I get 1. Now, let us see below this just below, so just below I have theta equal to twice pi because theta has gone, theta has gone full round and theta is 2 pi here, ok. Similarly phi; phi goes full round and here phi is 2 pi ; phi is 2 pi, phi is equal to 2 pi. So, what happens? I get e to the power i twice pi, plus twice pi, by 2, for this argument, theta is 2 pi, phi is 2 pi by 2.

So, it is e to the power i 2 pi which is equal to e to the power i twice pi, so this is equal to 1, so I get 1 over here also, ok. r 1, r 2 to the power half is a positive number. It does not cause problems. Now, let us look at in between minus z 0 and z 0, ok. So in this region here, theta, ok. So, I am going to look at a point over here right on the x axis, ok. So, theta now turns and it acquires the value pi, so theta is equal to pi, ok; that means, if we if I draw a phasor from here to z 0, this is that phasor, so theta is equal to pi, ok. What about phi? Phi is the angle for z plus z 0.

So, if I bring this z here down here, then this is the phasor, so now phi is 0, so here phi is 0, ok. And therefore, e to the power i theta plus phi by 2 is equal to e to the power, let me write it here, e to the power i theta is pi, phi is 0, by 2 is equal to e to the power of i phi by 2 which is equal to i, ok. So, at every point we are looking at this function. You should not forget looking at this function how it is behaving.

Now, r 1, r 2 to the power half is a real number, so it does not create problems if the problem comes from this angles. So, here I get i above; here I get i, now if I look at just below, a point below over here, then from z to z 0, that phasor looks like this, just below. So, what would be that angle, theta is pi here again, theta is equal to pi here again, right here, but what about the angle for z plus z 0, this phasor has this position, but phi is measured from the x axis above.

So, we go around full and come back here, so phi is equal to 2 pi; phi is equal to 2 pi at this location. Then what do I have for this, e to the power i theta, plus phi. I have pi plus 2 pi by 2 which is equal to e to the power i 3 pi by 2 it is equal to minus I, ok.

So, I come from above, towards x axis I get i, I come from below towards x axis I get minus i. So, the function takes a jump, this function takes a jump along this line. Now, let us look at it here; let us look at it here, here I am here let us say just again just about the x axis, what is theta here; what is theta here? Theta is with respect to z minus z 0, so theta will be here. So, if I draw this line to z 0, theta is again pi, ok.

So, this angle is pi, what about phi? Phi is measured from here and phi will also be pi ok, what about just below here, not much change, just below here, this is the phasor from z 0, so that is theta is equal to pi again and what about from minus z 0, that is also pi again ok. So, here pi and pi, below pi and pi. So, you get e to the power i pi plus pi by 2, e to the power i pi, which is minus 1 and minus 1 here. So, there is no problem approaching from here we get minus 1, approaching from below we get minus 1, no problem. This is the only place where we have a discontinuity in the function, ok.

So, if we look at; if we look at that again, let us see look at it again here. So, here is z 0; here is minus z 0 and I am not allowed here ok, but here I am allowed; here I am allowed and here I am allowed. So, if there is some closed contour integral, so I cannot start over here go around and come back and join up because this line is prohibited. Similarly, suppose I decide to go around here ok, up till here the function has one value you cross this line the function jumps the value, so I cannot go around in a circle like this.

So, this line is prohibited. This portion is prohibited when doing closed contour integrals. I can go around here does not matter or I can even go around here does not matter, but I cannot cross the cut over here. It can go around this way does not matter, but I cannot do this I cannot cross the cut over here. So, that is a finite branch cut ok; that is a finite branch cut.

Now, I will just give you a picture, some pictorial view; pictorial view of the infinite branch cut for square root of z. It just occurred that it could have a pictorial view. So, let us see, so let us see, I have a, let us say I have a sheet; I have a sheet and I have another sheet. Just hold on. I have a sheet over here, let us say which comes up here and then I have another sheet which goes here. I hope it is somewhat clear, I do not want to complicate. Let us say this goes from top this is here and this is here.

So, if you look at it sideways, this is one top paper top sheet and this is bottom sheet. This a top sheet, this is a bottom sheet, ok. So, now here is the origin, here is or somewhere here is the origin. Let us say somewhere here is the origin, ok, let me know let me keep this as the origin. So, now, for square root of z, you start over here you go around, you go around on the top sheet, by the time you reach this dotted line, you have switched sign, ok, if you recall for square root of z, if you started; if you started over here it came around full circle, you switch sign, that is why this line is a branch cut, ok.

So, you come back here, to this point you switch sign, ok, but here, we have a cut actually, ok, we have an opening; we have an opening. So, what we do is we enter the opening; we enter the opening and now we are on the bottom sheet. So, for example, we go around once, we come back here and here we enter the bottom sheet. And once you go around the bottom sheet, so you have entered the bottom sheet, this is the lower sheet is upper sheet, the lower sheet. So, now, you go around, through the opening you enter the lower sheet and once you go around and come back over here, then through the opening you come back to the upper sheet.

So, you go around here on the lower sheet once and by the time you are here you enter the upper sheet. So, this way you enter the lower sheet and this way from the lower sheet you enter the upper sheet. So, the square root of z function has divided the complex plane into 2 sheets, ok. On the first sheet it is continuous, so if you remain on the first sheet you are continuous, else you enter the bottom sheet, you can remain in the bottom sheet then you are continuous, but if you want to come back and touch this line, you will enter one sheet or the other, ok. So, because you have square root, twice you go around, you come back get back the same sign, ok.

Once you go around you switch sign to minus 1; second time you go around you again switch sign, you get minus 1, so product of these two gives you 1. So, two rounds you go you get back the same sign. So, that is how the complex plane now is broken into sheets by square root function and so its called a branch cut, so you have actually a cut, ok. So, you start over here, you go around now you are on the bottom sheet, on the bottom sheet you go around come back here, you enter the cut, you are on the upper sheet, ok, the same thing is happening over here also, ok.

Now, for a certain choice of theta and phi limits; theta and phi had these limits 0 twice pi. May be the next class I will show you actually a cut out with 2 pieces of paper, then you will see this part better, see how the cut looks. So, now, getting back we had started and assumed a certain limits for theta and phi and that gave us a cut in the middle, ok. Now, let us see if we take the definition, let us give the definition, theta between 0 to 2 pi and phi between minus pi to pi let us see what happens, ok.

So, now I have again the complex plane, here is my z 0, here is my minus z 0, here is my minus z 0 and my z is over here somewhere. So, let me again make it light as there will be a lot of writing here. So, z is here and this is my z minus z 0 and this is my z plus 0 that is over here. Again we measure theta 0 and phi measures minus pi, phi is minus pi, phi is 0 here and phi is pi over here.

So, let us we will be looking at the same behavior of square root of; square root of z square minus z 0 square, which is equal to r 1, r 2 to the power half, e to the power of i theta plus phi by 2, but theta, phi have these definitions now, ok. So, we will examine the function here; here and here, just as we did before, with respect to square root of z square minus (Refer Time: 30:44) square. The time is running out, so I will close here, we will continue from here next class.

Thank you.