A short lecture series on Contour Integration in the Complex Plane
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Lecture - 13
Infinite and finite branch cuts

Hello, good morning. Welcome to this lecture on Complex Variables. Last time if you recall, we were doing a problem, which had an integral going from 0 to infinity and we have gone very far, we are at just to the last step, ok.

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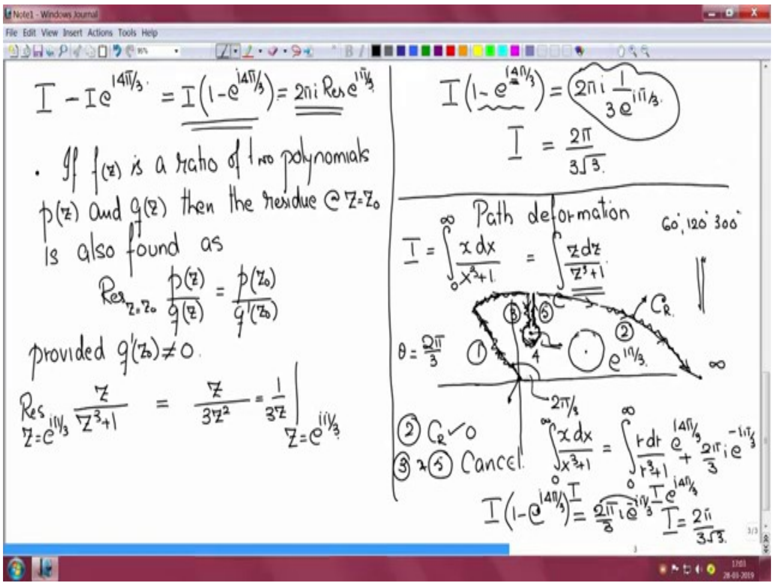

So, we have I times, 1 minus e to the power i 4 pi by 3 is equal to this residue contribution on the right-hand side. Now, a word about residues; in your complex variable classes you know how to compute residues for standard functions, but here is another way, ok.

If f of z is a ratio of two polynomials. What are the two polynomials? p of z numerator and q of z in the denominator, ok. Then the residue, the residue at z equal to z naught is also found; is also found as, residue at z equal to z naught of p of z, by q of z, is equal to p of z naught, by q dash the derivative, q dash of z naught, ok. Provided q dash of z naught is not equal to 0. It is another very useful way of computing residues. So, now what do I have, I have p of z is z and q of z is z cube plus 1, ok. And I have to evaluate my residue, where? residue at z equal to e to the power i pi by 3, ok. So, I first do is z and I take the derivative of the denominator, 3 z square, ok, which actually gives me 1 over 3 z and I evaluate at z equal to e to the power i pi by 3, ok.

So, see what happens here. I have I ,1 minus e to the power i 4 pi by 3, that is equal to twice pi i times the residue, which is 1 over 3 e to the power of i pi by 3, ok. If you compute this, you will not spend too much time organizing this; if you bring this in the denominator and make some arrangements and calculate. The answer is twice pi by 3 root 3, ok. You can take out e to the power i 2 pi by 3 common, then you get e to the power of minus i 2 pi by 3 and so forth, then you can calculate it, ok. So, that is one way of doing it, ok.

Now, what we are going to do is, we are going to do again the same problem using path deformation, ok. I am showing you the advantage or magic of path deformation, ok. So, I have this integral, I let us say going from 0 to infinity, x dx over x cube plus 1, ok. So, I will replace this because I am going to go into the complex plane, I will replace this with integration over some contour, not necessarily closed now and z dz by z cube plus 1.

Now, I know the singularities of this function where this blows up. So, these are given by, at 60 degrees, 120 degrees and 300 degrees, ok. So, as long as I do not touch these singularities the poles, as long as I do not touch these poles, so my starting integral is from 0 to infinity and my one singularity is at e to the power i pi by 3. Let me make it bring it closer. My singularity is at e to the power i pi by 3.

So, what now I will do, let me shift it, what now I will do is my singularity is I will deform this path, I will hold the edges fixed; this point I will hold fixed, that at infinity I will hold fixed. I will deform the path. How do I deform the path? I will deform the path a little bit ok; a no singularities, I am not touching any singularities not crossing any and this is a well behaved function away from its singularities, analytic and then I move further along the straight line; again I deform, I move along the straight line again I deform, ok.

And this angle is the same angle as before, theta equal to twice pi by 3. This angle is twice pi by 3, ok. So, I come here now. So, it is as though the string is being stretched or this path is being, path is like a rubber band and its being stretched, ok. So, I come over here be the stretched path is over here. I come down ok, but now I cannot cross the singularity I have to go around it. So, that singularity is like a nail, if my path is like a string the singularity is like a nail that is jutting out of the board and so I cannot cross it, so I come up.

So, the path is like this. So, I come down, I go up; so, this is the stretched deformed path. This is the semicircle, semicircle ok. So, the stretched path is like this. The original path was like this, ok. The stretched path is like this, I come down, I go around, I come up and then go and meet at plus infinity, ok.

For clarity if I, if you want me to remove the unnecessary paths ok, the unnecessary paths; actually I can remove this also except that let me keep the axis, real axis of the complex plane. So, this is the path, ok. The darkened portion is the path and it happens to be in the well-behaved region of the function. So the functional value should not change according to our theorems, ok.

Now, this is over CR. Its an arc with radius infinity the degree of polynomial theorem comes in, so on this arc the integral 0, ok. So, let us keep track of number of portions. So, I have this portion and I have this whole portion which I will call CR, which is portion 2 let us say. Then I have portion where I come down, portion 3, I go around portion 4, ok. Here I am going anti clockwise, then I go up portion 5, this is the 5 portions. The portion 2 on CR goes to 0, using the degree of polynomials theorem, z cube and z they differ by degree of 2 at least, ok, so this CR goes to 0.

Then I have 3 and 5, 3 and 5, ok. 3 and 5 are in a region where this function is well-behaved, the only singularity in our view is this point here 3 and 5 are away from it. So, there in that region I have a downward coming integral and there alone I have an upward going integral. So, they cancel out. So, 3 and 5 cancel out, 3 and 5 cancel out, ok. So, what am I left with? I am left with this integral 0 to infinity, x dx by x cubed plus 1, is equal to the portion 1, which is actually going in the positive direction.

So, this is I will write it as r dr by r cube plus 1, 0 to infinity, e to the power i 4 pi by 3, plus I am going counterclockwise around a pole, ok. I am going here counterclockwise round a pole at e to the power i pi by 3. So, I will have a residue contribution, ok. So, that residue do contribution is already there up here, which is twice pi by 3, i e the power of minus i pi by 3.

So, if you see I have acquired the same equation. So, this is i into i 4 pi by 3. So, if I bring it on to this side, this is also I, so I get i into 1 minus e to the power of, e to the power i 4 pi by 3, is equal to this residue on the right-hand side; twice pi by 3 i e to the power of minus i pi by 3. And therefore, as before, I is equal to twice pi by 3 root 3, ok. So, you can see the power of path deformation, ok, its a very powerful technique.

If you feel safe; that means, if you do not feel very confident with path deformation, the safest is a closed contour, ok. It is safe to take a intelligent closed contour and do the problem. Once you gain some insights and some confidence, then, you can move to try problems using path deformation, ok.

Let us move to the next idea here. So far, the problems we have done are quite standard, ok, here and there in text books on complex variables you will see, ok. Although, in any one particular textbook you will not see too many, ok. So, these are quite standard problems.

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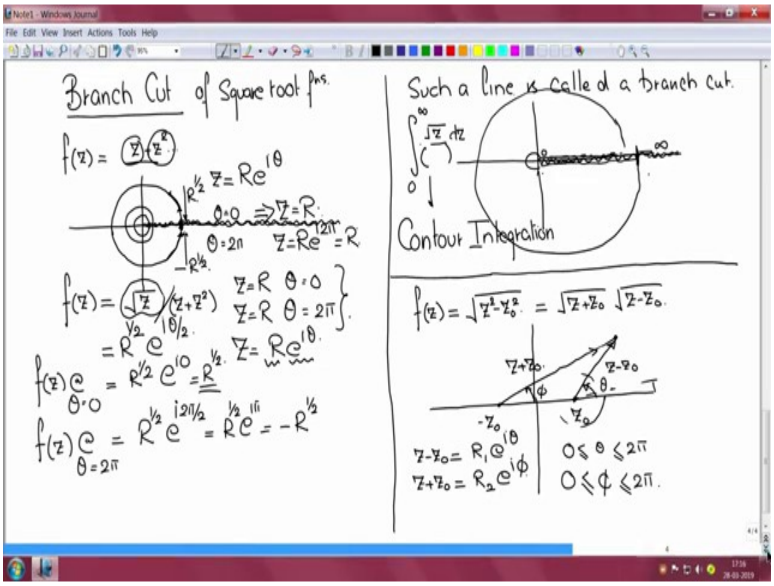

Now, we move to those problems where difficulties arise and where the regular classes do not emphasize, ok. So, we examine branch cuts; we examine branch cuts. What is a branch cut? And we will deal, because of the short number of lectures that are going to be offered, we will look at one particular type of function which brings about a branch cut; square root functions; branch cut of square root functions, ok.

Now, suppose I have a function of z, which has normal zs and z squares etc. everywhere whatever and I think of a contour, closed contour, let us say I want to start over here and I want to go around a circular arc of some radius and I come back over here, ok. On this circular arc, z can be parameterized as R e to the power of i theta. So, let us say I measure theta, theta equal to 0 over here and theta is equal to twice pi over here. I start at theta equal to 0 and I come back here theta equal to twice pi.

Then, at theta equal to 0, z is equal to R e to the power i 0 and hence R. And at theta equal to twice pi, I am right here, right below the real axis, z is equal to R e to the power of i twice pi, which is R, ok. So, z is continuous. I go around I come back to the same point, same physical point in z and z also has the same value here R and R. Now, if this function has all zs and z squares and so forth then all of these functions also will acquire the same value at this physical point, when the function goes around the arc, circular contour and comes back down here, the function also acquires the same value, ok.

But imagine f of z, in the simplest case has square root of z, or z to the power half, the function is that, ok. Now, as before, the circular movement of z brings me back to the same point z is equal to R when theta equal to 0 and z again is equal to R when theta is equal to twice pi, there is no problem, ok. Now, watch, z is being parameterized as R e to the power of i theta, ok. So, if I now look at square root of z, this is the amplitude and this is the phase. So, square root of z will be R to the power half e to the power of i theta by 2, ok.

So, now, what is the function now? This function f of z at theta equal to 0, this function f of z at theta equal to 0 is equal to R to the power of half, e to the power of i 0, which is R to the power of half, a positive number, ok. Now, z goes on this circular contour and comes back to the same point f of z, at theta equal to twice pi, where z acquires the same value. What is the function? The function again; R to the power half e to the power of i theta by 2, theta is 2 pi, 2 pi by 2, and hence this is R to the power half, e to the power of i pi and hence, minus R to the power half.

So, the moment you have square root in this function, maybe with along with other forms, ok. So, let us say the denominator has z plus z square and so forth, but the moment you have square root of z in the function, that part of the function takes a jump. You go around z and come back to the same point when the function takes a jump. It is you approach from the top it is R to the power half, you approach from the bottom it is minus R to the power half, the function takes a jump, ok. And therefore, the function is not uniquely defined at the same point the function has a jump. So, it is not uniquely defined. You approach from the top its R to the power half, you approach on bottom minus R to the power half.

So, with respect to square root you can see that a circular contour of any size will have a problem along this line. Along this line if you have square root of z in the function, the function will have a jump along this line from 0 to infinity, such a line is called a branch cut, ok. Such a line. What does it do? It separates the function into various planes where the function has a unique value, a continuous unique value, such a line is called a branch cut, ok.

Now, what does it mean to have a branch cut? It means that suppose I would like to do an integral, a real line integral that goes from let us say 0 to infinity, I have some integral that goes from let us say 0 to infinity and it has some functions and there is a square root of z in there and other things here. And I am going to use principles of contour integration to evaluate it, I am going to use contour integration to evaluate it, I need a closed contour, ok.

So, now with respect to square root of z this line cannot be crossed in trying to form a closed contour. So, one of the contours I will tell you right in the beginning, you may go above this line to infinity, go around at infinity come back down, come from beneath at infinity. Now, I cannot cross this line ok, so I go along the cut, I go around the 0 and join again. So, that is how I form a closed contour, this is again a closed contour and I have to avoid the cut I cannot cross the cut, ok.

So, we will see now the next level, I mean this branch cut extends between 0 to infinity, the next type of branch cut is between two finite points and called a finite branch cut. So, let us say I have some function which has a square root of z square minus z 0 square, which is equal to square root of let us say z plus z 0 and square root of z minus z 0, ok.

So, let us examine what sort of a branch cut we will get if we have this sort of a function. This is the complex plane, let us say my plus z 0 is over here and my minus z 0 over here minus z 0 is over here and let us say my z is some point, ok. So, this is the phasor z minus z 0 and this is the phasor or vector z plus z 0, ok. So, let me define z minus z 0 as some amplitude R 1, e to the power of i theta, ok and then z plus z 0 and some other R 2, e to the power i phi, ok.

For now, I am choosing the limits of theta and phi, let me say 0 less than equal to theta less than equal to twice pi and similarly 0 less than equal to phi less than equal to twice pi ok. So, we measure 0 here and we go around and we come back to 2 pi, for both theta, theta is this angle, phi is this angle, this angle is phi, this angle is theta, ok.

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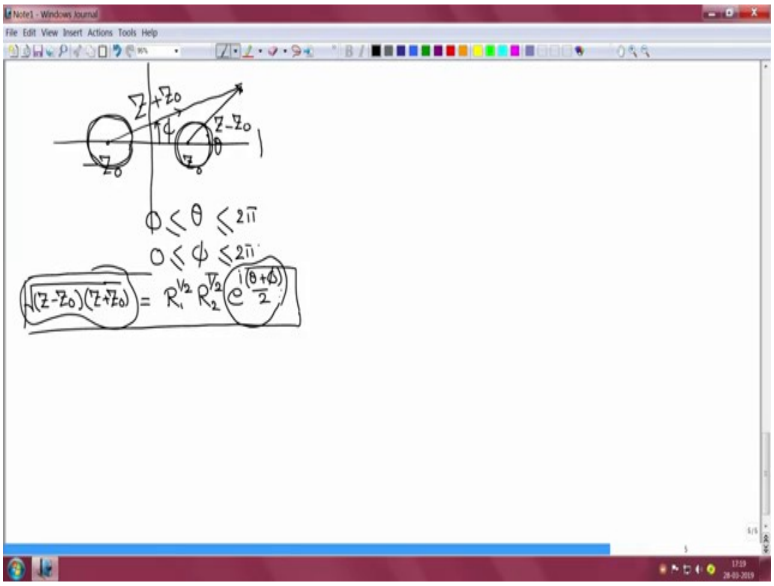

Now, so I have this picture over here. This is z 0, here is minus z 0, here is a z, this is z minus z 0 theta measured here, this is z plus z 0 phi measured from here ok. Theta limits are 0 twice pi and phi limits are the same twice pi.

So, now, square root of z minus z 0 into z plus z 0 gives me R 1 to the power half, R 2 to the power half, e to the power i theta plus phi by 2, ok. So, this function now, this function now should be single valued, this function should be single valued in the region that I want to cross or in the region that I want to form circular arcs. We want to form circular arcs in various places. So, in forming these circular arcs I should make sure that the function comes back to the same value, wherever I start, ok.

So, we will continue in the next class. We will see that this will have problems. You cannot cross the real axis wherever you want. We have to examine the continuity at this complex argument, ok. So, we will do that in the next class. We will start from here, we will continue ok. This is a finite branch cut.

Thank you.