A short lecture series on Contour Integration in the Complex Plane
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Lecture - 12
Method of path deformation

Good morning, welcome to this next lecture on complex variables. We have started doing examples in Contour Integration and we have done a few. In the last class we were doing this particular problem.

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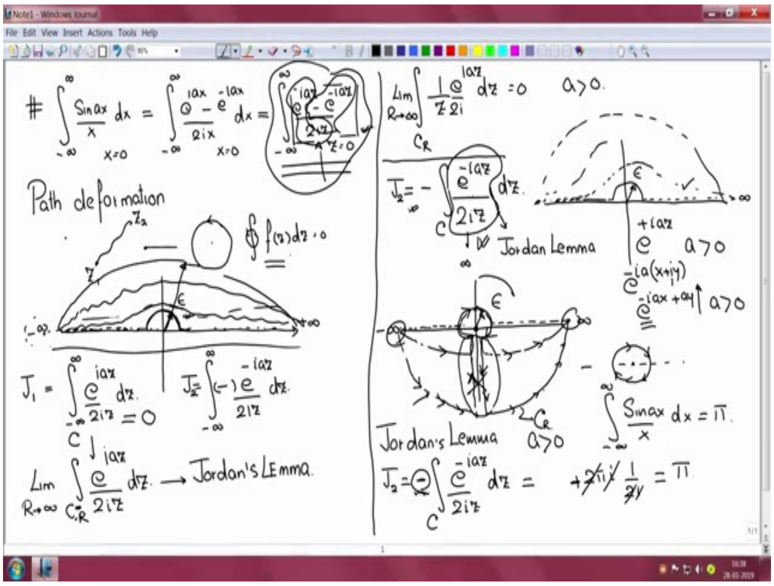

Integral minus infinity to infinity, sin a x over x dx; we have done this using a few methods and this is the last one which I said is based on path deformation, ok. So, this I will write as integral minus infinity to infinity; e to the power i a x, minus e to the power of minus i a x, over twice i x dx. And this will be written as minus infinity to infinity, on the real axis; e to the power i a z, minus e to the power minus i a z, by twice i z.

Once we stay on the real axis; all three are equal and here neither x equal to 0 is a singularity, nor z equal to 0 is a singularity, in the way we have formed it, ok. Now I was mentioning path deformation; I will repeat that. Our first theorem said three statements and if any one was true, the others were true. So, one of the statements was that integrals between two points are path independent. We integrate from z 1 to z 2; the integral is path independent and that is equivalent to saying that integrals round closed contours are 0, ok.

Integrals of f of z; round close contours are 0. Now we have theorems which say that if a function is analytic on an inside a curve, integral is 0. And we also have theorems which say if in a simply connected domain this following integral is 0, then f of z is analytic everywhere, ok. So, now if f of z is analytic in that region you will have integrals of closed contour being 0. And from the first theorem, integrals between 2 fixed point points being independent of path, ok. Now this function is such a function; unless we go to infinities, this function is well behaved.

So, we are going to consider that integral starting from minus infinity; we move along the real axis, there is no problem at z equal to 0 and we should be moving straight away. But what I propose is that I will hold the edges fixed; just as we do in this case z 1 and z 2, and I consider the following path which comes from minus infinity, but around 0, it takes an epsilon radius path and again proceeds to infinity. I can do this; why? Because the function is analytic in the region we are considering, ok.

Why? Because that is equivalent to sin a x over x; sin a x over x is a well behaved function, ok. So, I consider that this integral, this particular integral will be same whether I come from minus infinity and go straight to infinity or I take a slight detour around x equal to 0, with a radius epsilon. So, now I consider, now I break up, I will consider J 1 which is this integral, minus infinity to infinity, ok e to the power of i a z by twice i z d,; along this contour, along the contour I have found, ok.

And then I will also consider J 2, so we will have to make it C because it is not a straight line, we will make it C. I will also consider this integral, minus infinity to infinity, now with a minus, ok; with a minus, e to the power minus i a z over twice i z dz, again along the same contour. The sum of these two should equal this integral that is the contention; why? path can be deformed, ok.

Now let us look at J 1 ok; now what I do with J 1 is that I will continuously deform this path; I will continuously deform this path. And I can do so till I hit a singularity and I will finally make it; make the path a circular arc of radius infinity in the upper half plane. So, this integral is now over C R; C R e to the power of i a z by twice i z d z, ok, as limit R goes to infinity, ok. Now you can watch that the conditions of Jordan's Lemma apply to this; conditions of Jordan's Lemma apply to this. We have a function 1 over z which uniformly goes to 0, as the arc circular arc acquires the radius R infinity. And hence e to the power i a z is, 2 i just a constant; integral over that circular arc, limit R tending to 0; I mean sorry infinity dz must be equal to 0 and a is positive; a is positive.

So, this part is 0; one part is 0, one part is 0, ok. Now the other part J 2, ok; J 2 has a negative sign in it, J 2 is the negative sign. So, let us keep it minus I again consider this arc C, e to the power of minus i a z by twice i z d z and I start with the same arc; I start with the same arc I considered or same path I considered; which is coming from minus infinity, taking a detour, epsilon radius and moving forward to infinity.

Now, if I start deforming this path in the upper half plane and I move towards infinity ok; what happens is that this form does not suit the Jordan lemma; this form does not suit the Jordan lemma. Jordan lemma has e to the power plus i a z with a positive; if I have e to the power minus i a z and z is x plus i y ok; then this i and minus i gives me a positive. So, I get e to the power of minus i a x plus a y and if y goes to infinity; this function will blow up for a positive.

So, it does not fit the Jordan lemma and in fact, in the upper half plane this will go to infinity; this goes to infinity, ok. So, then what do we do? What we do is we deform the contour, so that means, that means I am deforming the contour, till I hit a singularity; that is why the function went to infinity. So, remember that in deforming paths I cannot go and touch a singularity till I touch a singularity; I am allowed. So, the upward deformation of this path is not allowed, ok. So, let us look at it now; we will take this path, starting path with an epsilon detour around the origin, but now I deform downwards ok; now I deform downwards.

But remember when I deform this path downwards; suppose I go like this come back like this and join; I could go like this come back like this and go off to infinity, the two endpoints being held. I cannot cross the 0 over here; that means, my path cannot cross the 0 over here. Why? Because when I separate this integrand out from the total, ok; from the total when i separate it out, here z equal to 0 was not a problem ok; it is a removable singularity.

But here z 0, z equal to 0 is a pole is an isolated pole for this individual part ok, the J 2 part, z equal to 0 is an isolated pole. So, if I deform this path downward; I cannot cross it.

Those are rules of path deformation. So, now what do I do? I deform, I take this to infinity; from infinity I come down straight, I go around the origin, go back up straight to infinity and then close this; this is my C R in the lower half plane.

Now, again on C R, Jordan's Lemma is applicable; Jordan's Lemma is applicable for a being positive, the downward negative half plane is applicable, ok. So, the integral on this part goes to 0. This integral, this integral that we are doing in the deform path that has gone into the lower half of the complex plane, the C R part is 0. And if function is well behaved everywhere; so this path, so this was the path I was taking. So, now this is the path I am taking; I go down here, go around the pole; come back up here and go to infinity; keeping these endpoints fixed.

Now, C R is 0, the function is well behaved everywhere except at x equal to 0. So, this upward integral and downward integral are in the region where the function is well behaved and analytic; so this also goes to 0. So, this integral J 2; let us say we keep the minus over this contour, e to the power minus i a z by twice i z d z, ok. The minus portion we will include later. This becomes equal to now. So, what is left in this integral; the integral is just going around the singularity at z equal to 0. That is all that is left.

So, that is equal to twice pi i times the residue; residue is 1 by twice i and we are going in the counterclockwise direction. So, that is this part is ok and but there is a minus in front, sorry this is clockwise my mistake; my mistake, this is clockwise, this clockwise, anti clockwise does get confused; so this is clockwise. So, we are going down the pole in the clockwise direction 1, 2, 3, 4. So, the residue first of all is negative; first of all the residue is negative, then there is this negative sign in front; so that makes it positive.

So, I get pi over here, I get a pi over here. So, now what? The first part gave me 0; this part gives me 0 and this part gives me a pi. So, that finally, my integral minus infinity to infinity sin a x over x dx is equal to pi, as before. So, we have done this problem; a very standard problem in textbook using four different methods or in four different ways, ok. So, now we will continue we will take up a next problem, next problem.

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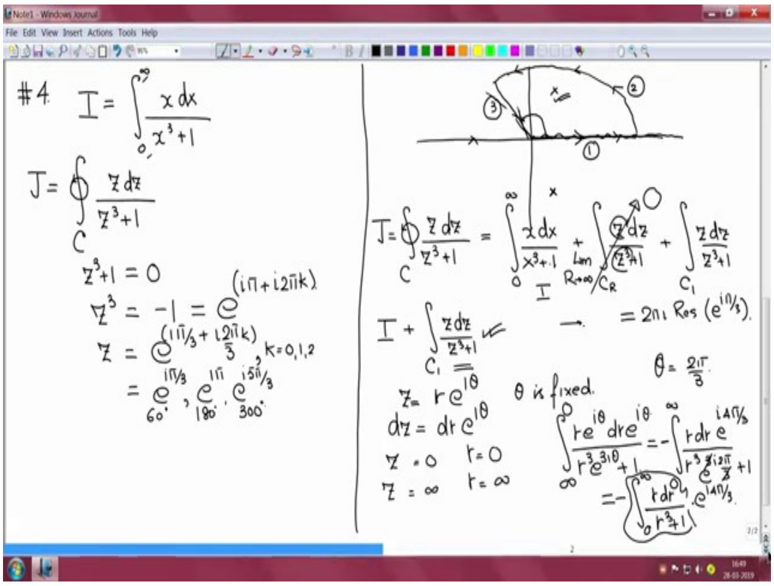

I have lost the count; let us call it the fifth problem or maybe the fourth problem. It is this, I have a real valued integral, I is equal to integral 0 to infinity x dx over x cube plus 1, ok. So, we are going to apply contour integration to this, ok.

So, as before we replace this with J and an integral over a closed contour; taken in the positive sense, we will call that C; replace x with z. I have z dz over z cube plus 1. Now, let us first look at the singularities of this function; poles of this function. So, I have z cube plus 1 is equal to 0 or z cube is equal to minus 1, which we write as e to the power i pi, plus i twice pi k, ok. So, that now my z ends up looking like e to the power i pi by 3, plus i twice pi by 3 k; k going from 0, 1, 2.

So, this gives me what? For k is 0; I get e to the power of i pi by 3, there is a pole there, ok. For k equal to 1, I get e to the power i pi and the next one is e to the power i 5 pi by 3, ok. So, this is actually 60 degrees; this is 180 degrees and this is 300 degrees; you want to look at it in terms of degrees; so let us see that. So, I have this complex plane and I have a pole at 60 degrees, I have a pole at 120 degrees and I have a pole as at let us say 300 degrees ok; 120 degrees apart, I have poles, ok.

So, now I have this integral it has to be done from 0 to infinity, ok; I have to move from 0 to infinity; that is a must that the limits here I have. So, what is the contour I choose? I choose the following contour, ok. I move from 0 to infinity, ok; then I take a circular arc, we will consider this to be circular; I come up till a certain point at infinity and come down to the origin; so, the direction is this, ok. So, the closed contour includes one pole which is e to the power i pi by 3; so let us see now J. So, there are three portions, there is one portion, there is second portion, there is third portion ok.

So, J is equal to integral counterclockwise over path C z dz over z cube plus 1, which is now equal to the portion I want; integral I write it straight away 0 to infinity x dx by x cube plus 1, basically I, plus an integral over a circular arc C R with limit R going to infinity, ok. Let me say z dz by z cube plus 1 and then an integral along a straight line, let me say some C 1, along C 1 which is actually a straight line from infinity z dz by z cube plus 1. And we are including one pole. So, there will be a residue coming; twice pi i times the residue at e to the power of i pi by 3, ok.

Now, straight away by using the degree of polynomials theorem, I have a numerator polynomial of degree 1, denominator of 3; there is at least a difference of degree 2, ok. So, this goes to 0 as R tends to infinity; this goes to 0. I am left with 2 pieces. So, one piece is my I; let me call it I, the other piece is along some arc which is a straight line z dz over z cube plus 1, ok; that is equal to the residue, ok.

Now, let us look at this particular term, let us look at this term. So, we are going to come down a straight line, ok. So, let z along that straight line be some r e to the power of i theta, because it is a straight line now; theta is fixed, theta is fixed, ok. Then dz; dz is equal to here theta is fixed; so it is dr e to the power of i theta. When z is equal to 0, r is equal to 0 and when z is equal to infinity r is equal to infinity, ok.

So, now if we put it in terms of r and theta, my integral, this integral is given by integral, z is r e to the power of i theta, dz is dr e to the power of i theta, divided by z cube which is r cube e to the power of 3 i theta plus 1, ok. Now here is a clever choice of theta, there is freedom in theta, I want to include only one pole. So, there is which choice of theta I have; I could come this way, I could come that way. So, here is the choice of theta, if I choose theta is equal to twice pi by 3, 120 degrees; if I choose. Look at what happens here.

This ends up more; more important the limits I am coming from z equal to infinity z equal to 0. So, I am going to come r equal to infinity to r equal to 0 ok. So, I will; so let us look at theta equal to twice pi by 3 and at the same time I will switch the limit. So, I have minus integral 0 to infinity; I have r d r; e to the power of i 4 pi by 3; divided by r cube e to the power 3 i theta; e to the power 3 i theta, twice pi by 3 which is e to the power i twice pi; which is 1; plus 1.

So, how does this look? This looks like minus, integral 0 to infinity; r d r by r cube plus 1, into e to the power of i 4 pi by 3. So, you see this part is my I; this part is my I, this part is my I, x dx by x cube plus 1, 0 to infinity is my I; r d r by r cube plus 1; 0 to infinity. So, r is just the dummy shift ok; so this is I. So let us see what happens.

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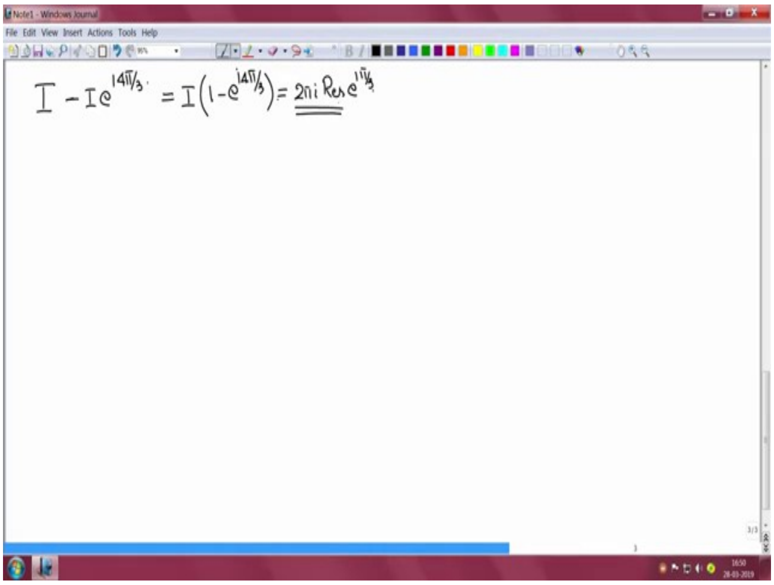

So, this ends up looking like, minus I e to the power of i 4 pi by 3, ok, but I have I in front, ok. So, I have the total equal to I into 1 minus e to the power of i 4 pi by 3; that is the left hand side, ok. So, we need to compute the residue now; the right hand side is residue, twice pi i times residue at e to the power of i pi by 3, ok. We will do that in the next class; time is running out, we will do that in the next class; we will start with this, ok.

Thank you.