A short lecture series on Contour Integration in the Complex Plane

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Lecture - 11
Method of path deformation

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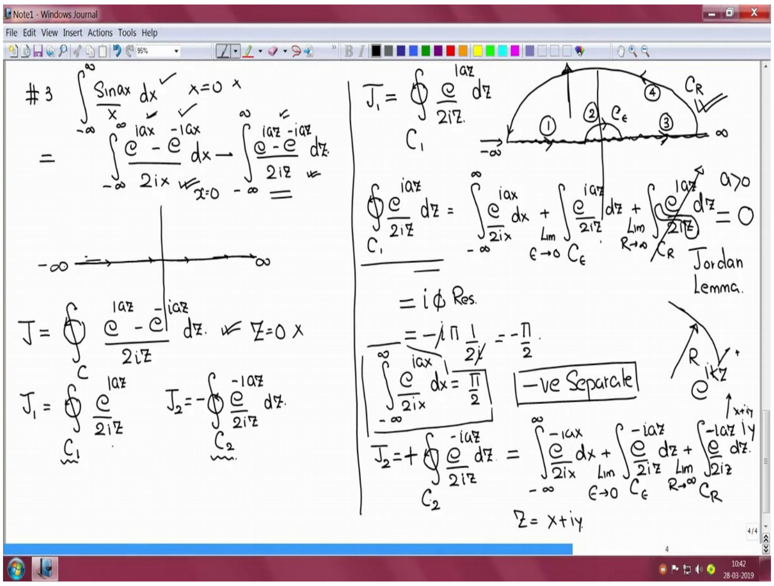

Good morning. Welcome to this next lecture on complex variables and we were looking at this problem: integral minus infinity to infinity, sin ax over x. We have solved these using two methods ok. So I am going to show you another method over here. So, this has been written in that exponent form, which has been written, rewritten using the complex variable z, ok. Now this is the form we will use for the contour integration.

Now, here, in this form, x is, x equal to 0 is not a singularity. It is not a problem and this being identical to sin a x or x here also there is no problem at x equal to 0. Same as here; here also there is no problem, ok. Now, if we begin the problem using this, a closed contour integral C, e to the power i az, minus e to the power of i az, over twice i z dz, ok. We can now for a, we can go through the singularity. There is no problem here with z at z equal to 0. So, we can move straight and go through z equal to 0.

But what we are going to do is because there is a you know subtraction over here, we are going to break it up into two pieces. So, I am going to let us say break it into J 1 which is integral, e to the power i a z by twice i z. And let us say J 2, which is equal to minus integral contour C. So, we will give different contours also C 1 and C 2, e to the power of minus i az by twice i z dz, as long as that this portion is included; as long as this portion going from minus infinity to infinity is included, we are allowed to keep two different contours, ok. So, let us see with each what happens.

So, let us look at J 1. J 1 now is equal to integral C 1, e to the power of i az by twice i z dz and in the contour we are supposed to go along the real axis and create a contour. But, the problem is that, in the combined form z is not a, z equal to 0 is not a problem, ok; however, once we separate, z equal to 0 is a singularity a simple pole for both functions.

So, if we have separated then we can no more go straight like this. We have to form this epsilon contour; we have to create a C epsilon contour; C epsilon contour. So, the contour I choose for J 1 is you come from minus infinity, I come from minus infinity around z equal to 0, I go on C epsilon, I go to plus infinity and then I go semicircle and join back. It is a counterclockwise contour. So, here I have C R again I have portion 1, portion 2, portion 3 and portion 4, ok.

So, I have integral, 0, integral over a closed contour C 1, e to the power i az by twice i z dz and I will say that first in third portions give me what I want, the first and third portions give me minus infinity to infinity, e to the power i ax by twice i x dx. I can write it that way, ok, then I have a C epsilon portion, ok. So, limit epsilon tending to 0, the C epsilon portion, e to the power i az by twice i z dz and the C R portion, limit R tending to infinity, C R, e to the power i az by twice i z dz.

Now, the C R portion, by Jordan lemma goes to 0. We should get used to this without repeating the theorem. A function 1 over z, uniformly goes to 0, as a circular arc grows in its radius, And therefore, the function f of z, into e to the power i az dz integrated on the R goes to 0. So, C R goes to 0, for a positive that is important, a being positive. Now, we have C epsilon contour; C epsilon contour, the second C epsilon theorem says that if you have a simple pole at z or z we have, i times phi times the residue and taken in the positive sense.

So, we are going again in the clockwise sense, so there will be a minus. So, this is going to be equal to, minus i times pi and the residue is multiply this by z equal to 0, set z equal to 0, that is 1 by twice i, ok. So, this is equal to, minus pi by 2 and we are not including any pole in our calculation. So, this right hand side, no contribution from residues, so this goes to 0, ok. So, what I now have is, integral minus infinity to infinity, e to the power i ax by twice i x dx, is equal to pi by 2. This is one half of that problem, this part, ok.

Now, for the next part J 2; J 2 we promised as a minus a closed contour perhaps different, e to the power of minus i a z by twice i z dz, ok. Now, if we write it in this same fashion; if you write it in the same fashion, I will have, I will keep this negative separate, ok. So, we should remember that negative sign we keep separate. We will add it later.

So, without the negative sign if I look at this, so let me make it plus without the negative sign, then I get again minus infinity to infinity, e to the power minus i ax over twice i x dx. And plus the C epsilon; C epsilon, limit epsilon tending to 0, I have kept the same contour. I am looking at the same contour; same closed contour. So, C epsilon, e to the power of minus i az by twice i z dz, plus integral over C R, limit R goes off to infinity, e to the power of minus i az by twice i z dz.

However, here because I have a positive and I have a negative in front, this is different from Jordan lemma. In Jordan lemma, we had e to the power i k z whatever i z be, k was positive. Here, I have an extra negative sign. So, if I look at what happens as z goes to infinity, as x plus iy goes to infinity, you can see that z being x plus iy, z being equal to x plus iy, ok.

So, I have an iy here, this i and this I, i square is negative and this negative become positive. So, it y goes to infinity, so I may move further and further above the top half plane, the function blows up, ok. So, Jordan lemma will not work in the upper half plane, ok; however, it will work in the lower half plane. So, we will take it into the lower half.

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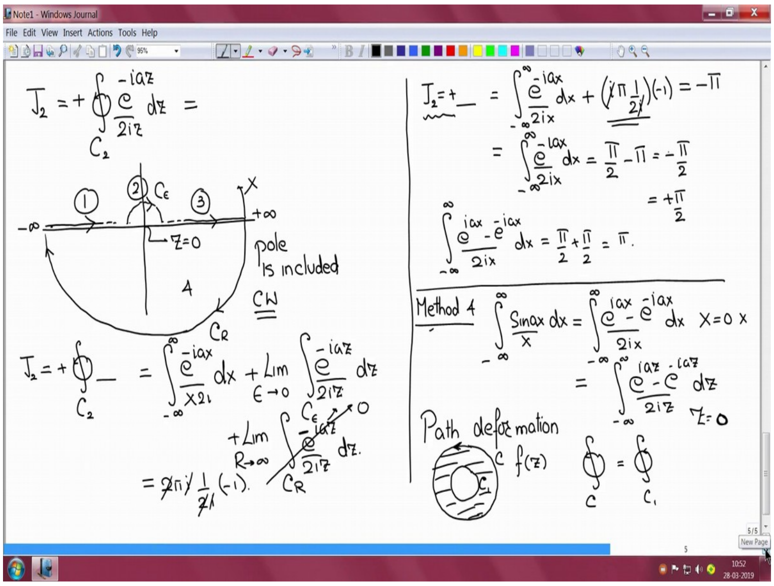

So, what now we have is, J 2 is equal to I said we will use plus for now, integral over a different contour, e to the power minus i az by twice i z dz, ok. In the contour now, I am going to choose is this. The portion from minus infinity must be there, going off to plus infinity, otherwise we do not have what we need. And, so we come from minus infinity on the real axis, we circumvent the pole using C epsilon, move forward to infinity, but I am not allowed to go this way. If I go this way Jordan lemma does not help me. So, I go this way; I go this way and I go this way, the pole gets included, pole is included; pole is included in a clockwise counter.

So, the residue contribution will be negative from z equal to 0, ok. So, the first and third portions, now I have 4 portions 1, 2 the C epsilon, 3 and 4 which is C R, ok. So let us write it. So, J 2 with a plus sign, plus you should not forget there is a minus sign earlier, integral over this contour C 2, whatever it is equal to the in the portion I want which is minus infinity to infinity, e to the power minus i ax over x dx plus, let me see, limit epsilon tending to 0, C epsilon e to the power minus i az twice ix, by twice i z dz, ok.

Now, I have the C R portion: limit R tending to infinity integral over C R, ok, e to the power of minus i a z by twice i z dz. Since we have moved to the lower contour, lower half plane, lower half complex plane, the Jordan lemma again again works, i being, a being positive, provided if we have a minus the same integral in the negative half plane gives a 0. So, this goes to 0 now again based on Jordan lemma ok. Now, that is equal to the residue, twice pi i times the residue is z equal to 0.

So, setting z equal to 0, I mean canceling z, setting z equals 0, I get a 1, 1 by twice i; 1 by twice i, but we have gone round the pole in the clockwise direction. So, there is a minus one ok. So, we still have C epsilon to do, which is not a mystery for us now, so what I have is J 2, with a plus sign, is equal to integral minus infinity to infinity, e to the power of minus i ax by twice i x dx, plus what goes with C epsilon, let us evaluate C epsilon, there is a pole at z equal to 0, ok. So, we get, i times a pi and we are going round it in the clockwise direction. So, we get 1 by twice i, with a minus 1 and that is equal to this value over here which is minus pi, which is equal to minus pi, to minus pi.

So, here I have an integral minus infinity to infinity e to the power of minus i ax by twice pi x dx, is equal to here this is a, i cancels minus pi by two, it goes to the other side, pi by 2 minus pi which is equal to minus pi by 2. But you remember our original J 2 had a minus, so this is plus it is a plus pi by 2. And therefore, integral minus infinity to infinity, e to the power i ax, minus e to the power minus i ax, by twice i x dx is equal to pi by 2, plus pi by 2.

So, this is the 3rd method. We will do this using a 4th method also that will involve path deformation, ok. So, let us begin the 4th method 4. So, we now have integral sin a x over x dx, minus infinity to infinity and this we are going to write as, minus infinity to infinity, e to the power of i ax, minus e to the power of minus i ax, by twice i x dx and that is equal to minus infinity to infinity, e to the power i a z, minus e to the power of minus i a z, by twice i z dz on the real axis, on the real axis this is identical.

Now, in this form, in this combined form, z equal to 0 is not a problem; z equal to 0 is not a problem, ok, here x equal to 0 is not a problem; that means, it is not a singularity, z equal to 0 is not a singularity. So, what I do is I will use the idea of path deformation. What did we say, if we have an integral, ok; if we have an integral, going around a contour of f of z and f of z was analytic in an annular region between two curves, this is C and this is C 1, the closed contour integral, the closed contour integral, around C is the same as the closed contour integral around C 1. So, the function is analytic between these two curves and on these two curves the function is analytic over here, ok.

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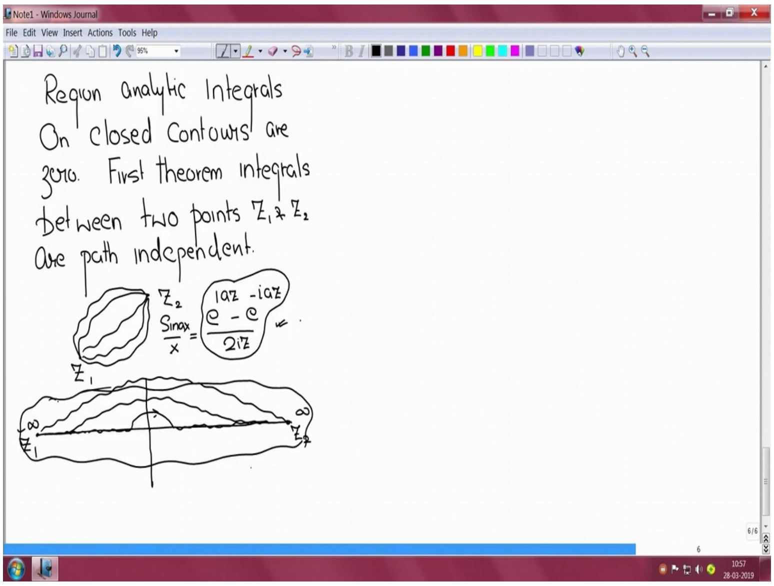

And, hence what we have, let us continue, what we have is the, in the region of analyticity, in the region where the function is analytic; in the region where the function is analytic, we know that integrals on closed contours are 0; closed contours are 0, ok. In the very first theorem, very first theorem says, what? if integrals on closed contours are 0, then integrals between two points; two points z 1 and z 2 are path independent; are path independent; that means, if we have an integral going from z 1 to z 2, I can go in any manner in the region that the function is analytic.

So, I have a function which looks like this: e to the power i az, minus e to the power of minus i az, by twice i z dz and this is on the real axis. This is equal to sin a x over x, sin and sin a x over x along the real axis is very well behaved and this function. I am sorry there should be no dz; this function is well behaved at least close to the real axis, ok. So, let us see what I mean.

So, we have a function given by this and we are supposed to integrate along the real axis from minus infinity to plus infinity and this function has no problem, no singularity at z equal to 0 if. So, we move from minus infinity to infinity, but we can also deform the path, I will fix the two endpoints.

So, this is my z 1 let us say and that is my z 2, so I will fix the two endpoints and I come here and I deliberately take a detour; I deliberately take a detour, ok. So, this value of this integral should be the same, because the function is analytic here, in some region the function is analytic, I should get the same answer, ok. So, this is my starting point and I will further deform this; I will further deform it in the regions where the function is well behaved, is analytic and I will do that with respect to both the terms now, the plus term and the minus term. The time is up. So, I will continue with this problem in the next class.

Thank you.