A short lecture series on Contour Integration in the Complex Plane
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Lecture - 10
Contour integration of sinc function

Good morning. Welcome to this short series on complex variables. Last time we started looking at specific problems in Contour Integration. We will continue that. But before we do that we need a few theorems ok; we need a few theorems which we will merely state and accept, ok.

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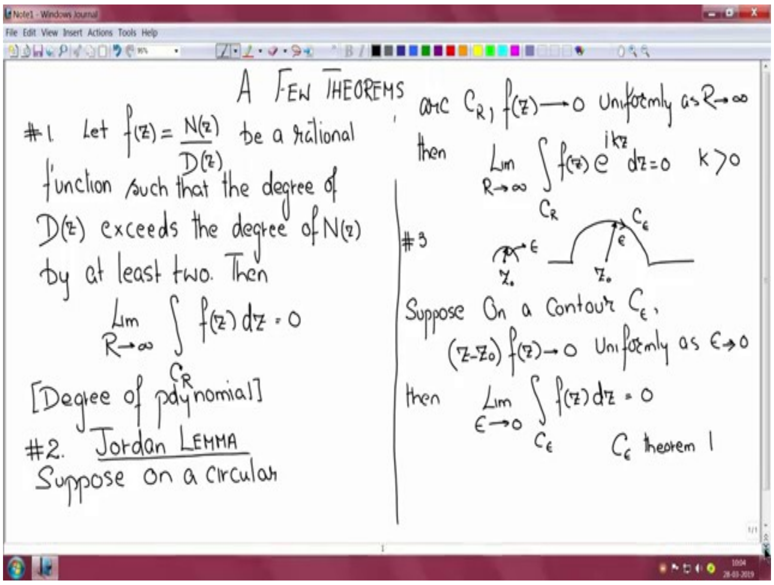

So, they are as follows: let f of z, be given by, N of z, by D of z, be a rational function, such that the degree of D z exceeds the degree of N z by at least two. Then, limit R tending to infinity, integral over a circular arc C R, f of z dz equal to 0, ok. We have already used this idea.

Now, we will refer to this as the degree of polynomial theorem, ok. Whenever we need it will call it that degree of polynomial, ok, degree of polynomial theorem. Next, we will need another very important theorem. It is called Jordan lemma. What does it say? Suppose on a circular arc C R; on a circular arc C R, the complex function f of z tends to 0 uniformly as R tends to infinity, then limit R tending to infinity, integral over the circular arc, f of z which is uniformly tending to 0 e to the power i k z dz equal to 0 and it is important that k be positive.

So, we will refer to this as the Jordan lemma, so we will use it several times any problems to come, ok. Next, very often we will be moving round a point z 0 using a very, using an arc which has a very small radius epsilon, ok. We exaggerate this. There is a point z 0 and we will come more around the point z 0 using a small circular arc, the arc radius will be epsilon, ok. So, we will call this the C epsilon arc ok.

So, theorem number 3 says: suppose on a contour; on a contour C epsilon, z minus z 0, f of z goes to 0 uniformly as epsilon tends to 0, epsilon tends to 0. Then, limit epsilon tending to 0, integral over C epsilon, f of z dz is equal to 0 ok. We will call this the C epsilon theorem 1, when we use it next.

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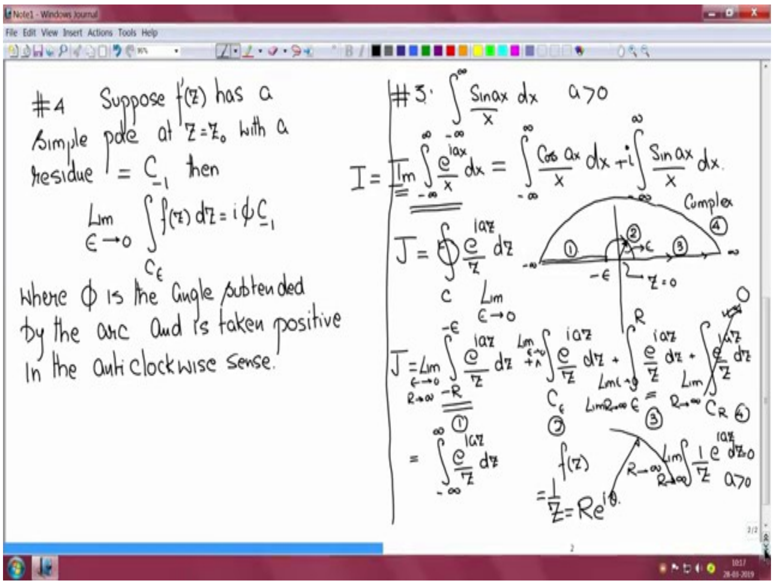

Number 4, related to the same figure: suppose f of z has a simple pole at z equal to z 0 with a residue; with a residue equal to C minus 1. Then, limit epsilon tending to 0, integral C epsilon, f of z dz is equal to i times phi, times C minus 1, where phi is the angle subtended by the arc and is taken positive in the anti clockwise sense. These are some theorems that we will be using, ok.

So, now let us continue with our examples on contour integration, ok. This was probably example 3. We started looking at a very interesting example which is integral minus infinity to infinity, sin of ax by x dx, with a positive. And, the idea we had proposed was that if we look at this integral, minus infinity to infinity, e to the power of i ax over x dx, that can be written as, integral minus infinity to infinity, cos ax over x dx, plus integral, minus infinity to infinity, i comes here, sin ax over x dx. So, we will deal with this integral as though our objective is this integral and when we get the answer we will take the imaginary part. That is the idea, ok.

Now, as we have said before if this is what I call; now this is what I call I, I is my starting integral without this imaginary. So, I formulate now an integral on a closed contour C and I now change the variable i a z by z dz. The contour I choose is this. The original integral goes from minus infinity to plus infinity, ok. But here as long as it was sin ax over x, x equal to 0 was not a problem, because this is the sinc function and it has finite limits as we approach x equal to 0. But in this form e to the power i ax over x or e to the power i az over z and this is a complex plane now. And z equal to 0, this function blows up.

So now, the contour we choose is, we come from minus infinity on the real axis. At minus epsilon, we choose a small circular arc. We circumvent the singularity, then we proceed towards plus infinity and close it with a circular arc. So, this is portion 1, this is portion 2, this is portion 3, this is portion 4. If the integral you want passes through a singularity then you have to evaluate it gently by taking a small circular arc of radius epsilon and then sending epsilon to 0. There is no choice of completely avoiding this singularity. But we will evaluate what happens as we pass through it gently and that is done by initially taking a small contour of radius epsilon along the singularity and then sending epsilon to 0, as a limiting process.

So now, we have 1, 2, 3, 4 portions, so my J I write as integral, I can write it as a minus R to minus epsilon, e to the power of i az by z dz, plus integral over C epsilon contour, e to the power of i az by z dz, plus integral again going from epsilon to R, e to the power of i az over z dz, plus integral over C R, e to the power of i az by z dz. In future, I will shorten this. We are writing it because we have just started and giving in the full form.

Now, I have not left space to put limits here, limit epsilon goes to 0 and R goes to infinity, ok. Here epsilon goes to 0, here again limit epsilon goes to 0, limit R goes to infinity and here limit R goes to infinity, ok. So, these are now as we have seen earlier in the sense of Cauchy principle value, ok. Now, this first portion or first leg and third portion together will give me the integral: minus infinity to infinity e to the power i az by z dz. But I have the 2 portions, I have the second portion or second leg and a forth portion forth leg.

Now, let us look at the forth leg, forth portion, we have said in the Jordan lemma that if I have a function f of z on a circular arc that goes to 0 uniformly as the arc radius goes to infinity, my f of z in this case is 1 over z. So, 1 over z goes to infinity as R goes to infinity because I will write z as R e to the power of i theta and so now, my integral f of z which is 1 over z e to the power of i az dz equal to 0, as limit R tends to infinity. This is Jordan lemma where a is positive. So, forth portion follows Jordan lemma and it goes to 0, ok.

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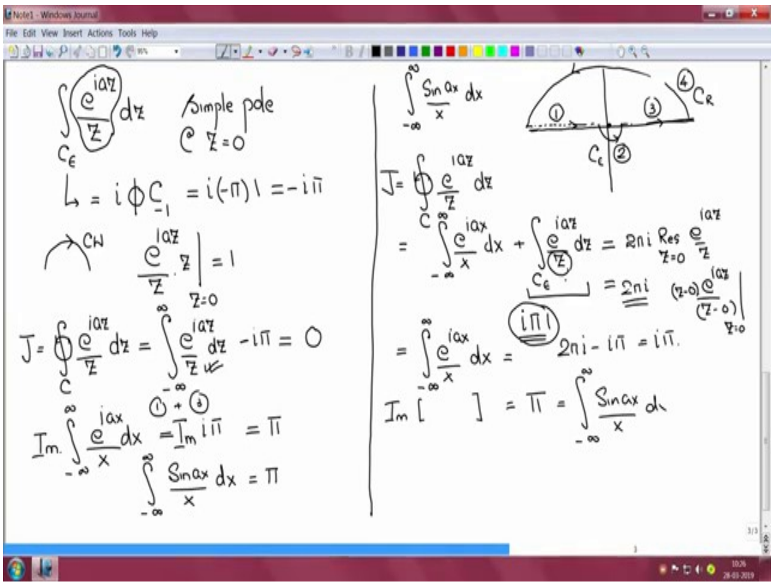

Now, we have the third portion. The third portion is this integral over C epsilon, e to the power of i a z by z dz. Here the C epsilon second theorem comes into play. We have a function which is e to the power i a z by z, which has a simple pole; which has a simple pole at z equal to 0, ok. Hence, this integral is equal to i times the angle of the arc phi times the residue at that point, ok. So, that angle is phi, angle is pi, the phi value is pi, because we are going through a full half circle, but we are also going in the clockwise direction.

So, I have i, I have minus pi and C minus 1 is the residue. So, residue for this is what? e to the power of i a z divided by z, into z, evaluated at z equal to 0, which is equal to 1, so that is 1. So, this whole value is minus i pi, ok. So, where do we stand? We started with J, which is equal to closed contour integral on this C, e to the power of i a z by z dz. That is equal to now the integral I want, which is minus infinity to infinity e to the power i a z by z dz. This is coming from first and third portions. Then the C epsilon gave me minus i pi and C R gave me 0. Now that is equal to what? If you noticed, our contour does not include any singularity and therefore there is no residue contribution from any pole inside C. So, this is going to be equal to 0, ok.

So now, what do I have? I have integral minus infinity to infinity. Now this z is on the x axis. So, z equal to x, so e to the power i a x by x. This is moving on the real axis, this part is moving on the real axis. So, therefore, z is x, dx is equal to i pi and I said the portion we want is the imaginary part of this. So, we take the imaginary part of this and we get pi. So, integral minus infinity to infinity, sin ax over x dx, equal to pi. We will do the same problem now, but this time in the contour we will include the pole, ok.

So, students often have this immediate doubt as to by avoiding the pole within the contour we got this answer. Do we have a same answer if we include the pole. Yes, ok. So, we will look at that problem, ok. So, the problem is the same: integral minus infinity to infinity sin ax over x dx ok. But this time we will include the pole in the contour. So, how does the contour now look like? So, this is a complex plane. I will come from here. Singularity is over here. So, I am moving on the real axis. I am showing it above, because the path would not be clear, so now, we go below. You move forward and close the arc with C R.

So, this is portion 1, this is portion 2, this is portion 3, this is portion 4, ok. So, this is C R, this is C epsilon, now the singularity is inside the contour. Why ? because we will start again with J is equal to closed contour C, e to the power of i a z over z dz. This is singular at z equal to 0 ok. So, we will do this now very quickly. The first and third portions give me what I want. So, I will write it: minus infinity to infinity e to the power i a x by x dx. And then we have the C epsilon portion, plus integral C epsilon portion, e to the power of i a z by z dz. And the C R goes to 0. Here also using Jordan lemma, so we would not include that.

However, now we have included the pole. The pole is inside, ok. So, I will get a residue from that: twice pi i times the residue at z equal to 0 of e to the power i a z by z, ok. Now, we are moving in the contour clockwise direction. So, I will get twice pi i and the residue here is 1, the residue here is 1. Why? Because e to the power of i a z, by let us say z minus 0, multiply by z minus 0, and evaluated at z equal to 0, so that is equal to 1; so I get twice pi i, ok.

Now, here again the second C epsilon theorem holds, ok. So z is a simple pole and we are going around a C epsilon contour, ok. So, what happens is I get i times phi; phi is the angle covered, which is pi, times the residue is 1; however, this time I am moving in the counter clockwise direction, so the sign is plus.

So, I get i pi from here and I write, I have 2 pi i on the other side. So, what I get now is integral minus infinity to infinity e to the power of i a x over x dx, is equal to, I will shift this i pi to the other side, twice pi i minus i pi which is equal to i pi. Again if we take the imaginary of this side, I get pi which is integral minus infinity to infinity sin ax over x dx.

We will do the same problem using two more methods and one of them will include the method of deformation of the path ok, so let us look at that.

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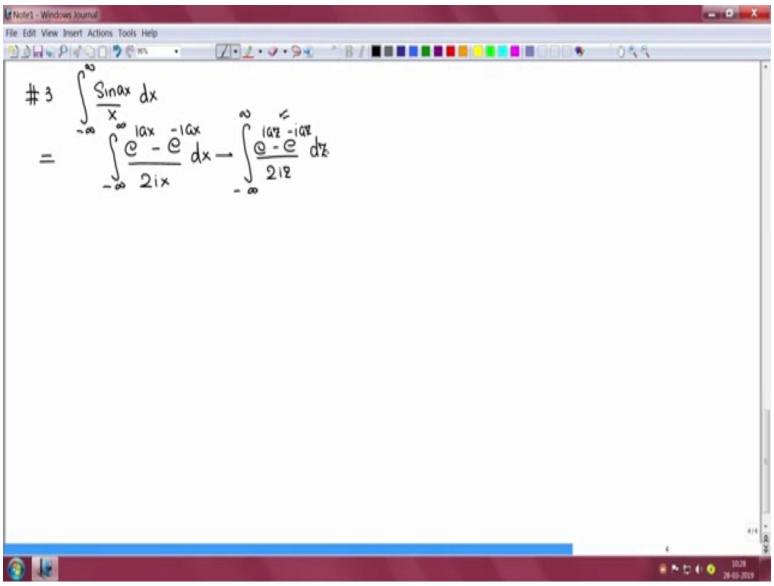

The next method of doing the same problem is this, so the problem is same the third problem: integral minus infinity to infinity sin ax over x dx. So, let me just make a statement and the time is running out. So we will stop and continue. So, this is exactly equal then identity equal to e to the power of i a x, minus e to the power of minus i a x by twice i x dx. And, we will write this as, integral minus infinity to infinity, e to the power of i a z, minus e to the power of minus i a z, by twice i z dz ok. This is exactly equal to this on the real axis. So, we will continue with this problem in the next lecture.

Thank you.