A short lecture series on Contour Integration in the Complex Plane

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Lecture - 01
Introduction to complex variables

Good morning to all of you. Welcome to this series on Complex Variables with specific applications to contour integration.

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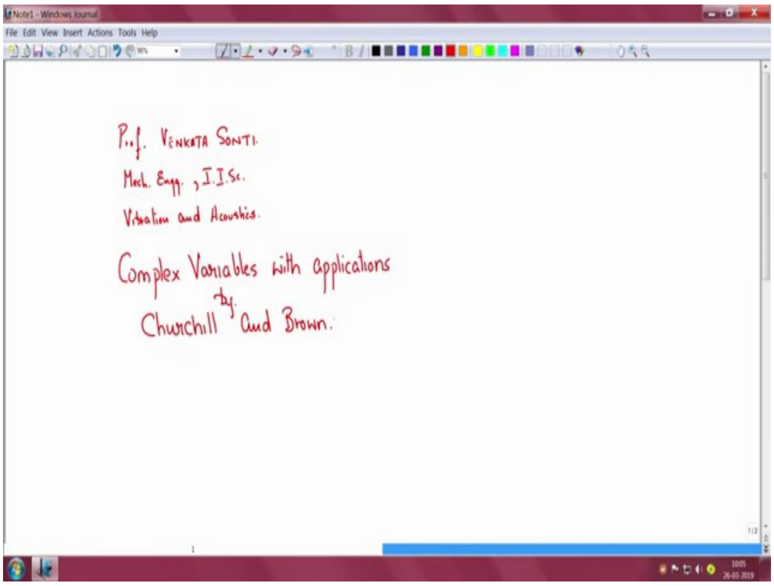

My name is Venkata Sonti, and I am a faculty in Mechanical Engineering at I.I.Sc. My area of work mainly is Vibration and Acoustics. So, one may ask why am I offering this series of lectures on complex variables. The main reason is, within complex variables there is a very useful tool called contour integrations and several branches of engineering do require this particular topic. And yet, it is not so much emphasized in a regular course on complex variables and it is very useful.

And so, I have put together a series of lectures 10 to 12 of them which are offered in our department as part of our engineering mathematics course. And so, I am thinking of sharing that on NPTEL, so that it is beneficial to a wider audience. Now, the number of lectures will be between 10, 12, 13 and so with very small. So, one cannot expect the same rigorous treatment that is given in a regular complex variables course. So, there will be a general rough treatment with not so much rigor and the student is expected to have had some form of exposure to complex variables, either through a full formal course or again as part of some mathematics where many topics are covered. So, some familiarity is assumed, so that it is beneficial to him.

Now, largely the course the material is my own. However, there is one particular book I will follow in the beginning few lectures, it’s title is somewhat like this, ``Complex Variables with Applications” by Churchill and Brown, a very popular book. There are other books which I may mention during the course of the lectures. And so, let us begin by giving a motivation. Let us motivate this course.

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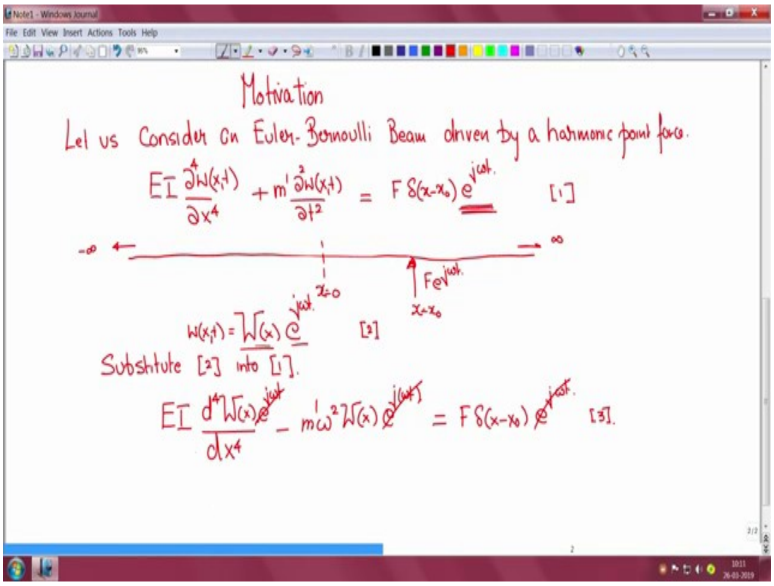

Since I work in the area of vibration and acoustics, I will take the example from vibrations. So, let us consider an Euler Bernoulli Beam. Those of you have had a course in vibrations would know this. This beam is now driven or excited by a harmonic point force. So, the equation of motion can be written as EI del 4 w del x 4 plus m dash del 2 w del t square and there is a point force applied at x equal to x 0. It is harmonic, so we have the time dependence and this is an infinite beam.

So, if we want a picture, this is an infinite beam and let this be the origin. So, this goes off to infinity this way, it goes off to infinity this way. And at some x equal to x 0, there is a point force F e j omega t. That is the pictorial depiction, ok. Now, this is a linear system and so, if we drive it with this harmonic frequency that system is going to respond at that frequency. So, the displacement w of x comma t can be written as some bigger W of x multiplied by e j omega t, ok. We have separated the two variables.

Now, if we substitute this let me give this some number 1. If we substitute it in 1, substitute 2 into 1, ok, then we find that the spatial derivatives will act on W(x) and the temporal derivatives will act on e to the power j omega t and so we get EI d 4 W x dx 4 minus m dash omega square W of x e to the power of j omega t. There will be an e to the power j omega t here also, and that is equal to f delta x minus x 0 e to the power j omega t. We call this 3. Since e to the power j omega t cannot be 0, we will divide the whole equation by e to the power j omega t and we cancel it out, ok.

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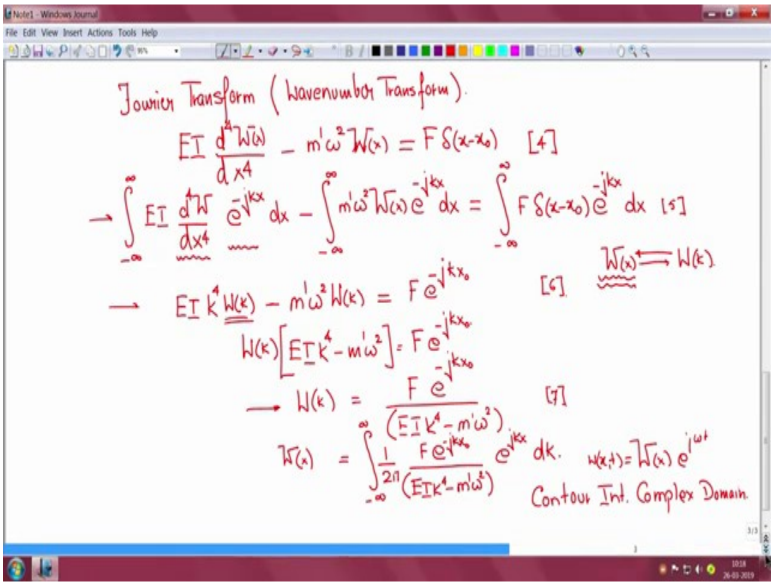

So, now I have this ode, the dependent variable is w(x), and the independent variable is x. Now what we do, is we take a Fourier transform. It is taken in the spatial domain, so there is a special name for it. We call it the wave number transform.

Since, we need that equation back again, let me just take a look. EI d 4 W d x 4 minus m dash omega square w x is the forcing function. So, let us get back here. EI d 4 W of x dx 4 minus m dash omega square W of x is equal to f delta x minus x 0, ok. We will give this number 4 perhaps.

Now, we take the Fourier transform, ok. So, what do we do? We integrate from minus infinity to infinity EI d 4 W dx 4 e to the power of minus j k x dx minus again integral minus infinity to infinity m dash omega square W x e to the power of minus j k x dx is equal to integral f delta of x minus x naught, e to the power of minus j k x dx, ok.

Now, I will assume some familiarity on your part with respect to this equation, ok. We will essentially transfer the derivatives that are there on W to e to the power of minus j k x, ok. So that this ends up looking like EI k 4th W of k minus m dash omega square W of k is equal to now the delta function property comes in F e to the power minus j k x naught, ok. Since, this is a motivation, I am not going too deep into it. It is enough to know that this equation Fourier transform leads to this, ok. Many of you will have seen this in your classes.

Now, the differential equation has been converted to an algebraic equation here. So, I can take W of k common and that gives me EI k 4th minus m dash omega square equal to F e to the power of minus j k x naught or W of k is equal to F e to the power of minus j k x naught by EI k 4th minus m dash omega square, ok. So, let us call this 7, ok. Now, W of k here is the Fourier transform or wave number transform of W of x, ok. W of x has been transformed to W of k, ok. And you can go back also using an inverse Fourier transform.

And W of x is what we want because our answer is W of x times e to the power j omega t and therefore, we have to go back take an inverse Fourier transform, ok. So, that inverse Fourier transform is now getting back to again W of x is equal to this integral, with a 1 over twice pi minus infinity to infinity F e to the power of minus j k x naught by EI k 4th minus m dash omega square and here e to the power plus j k x, but this time d k, ok.

So, now, the k variable, the wave number variable k extends from minus infinity to infinity, k gets integrated out and we have W of x. And my final answer is this small w x comma t which is this W of x e to the power of j omega t, ok. Now, this integral over here that we have can be integrated using contour integration in the complex domain, contour integration in the complex domain, ok. And in many branches of engineering similar integrals one does come across, ok. So, this will serve as motivation.

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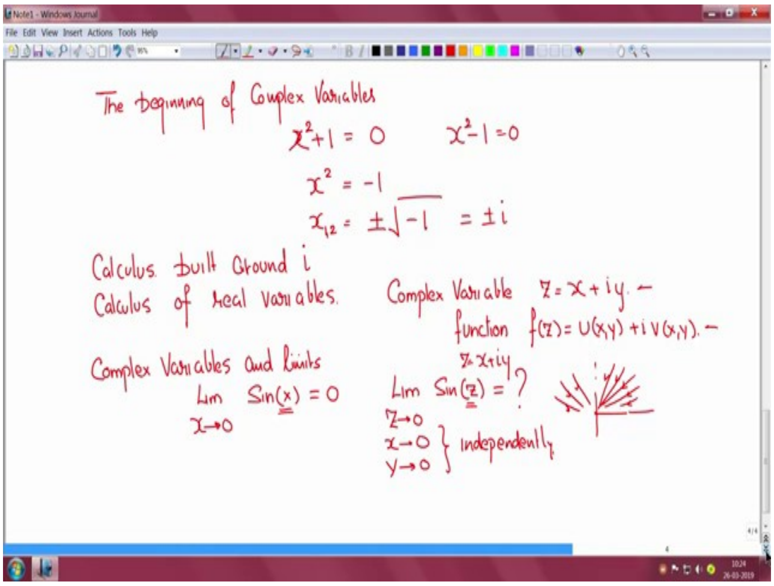

Now, very brief starting of complex variables. Most likely the beginning, I am not sure what actually happened, ok. Beginning of complex variables, the thought perhaps came up with an equation of this form where you have x square plus 1 is equal to 0, ok. Normally, one would go only to this extent x square minus 1 is equal to 0, but there was someone who perhaps thought of x squared plus 1 equal to 0, so that you have x square is equal to minus 1 which is quite strange and then x, the solution 1 comma 2 happens to be plus or minus the square root of minus 1 which we now understand as plus minus i, ok.

Now, an entire calculus or branch of mathematics had to be built around this, to be built around this new discovery, ok. And we are familiar with calculus of real variables, and so, we would be most comfortable, we would be most comfortable if the calculus of complex variables also follows the same laws and rules, ok. So, now, the new variable, the complex variable is now z which is x plus iy, and the complex function, the complex function is f of z which we will tend to write as u of x comma y plus i v of x comma y, ok.

Now, these two together we have to check if the regular laws of real variable calculus are applicable to the complex domain. So, let us do some simple checks. Let us discuss limits, ok; complex numbers, complex variables and limits. Now, we are aware that let us say this limit x tending to 0 sin of x is equal to 0. So, the question we ask is limit z tending to 0, sin of z is it equal to 0 or what is it, ok. Now, it is not automatic, that is z plays the role of x here, z going to 0 implies that x goes to 0 and y goes to 0 independently, independently.

So, if you consider x to be the real axis and y to be the imaginary axis and you could be arriving at 0 at any angle that is the whole idea, ok. When z goes to 0, that is the whole idea and so, therefore, it is not that you have x over here just put z over here, now that is not the case. So, we have to inquire whether limit x going to 0, y going to 0 independently sin z, where z is again x plus iy goes to 0. So, let us see if that happens.

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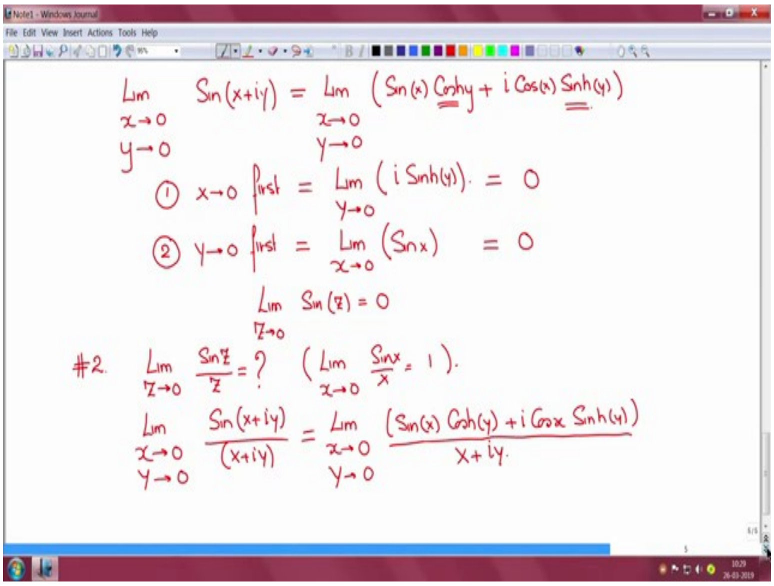

So, now, we will start. So, we say limit x tending to 0, y tending to 0 independently, sin x plus iy. What is it? So, it is going to be limit x tending to 0 in any manner, y tending to 0 in any manner, we open this out, we will use sin a plus b law. So, we get sin x, cos of iy is cos hyperbolic of y plus cos of x, cos of x and sin of iy is i times sin hyperbolic of y, ok. These are some functions one has to be familiar with, hyperbolic cos and hyperbolic sin.

Now, we will take x going to 0 first, ok. So, here for a case one is we will take x goes to 0 first, so x is set to 0 first, ok. Then I get the remaining which is limit y tending to 0, x is 0. So, this term is 0, x is 0. So, it is i sin hyperbolic y, ok. Now, if we apply limit y tending to 0, sin hyperbolic y, then sin hyperbolic y goes to 0. So, I get a 0 over here, ok.

In the second case, what we do is, second case, we set y to 0 first and we get limit x tending to 0, y is 0 and therefore, this term is 0, sin hyperbolic y is 0 this term is 0 and here we get 1 and we have sin of x, ok. For y equals 0 cos hyperbolic y is 1 and therefore, we get 0 over here, ok. So, we get 0 in both cases. So, what do we see? We see that limit z tending to 0, sin z is indeed 0. Just like the real variable calculus. So, we feel very comfortable with this, ok.

Let us examine a next example. Let us say we have another example, ok, example 2. We look at limit z tending to 0, sin z over z. This is the familiar sync function, ok. We do know that limit x tending to 0, sin x over x is equal to 1. It is not singular at x equal to 0. So, we want to ask what is this, ok. So, let us open this out.

So, we have limit x tending to 0, y tending to 0, I am opening this out sin x plus iy by x plus iy, is equal to, we further open this out. We have, limit x going to 0, y going to 0, ok. In the numerator I have, sin of x cos hyperbolic of y plus i cos of x sin hyperbolic of y divided by divided by x plus iy, ok. Now, we do not like that there is a complex variable in the denominator. We will do what is called rationalization, ok. We will remove the complex number in the denominator, ok. How do we do that?

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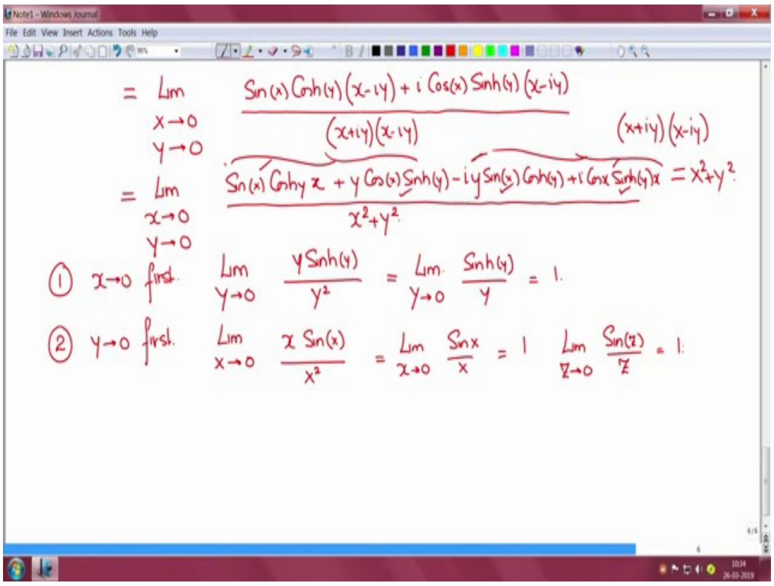

We have limit x tending to 0, y tending to 0. We multiply the numerator and denominator by x minus iy, so we get sin of x, cos hyperbolic y into x minus iy plus i cos of x, sin hyperbolic y, x minus iy and in the denominator we multiply x plus iy and x minus iy, ok. x plus iy into x minus iy is equal to x square plus y square, ok. So, then what we will do is, we will collect now the real parts, ok. We will collect the real parts and the imaginary parts, separately. So, we have sin of x, cos hyperbolic y, into x, plus y cos of x, sin hyperbolic y, minus iy, sin of x, cos hyperbolic y, plus i cos x, sin hyperbolic y, into x. And the whole thing is divided by x squared plus y squared. So, this is the real part of it. This first portion is the real part, this next portion is the imaginary component.

Now, here again we will take x going to 0 first, so x will be sent to 0 directly. So, we call this 1, ok. Then, we have limit y tending to 0, if x is 0 this is 0, if x is 0 this is 0. So, what we have here is y and this is also 0, yes, if x is 0 this is also 0. So, y cos 0 is 1, sin hyperbolic of y by x is already 0, so y squared. And so, we get limit y tending to 0, sin hyperbolic y by y which is equal to 1, ok. One can use L'Hospital's Rule and prove to oneself that it is indeed 1.

Now, the second manner is we take y straight away to 0 first, and then we have limit x tending to 0. So, if y is already 0, this term goes to 0, if y is already 0 this term goes to 0 and this term goes to 0. So, we have x, sin of x by x squared or equal to limit x tending to 0, sin x over x, which is equal to 1, we already know. And that is what is the result, limit z tending to 0 sin of z by z equal to 1, ok. So, it appears that limits follow rules similar to real variable calculus, ok.

So, we will continue this thought in the next class, ok. I will stop here today

Thanks.