Convective Heat Transfer
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Lecture - 08
Similarity Solution- Momentum

Following up from our last class, where we showed that how scaling can be a very powerful argument. Let us now try to see, that whether we can do things in a more proper way more quantific way. What we are going to do over here in this particular part of the lecture is something called similarity solution.

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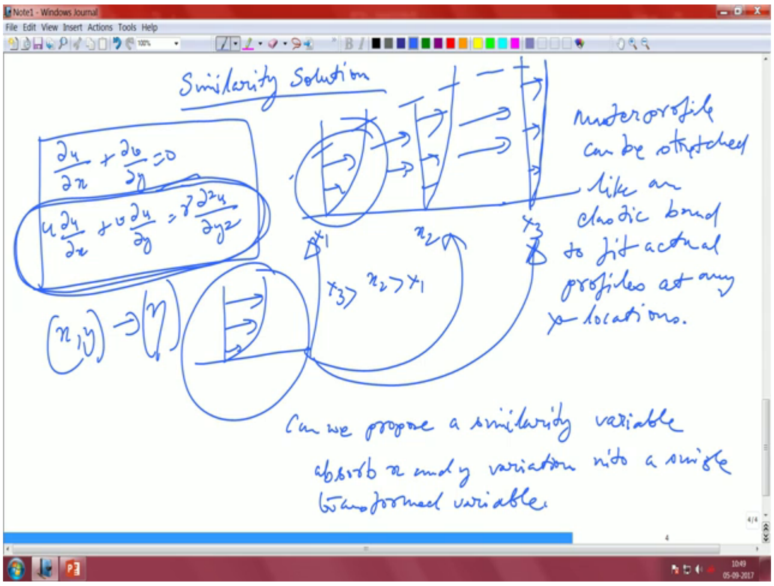

What is similarity solution to begin with?

If you are take a look. The basic idea is very simple, we will start with momentum and then we will try to see how it translates to the energy equation. Let us put the 2 equations side by side. In case we need to refer it, we can do it quickly. This is the way these are the 2 sets of equations of course, we have written just the x momentum equation. It is within the boundary layer concept. We are still using the same boundary layer equations with whatever the caveat. The boundary layer equations are, in this particular situation. The basic idea is very simple.

Now, if you look at it, you will also get the whole thing. You will see that the boundary layer at say a particular location X1 looks like this. It is say the momentum at a different location it looks very similar of course, it moves up like this right because the boundary layer is increasing. Basically, X2 is greater than X1 correct and if you go further downstream you will get that this profile will become more like that right for the forgive my, drawing, but it is you can get the thing that X3 is greater than X2.

All you can see that the profiles look very similar. Looks very similar to each other except that, they there are differences, but if you look at if you take this particular profile for example, and if you stretch it perhaps you can get this. If you stretch it further you can get this. It is almost like if we have a master profile like that something like this. This master profile can be stretched like an elastic band and we are we should be able to fit this master profile to all these profiles that we have.

The master profile basically if we have to put it the master profile can be stretched, it is almost stretched like an elastic band elastic band to fit actual profiles. Profiles at any X locations is not? That. And the same thing is valid for temperature also the temperature is the same thing the profile just looks a little bit opposite, but there also there is a master profile, which you can stretch like an elastic band and you should be able to fit the profile.

This gave an idea that they look very similar to each other. It is like those similar triangle concepts right that you have triangles which are similar, but they are not exact. What you do is you stretch it. That is how you get to make them exact. It is very similar. Based on this the idea came that these profiles all looked very similar to each other. Is there some way by which we can take the X and Y dependence? There is X dependence there is a Y dependence of the velocity. As you can see the velocity if this is the velocity profile this has got both X and Y dependence right because your momentum equation has got that kind of dependence.

Based on this X and Y dependence can we merge them into one single variable. That would mean. There will be no X and Y there will be only one variable across which we can describe this profiles. In other words, we can we propose? Propose what we call a similarity variable. The similarity variable what it should do it should absorb X and Y variations into single a transformed variable or in other words, we can convert this pds if you look at the pds like this we can convert this pds to an od. Because X and Y dependence essentially X comma Y is merged into a single variable eta.

If we can do this then this particular equation ideally should get reduced to an od. It can be non-linear does not matter, but we should be able to go and make it down to an od. There is one single dependence of u or v with respect to eta. The other variables and that is possible because of the reason that we can stretch this profiles across X across these are different X profiles. There may be a possibility by which we can actually achieve this. We would not go into the historical perspective of this whole thing. This was this is known at the blasius solution essentially.

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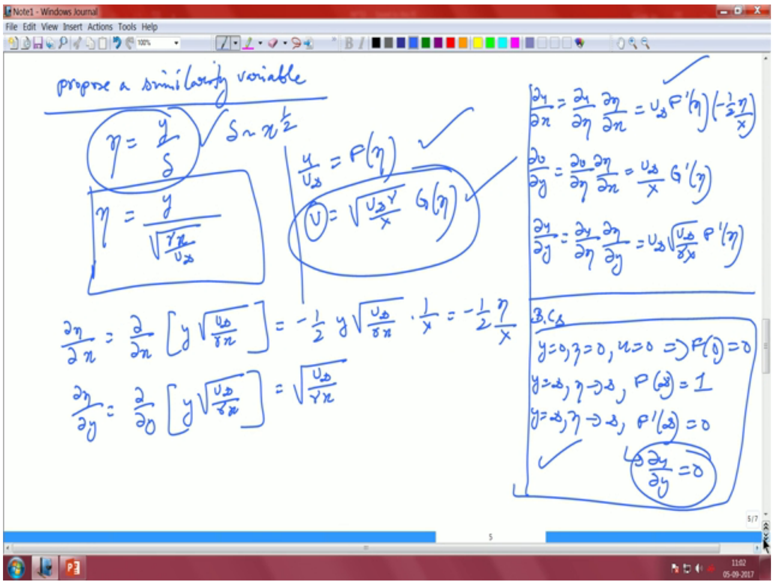

Let us propose a similarity variable. There are several ways of doing this. Not sure if blasius did it in this way, but we will propose it like this. Let us say y eta equal to y over delta and we already saw that delta scales as x to the power of half. This we already saw from our last scaling argument lecture. All we have done is nothing we have just taken a new variable eta and every y we are dividing it by this delta. Delta actually varies because delta is a function of x. As you see that it grows at every y location we are basically normalizing it by the corresponding boundary layer thickness at that particular location. Something like that, eta if you substitute also for the form of delta, from our scaling argument if you recall the scaling argument that is why the scaling argument was. So good. If you substitute delta in this particular way, delta if you substitute it in this particular way this is the form that you get. This actually absorbs X and Y variation into one variable eta, but we do not know whether this transformation will hold water or not. Till we work out the full solution and see that whether it really transforms a momentum equation from a pd to an od. That is that is the proof right will hold if and only if the momentum equation is reduced one single variable.

Let us assume a few things u by u infinity where u is the parent variable, is f eta. V is equal to u infinity into gamma by x into g eta. You look at it some transformation rules we can establish. It is very simple actually, just this is a lot of algebra. Just bear with me with the steps. Let us establish d eta by dx for example, similarly d eta by d y.

These are standard math. There is nothing much to explain except that these are the transformations that you need. To convert the momentum equation from it is x and y coordinates to the eta coordinate. That is what we are trying to do over here similarly d u d x u infinity. Similarly, du by dy these are some of the transformations which will come in useful because we have to convert it to the corresponding eta coordinate systems.

What will be the boundary conditions will look like? The boundary conditions will be say at y equal to 0 eta goes to 0 right because of this, pay attention there eta goes to 0. So, that would mean that you goes to 0 which will lead to f 0 becomes equal to 0. Correct that is the first argument, y equal to infinity, eta goes to infinity f, infinity becomes equal to one right.

As y goes to infinity, eta goes to infinity, f prime infinity goes to 0 and why this happens that is because, your du dy goes to 0. At the edge of the boundary layer because there is no more variation with respect to y at the edge of the boundary layer. These are your basic boundary conditions. Once they are converted into the corresponding eta coordinate systems.

Now, with these rule set this is the transform variable these are the u and v expressions, these are the corresponding transformations using chain rule, these are the corresponding boundary conditions. Now we are in a perfect situation to take the continuity equation first and then shift to the momentum equation.

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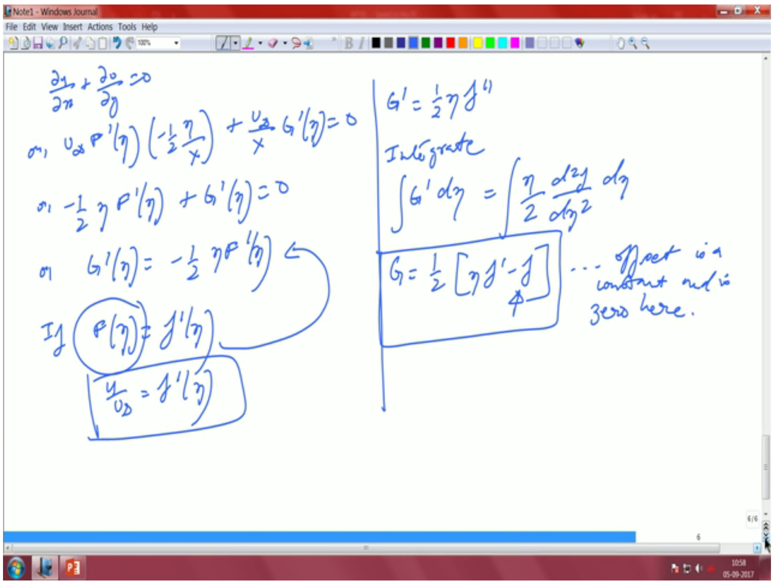

Let us do the continuity or because u infinity by x actually takes off. G infinity prime. You can see this is the expression now here let us propose something looking at this particular form of the equation that if f eta becomes equal to f prime eta this is just a casting different way of casting that thing and then if we substitute it here how should it look.

In other words, then u by u infinity becomes equal to f prime eta not f eta based on this particular transformation why we will do this it will become apparent in a second. Based on this transformation what we have is that G prime becomes equal to half eta f double prime right that is what it becomes.

Now, if we integrate this particular guy. G prime d eta equal to eta by 2 that is the integration this will give you G half n f prime minus f. You can see why this is this transformation has helped, that is because of the simple reason. We would have otherwise dealt with an integral of f eta over here. Capital f eta instead of this small f over here. It would have become there would have been an integral within this particular bracket.

The here of course, the offset or the constant of integration is a constant and is 0. Here got it? What we got from the continuity equation G equal to half of n f prime minus f. Where f prime eta is basically nothing but u by u infinity. U by u infinity was f prime eta.

Now, let us look at the momentum equation.

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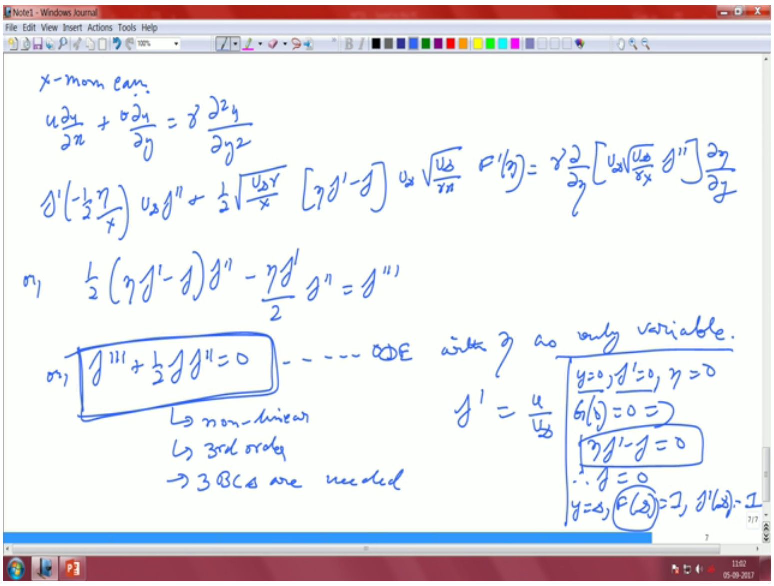

But. Look at that momentum equation. Let us put the terms in is a full expression. As you substitute all the numbers now you can work out the algebra I'm not going to do it over here. If you work out the algebra it will give you eta f prime minus f double prime minus eta f dumb or I can simplify this a little bit that is half f f double prime is equal to 0. This becomes an ODE with eta as only variable. This is correct. Then we have been able to reduce the momentum equation to an ODE with a single variable only right the nature of this equations are that it is non-linear. It is third order; that means, 3 boundary conditions are needed.

3BCs are needed got it understood. This is the parent equation, that is what we are going to solve. Remember always f prime is u by u infinity. Clear? Up to this particular portion. There are 3 boundary conditions which are needed. What are those 3 boundary conditions going to be let us write it down here y equal to 0. F prime is equal to 0 because u is equal to 0 and eta is equal to 0. Similarly, g equal to 0, equal to 0 implies eta f prime minus f is equal to 0 therefore, f is equal to 0.

Third boundary condition will be y equal to infinity means f infinity is equal to 1. F prime infinity is equal to. These are the 3 boundary conditions that we have. Y equal to 0 means essentially the no slip boundary condition that we have. It also implies that G equal to 0 because if you look at the definition of G right if you look at the definition of G here if you look at the definition of G that is right there.

When v is equal to 0 g also has to be equal to 0 right. Also comes from the no slip boundary condition. That would mean that G is given by this this is equal to 0. F prime is already equal to 0. F will be equal to 0 that is 1 condition then y equal to infinity that would mean that f this big f is equal to 1 because u approaches u infinity right. That would imply f prime infinity is equal to one got it?

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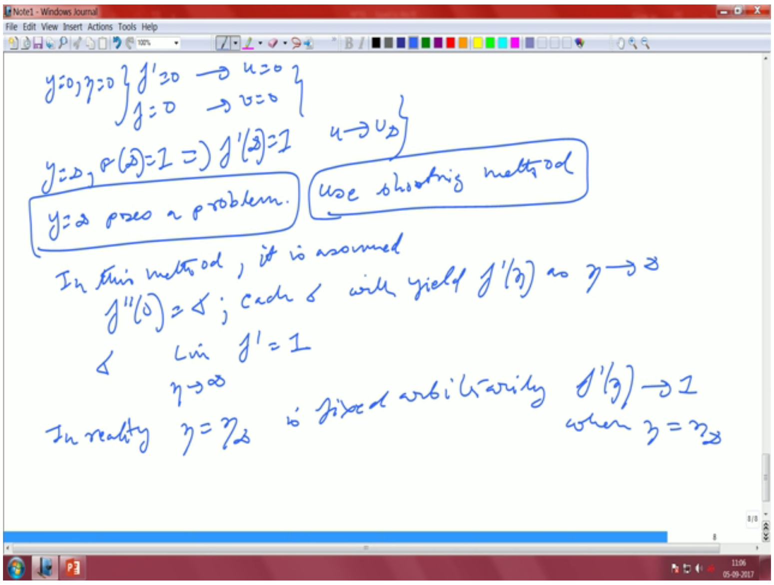

Basically, let me compile it in a more cohesive fashion. Y equal to 0 eta equal to 0 gives rise to f prime equal to 0 and f equal to 0, at y equal to infinity f infinity is equal to one which implies f prime infinity is equal to1 because, u approaches u infinity here of course, u becomes equal to 0 and v becomes equal to 0.

These are the boundary conditions that we have. Now y equal to infinity if you want to now solve this equation. There are this is an od with 3 boundary conditions which we already have one boundary condition is at the wall the other boundary condition is at the at infinity. Y equal to infinity poses a problem. Correct? Because this boundary is at infinity. That is the problem. What people use they use normally something called a shooting method.

What is a shooting method? In this method, use something like a shooting there can be other types of methods also it is a shooting method is particularly useful in this method what we do it is assumed actually assume, that f double prime 0 is equal to sigma and each sigma, this is not gamma though this sigma, where each sigma will yield f prime eta as eta goes to infinity.

This sigma is such chosen that in the limit, eta goes to infinity f prime becomes equal to 1,that is how we choose in this limit.

In reality of course, in reality eta is assumed to some value eta infinity is fixed arbitrarily. Arbitrarily such that, f prime eta approaches 1 when eta is equal to eta infinity. That means, once again is an asymptote. Because f prime eta is once again the same thing it is u by u infinity. When u approaches u infinity that is where you kind of truncate this whole thing off. But ideally what you do is that basically it is a guess approach; that means, you fix a value you do the do the solution.

That means you reduce the boundary to something like n equal to in infinity and see whether the solution kind of merges or not. It goes back and forth this is not a numerical class. You can find different ways, if you just do a thorough search you will find that how are the different ways by which we can actually.

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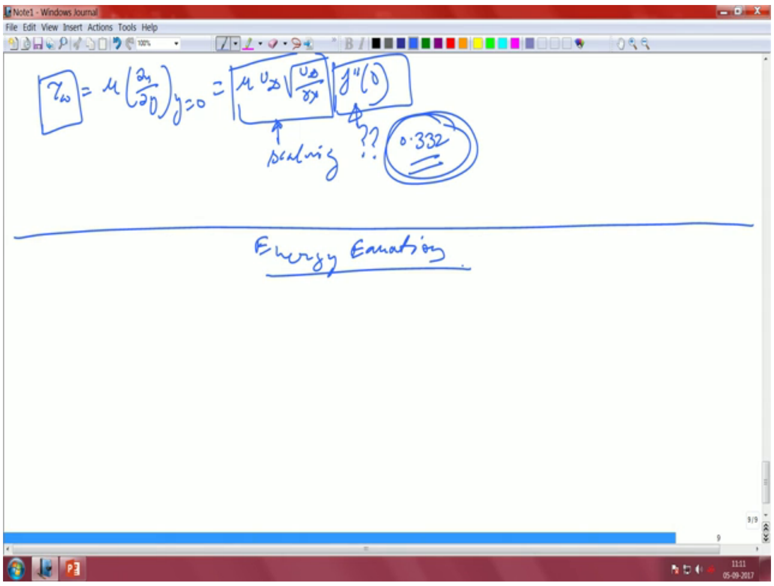

Solve this. Based on this tau wall is given by du by dy at y equal to 0.

That is the definition of the wall shear stress. Remember wall shear stress comes into account to find out the drag. That is the utility of the wall shear stress Mu into u infinity and 2 root over u infinity by gamma x into f double prime 0. Remember f double prime 0 is basically nothing but the slope on the velocity, because u by u infinity is f prime. F double prime is basically u by du by d eta essentially.

This part was already established if you look at it. It is already established through the scaling. We already know this part, correct this part is already there. We nailed it in a scaling this is the part that we need. This is the exact part remember that coefficient that we say that was missing that cannot be answered from scaling argument. That is this part. This part is value actually comes out to be point 3 3 2 if you solve that equation using the shooting method. That is how it comes out.

As you can see. By utilizing this you can find out that value of the wall shear stress and you can also find out the exact term. That is that point 3 3 2 which sits in front, now you know an exact answer to that query. Now let us see that how the solutions actually look like now that we have done all these things let us look how the solution looks like. Let us look at the ppt over here.

In fact, in this particular ppt, we can see a few things over here. So, let us look at the next slide perhaps because that would give you a nice idea.

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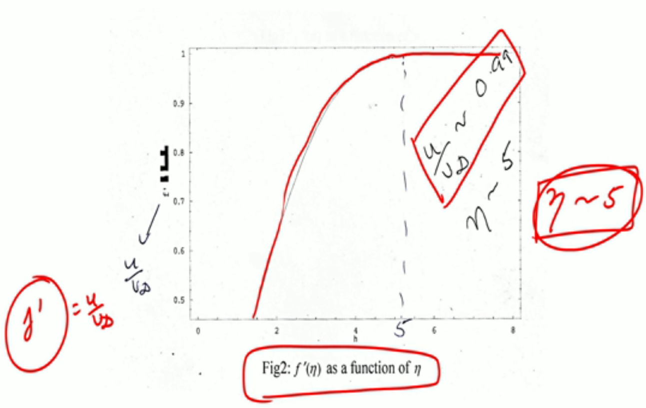

This is basically your f prime graph that we have plotted. F prime as you can see once again using the shooting method. F prime was what? F prime was nothing but u by u infinity that is f prime that is what I have written over there. And this is plotted as a function of eta a variable.

What do you get you get that after some time? they have prime gradually asymptotes like this. When it is equal to 0.9 or usual estimate right we basically say that that is the end of the boundary there. That value of eta comes out to be around 5. Got it? Last value of eta comes out to be around 5. It agrees. Once again, this particular factor we would not have been able to determine just by scaling argument. By using this particular form, we now know that that factor is nothing but equal to 5. That is your f prime eta.

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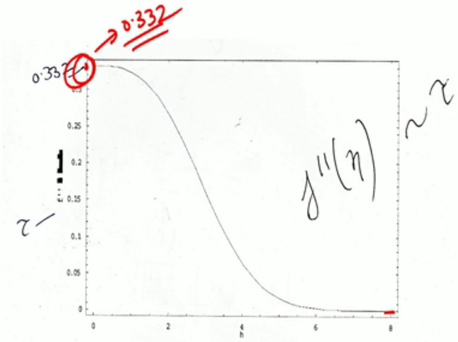

Fig3: $f''(\eta)$ as a function of η

Let us look at the second one which is basically nothing but the shear stress. That the second derivative is nothing but the shear stress. F double prime eta and it is evaluated at 0. This is the general thing for tau f double prime at 0 is basically is equal to a tau wall. The shear stress is always there that is tau, but this gives you the wall shear stress and that value as you can see over here is point 3 3 2, which we just now mentioned.

As you can see from here to here as we go on outside the boundary there what do you see about the nature of this particular profile? You can see that the shear stress approaches 0. Which is legitimate, because you are going to the free stream now, because there is no gradient of velocity that is remaining. It should be the opposite of the u as u approaches u infinity the shear stress approaches 0 and shear stress at the wall is nothing but point 3 3 2. That is the factor, that is that additional factor that comes out which cannot be found out through the scaling argument this is we already did we already showed.

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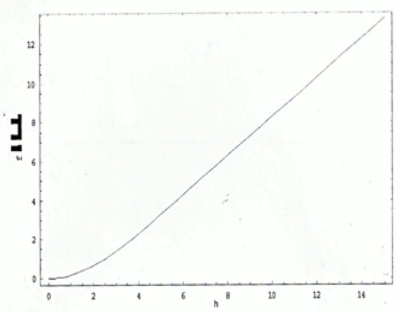

Fig1: $f(\eta)$ as a function of η

This particular thing and if you look at now the f. F is actually related to the v. We will consider this a little later, no need to consider it now, but we just gave a graphical overview that how this is actually done. Let us move into our windows journal form. When we actually say that this is what the profiles should actually look like.

Now, that we have done the scaling of momentum and we showed that what are the exact numbers and we have shown how the profiles actually look like. The natural next step will be to look at the energy equation and establish a similar scaling. Not a similar scaling similar similarity solution for the same. Let us look at this in the next lecture.