Convective Heat Transfer
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Lecture – 49
Turbulent boundary layer – Fully turbulent sub layer

So, in the class before we talked about the viscous sub layer part which was part of the inner region, now we are looking at the fully turbulent sub layer which is the part of the inner region, but the outer half of the same.

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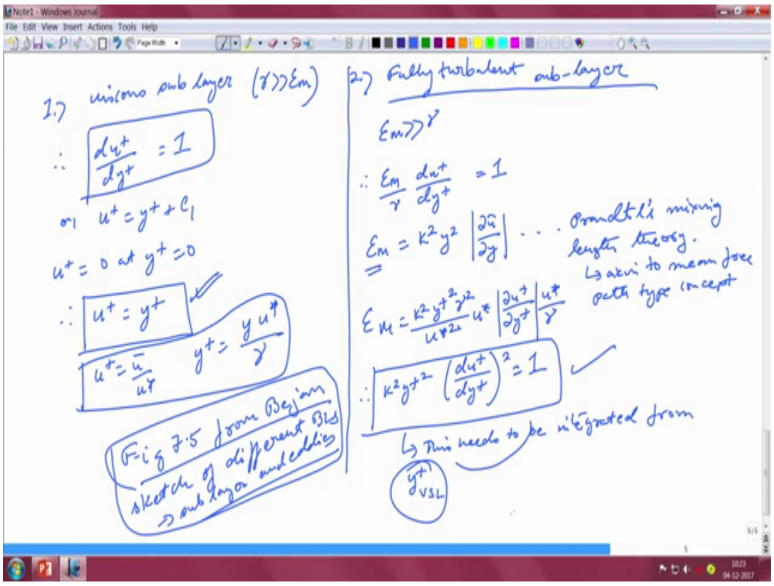

So, there what we said was that your epsilon m was much greater than your gamma correct. So, that was the initial point that we tried to make.

So, therefore, epsilon m by gamma du plus by dy plus is equal to 1, epsilon m we already knew from Prandtl's mixing length theory, this is comes from Prandtl's if you look at your old notes you will find Prandtl's mixing length theory which was if you recall was akin to it was akin to the mean free path type concept well this was not of course, 2 molecules but 2 blobs of fluid.

If you recall your old notes you will see that. So, that is what it is. So, therefore, the epsilon m now if you normalize the whole thing this is the normalization that we have to do knowing that epsilon m, remember there is no gamma now there is there is no kinematic viscosity well it comes through that.

So, therefore, k square y plus square is equal to 1. This is the total expression which is a little bit different from this, this was the expression for that, this is the expression for this. Now the integration this equation therefore, this needs to be integrated from the y plus VSL which is basically the viscous sub layer.

Because the integration cannot be from 0 it has to start from the VSL like the viscous sub layer correct. So, the integration has to be performed from the VSL got it. So, all we need to do is therefore, to integrate this particular expression from this particular limit as a limit of integration.

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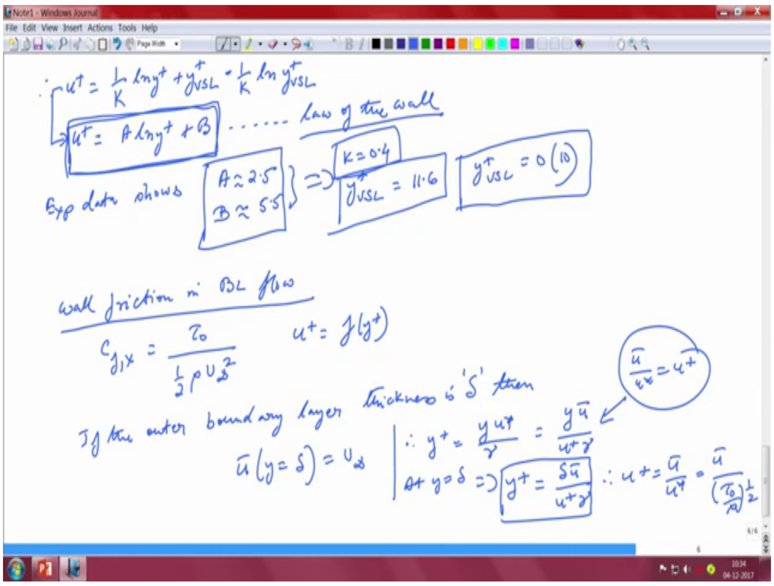

So, therefore, if you do the integration now u plus will be 1 over k you got it.

So now this u plus therefore, will be A ln y plus plus B which we call as a law of the wall it has got this generalized form from here to here, that is the generalized form that we have this is a generalized form, it is a log scale basically. So, the experimental data shows this A is approximately equal to 2.5, B is approximately equal to 5.5, this actually leads to k equal to 0.4 and y plus VSL is equal to 11.6.

So, y plus VSL is off the order 10 roughly. That is of the order 10 roughly. So, understood this particular profile is basically a logarithmic profile now which is one is called the law of the wall. Experimental data has been used to extract these and from there you get k is equal to this and y plus vs are equal to that and y plus VSL comes out to be 11.6 which is of the order 10, remember the existence of the viscous sub layer was severely questioned early on because nobody could see it is a very turbulent flow profile. So, near wall measurements it was very, very difficult.

But now of course, with high speed piv and laser Doppler anemometery and things like that one can actually measure the flow velocities and the stress gradients very close to the wall. So, that affords us with the flexibility that you can actually use you can measure and even validate that whether some of these quantities are really checks out or not. That whether there is something called a viscous up there because this is our assumption.

So, we need experimental data basically to validate some of these assumptions. So, if you look at this figure now.

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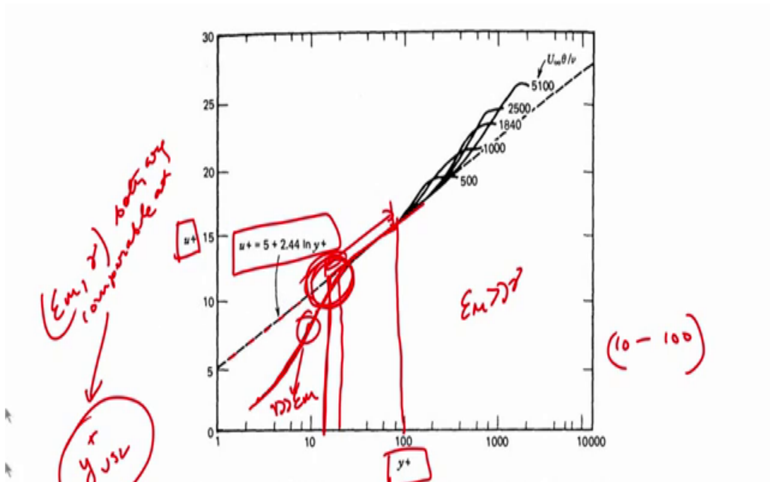

Figure 7.4 Example of $u^+(y^+)$ velocity measurements in turbulent boundary layer flow without longitudinal pressure gradient. Note the use of $A \approx 2.44$ and $B \approx 5$ to fit the data. (Reprinted with permission from L. P. Purtell et al., *Physics of Fluids*, Vol. 24, pp. 802–811, May 1981. Copyright © 1981 American Institute of Physics.)

Which basically gives you this is once again u plus, this is y plus. And we have been able to take some of these measurements all right in a turbulent boundary layer without any longitudinal pressure gradient which is exactly the flat plate problem and A of 2.44 and B around 5 which is the values that we said will match the data has been used basically to fit the profile.

So, got it so you can see that this is the profile that has been fitted, you can see that around this particular point onwards see the deviation starts to happen from this particular profile. As you go more and more towards the viscous sub layer. So, and it is kind of starts around here you can I mean get a reasonable fit in that particular region. And it kind of matches and then it starts to deviate as you go higher and higher in the y magnitude.

Because this is not supposed to cover the full boundary layer anyways, but up to about 100 as you can see it matches quite well. So, in that window of 10 to about 100 there is quite a bit of a good match between the wall profile in this particular window in this particular region there is quite a bit of a good match, beyond that it starts to deviate because of the presence of the viscous sub layer. I mean deviation from this logarithmic profile that we have outer side also it starts to deviate for obvious reasons.

That we are going away anyway from the we are going more into the turbulent core remember this was still the inner region, up to about 100 it does truly remarkable job, but; however, one thing that we have to mention that no profile can actually merge very close to here. So, when y plus VSL around that particular region no particular profile is actually going to do a good job. The simple reason is that basically you have their epsilon m and gamma both are comparable at around this particular limit.

Beyond that as you go away as you can see when you go to something like 20, you get almost epsilon m is really much, much greater than gamma. As you go down more towards lower than 10 you will find that this particular value here gamma a becomes much, much greater than epsilon m, but it is in this particular region which I am circling now that is the region where the 2 profiles are basically comparable to each other. So, therefore, you do not have a match there because these are 2 profiles which matches on the 2 limits, but it does not match quite.

Because the limiting conditions are never really satisfied limiting conditions are never really satisfied so many people have done many things like.

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Table 7.1 Summary of longitudinal velocity expressions for the inner region of a turbulent boundary layer

$u^+(y^+)$	Range	Reference
$u^+ = y^+$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 11.6$ $y^+ > 11.6$	$\boxed{\text{Prandtl and Taylor [10]}}$
$u^+ = y^+$ $u^+ = 5 \ln y^+ - 3.05$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 5$ $5 < y^+ < 30$ $y^+ > 30$	von Kármán [14]
$u^+ = 14.53 \tanh(y^+/14.53)$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 27.5$ $y^+ > 27.5$	Rannie [15]
$\dfrac{du^+}{dy^+} = \dfrac{2}{1 + \{1 + 4\kappa^2 y^{+2}[1 - \exp(-y^+/A^+)]^2\}^{1/2}}$ $\kappa = 0.4 \quad A^+ = 26$	All y^+	van Driest [16]
$u^+ = 2.5 \ln(1 + 0.4y^+) + 7.8[1 - \exp(-y^+/11) - (y^+/11)\exp(-0.33y^+)]$	All y^+	Reichardt [17]
$\dfrac{du^+}{dy^+} = \dfrac{1}{1 + n^2u^+y^+[1 - \exp(-n^2u^+y^+)]}$ $n = 0.124$ $u^+ = 2.78 \ln y^+ + 3.8$	$0 < y^+ < 26$	Deissler [18]
$y^+ = u^+ + A[\exp Bu^+ - 1 - Bu^+ - \frac{1}{2}(Bu^+)^2 - \frac{1}{6}(Bu^+)^3 - \frac{1}{24}(Bu^+)^4]$ (last term in u^{+4} may be omitted)	All y^+ $A = 0.1108$ $B = 0.4$	Spalding [19]

For example, you find these are many of the summaries of the of the longitudinal velocity expressions the first one is the one that we did. That it is y plus and it is this it was found by Prandtl and Taylor. So, up to about 11.6 u plus equal to y plus obeys all right where as greater than 11.6 you start to getting that logarithmic dependence.

Von Karman for example, this is a 3-part fit. So, it is valid up to 5 then 5 to 30 there is a log and then there is another variety of the logarithmic fit, which covers from y plus is greater than 30. Then there are other people who have done this there are many people who have tried this is for all y plus say for example, is like a combined fit which covers the entire range.

Then Reichardt also did it for the entire range, Diessler did it in a 2-range kind of a family and this is done by Spalding this is also done across a wide range for all y plus. So, some people have done devised correlations in which you do not have to have this 2-part fit you can directly go. So, basically you have a curve fit which basically takes into account which smoothens out as you approach from the 2 sides along this v y equal to y plus VSL line you basically smoothen out.

You basically smoothen out that particular region because if you can smoothen out that particular region, then you do not have to worry about which part of the fit works where. So, that is some people have done many of them are mostly empirical in nature by only the first one we did because the first one is something mathematically you can get some essence because it has got some physical insights. There are a lot of good work that was done in this particular category and you can read about it, but this is a table which basically summarizes the different attempts that people have made in different times.

Let us move back to our journal entry and try to find out now that the next important thing which is basically the wall friction. We all we did all these things for developing the wall friction. So, wall friction in boundary layer flow. So, as you know that Cfx which is the skin friction coefficient that definition does not change this is the definition that we have.

I have already showed you table 7.1 which basically shows that u plus is a function of y plus. This is the table that I just now showed attempts by different people including the attempt that we went through kind of rigorously. Now if the outer boundary layer outer boundary layer thickness is delta then u bar y equal to delta should be u infinity u bar at y equal to delta should be u infinity correct.

Similarly, therefore, your y plus is be given by y u star by gamma. In other words, this is actually given as y u bar divided by u plus by gamma. Keeping in mind that u bar by u star is equal to u plus. So, basically, we have used that over here directly. Now at y equal to delta leads to y plus is equal to delta u bar divided by u plus by gamma. This is y at delta. So, therefore, u plus which was equal to u bar by u infinity u star. So, this is the other parameter space this we already know.

But this was an important piece of argument that we are laying down over here now let us look at what can we do with this..

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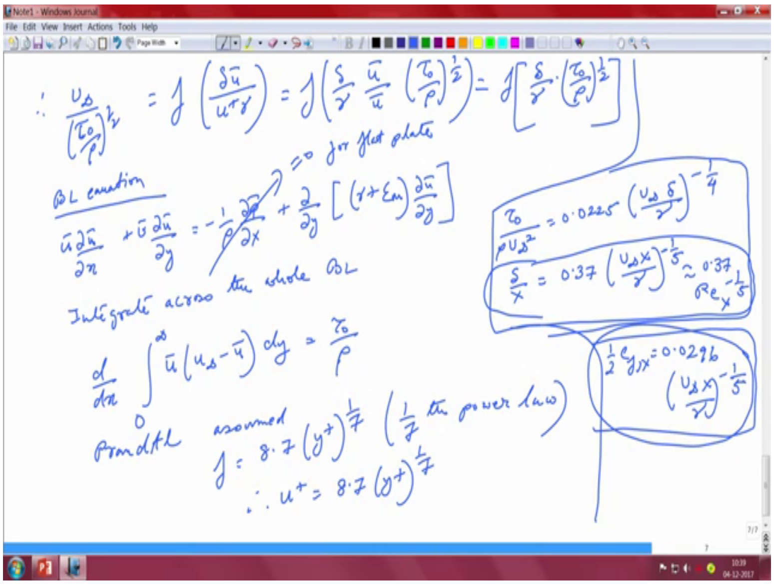

So, therefore your u infinity by tau naught by rho to the power of half because u bar is equal to u infinity at the edge of the boundary layer that is nothing but a function of delta u bar divided by u plus by gamma or in other words this is f delta by gamma.

So, in other words this leads to f delta by gamma. So, you basically get your u bar u infinity by the walls friction velocity is this is a function of delta by gamma this is delta multiplied by the again the wall friction velocity. Now let us look at the boundary layer equation once again u bar d u bar by d x. I am writing the full equation first.

We know that this particular part is equal to 0 for flat plate. Now we integrate it integrate across the whole boundary layer, integrate it across the whole boundary layer or in other words it is d by dx, if you recall your von Karman integration this is 0 to 0 to infinity u bar u infinity minus u bar dy is equal to tau naught by rho that part will remain the same.

So, you have integrated it across the whole thing. Now Prandtl assumed this is different from the 2-part solution that we did earlier. So, Prandtl assumed that this f which is basically the velocity scale is basically given as 8-point 7 y plus raised to the power of 1 by 7 this is a one 7th power law not different from what we had earlier.

So, therefore, your u plus is basically given as 8.7 into y plus raised to the power of 1 by 7. So now, using this we already know that what the relationship between u plus and all the other variables are you can write it here let us try to put everything in the same page that tau naught divided by rho u infinity square is given as 0.0225 u infinity delta by gamma raised to the power of minus 1 4th delta by x is given as 0.37 u infinity into x by gamma raised to the power of minus 1 5th which is basically also turns out to be 0.37 into Reynolds number to the power of minus 1 fifth it is a lot lower dependence on Reynolds number that you can see.

So, half Cfx it is given as 0.0296 u infinity x by gamma raised to the power of minus 1 5th got it. So, you can see that these are the 2 relationships that you get one is a 1 4th dependent 1 5th dependence of Cfx and you have the delta which is the boundary layer thickness grows as a Reynolds number to the power of minus 1 6th.

Now, 1 5th, in this particular context if you look at you can look at the Cfx plots and compare it with your with your laminar boundary layer case.

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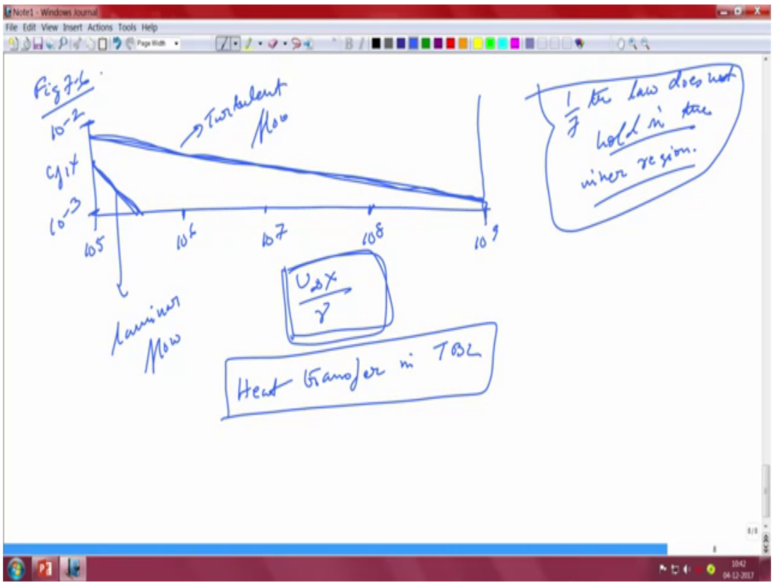

So, it is figure 7.6 in bejan, where you can see that this is how this thing will actually grow. So, you can see for the Cfx for a laminar flow will be something like this starts from 10 to the power of 5 this is about 10 to the power of 6 just putting the scales here 10 to the power 7, 10 to the power of 8, and 10 to the power of 9.

And this is of course, the scale caps of at 10 to the power of minus 2 this is 10 to the power of minus 3. So, this is the Cfx for the laminar this is Cfx and this is the corresponding u infinity x by gamma remember. We have not used the wall conditions over here and the profile that was chosen by Prandtl which is the one 7th law does not hold in the inner region it does not hold an inner region it does not satisfy the inner region..

We already you can plug it in and you can see it for yourself. So, this is it starts somewhere here and then it kind of grows on and on something like this. So, this is for the turbulent the dependence is much more feeble with respect to Reynolds number as you can see it is from the Reynolds number of 1 5th kind of a dependence.

So, the value is higher, but it is a lot less feeble if the dependence is very feeble with respect to the Reynolds number that we have over here. So, this is a one of the most important thing that one should take out the 1 7th law does not hold in a inner region, but this is an ad hoc approximation and of course, the wall region also does not hold at the edge of the boundary layer that is because you have assumed tau apparent to be constant.

In the inner region in a region we say that tau apparent is constant that will not be constant when you go up outside the inner region all right, but an ad hoc profile like a 1 7th power law seems to give you an idea that what will be the skin friction coefficient in this particular case. Now of course, with the case of turbulent flow you can see that is a lot feeble dependence on the Reynolds number whereas, for the laminar flow it kind of the dependence is right there because of the higher dependence essentially.

So, this actually brings out that what will be the heat transfer coefficient what will be the wall shear stress and we will see a variety of wall shear stress towards the end of this of the next lecture. So, we would just pose that we will now start with the heat transfer in the turbulent boundary layer all right so far, we are we have dedicated ourselves towards the towards the flow.

Now, let us look at the heat transfer very close to the. So, once again for the heat transfer let us write the basic equation. So, it is basically u.

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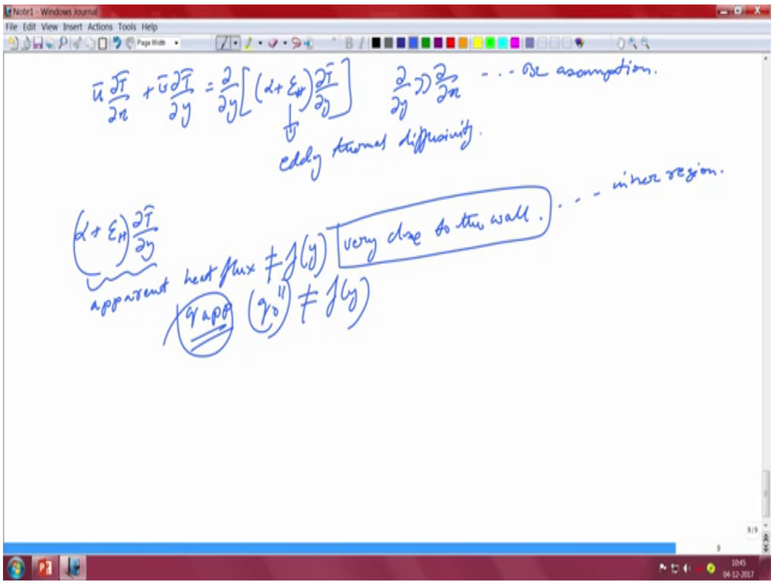

And we will pick it up start it here we will pick it up in the next class. So, this is the total equation once again we have assumed that d by dy is greater than d by dx is the boundary layer assumption. So, this exactly looks like a like your laminar boundary layer except that you now have this eddy thermal diffusivity coming into the picture.

So, eddy thermal diffusivity is important and you see that this eddy thermal diffusivity is the one that will play basically spoilsport here. So, also in this particular thing this is basically called the apparent heat flux. This is the apparent heat flux and once again we are conjecturing that it is not a function of y very close to the wall this particular heat flux like your apparent shear stress.

If you call this q app this like the shear stress it will or here we have mentioned it as q double prime, like the wall shear stress this is not a function of fy which essentially means the same thing that that apparent shear stress is constant, apparent heat flux is also constant in a region very close to the wall which is once again like a inner region all right, once again like a inner region.

So, next class what we will do is that we will see how this analysis can be now done for the heat transfer case and we will see that how the inner region like we have defined the inner region in the case of a laminar, in the case of the flow hydrodynamic part, let us do the same thing for the heat transfer part. So, we will see in the next class how that can be analyzed.

Thanks.