Convective Heat Transfer
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Lecture – 47
Reynold's Averaged Navier Stokes equation – II

So, similarly the y-momentum equation.

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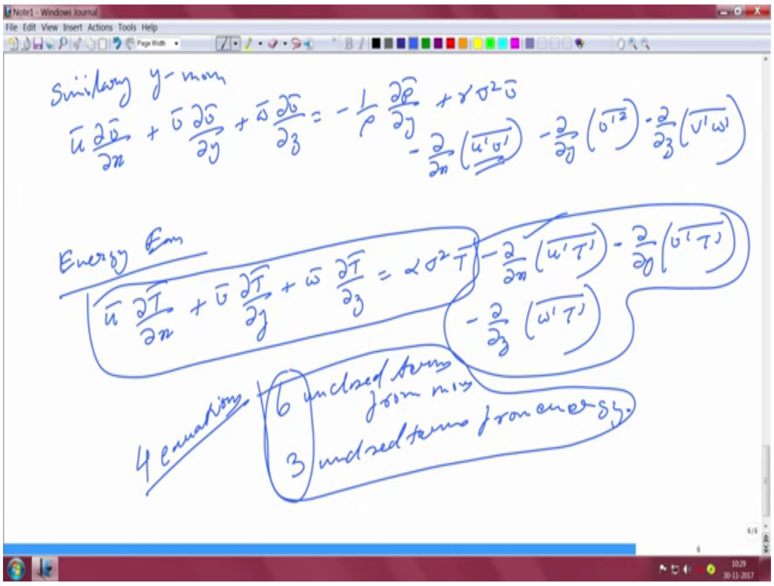

There also you have. So, this will be still the same. So, this looks exactly the same; out of this as you can see this particular term is the same u prime v prime. So, similarly there is an energy equation; energy equation is a scalar equation. So, you would not have that many terms.

So, if you do the same averaging technique. And I am not going through the motion once again. You will find that this is what you get. This is basically the temperature fluctuation correlation. So, out-pops 3 more terms from the energy equation. So, basically what we have? We have 6 unclosed terms from the momentums.

Terms from momentum and we have 3 unclosed terms from energy. Or in other words, we have created a problem in which using this statistical method, we have been able to get this kind of equations which has got now out of the 4 equations basically.

We have got 6 unknowns that has popped out. So, that is not good. So, you can put all these things in a matrix form also.

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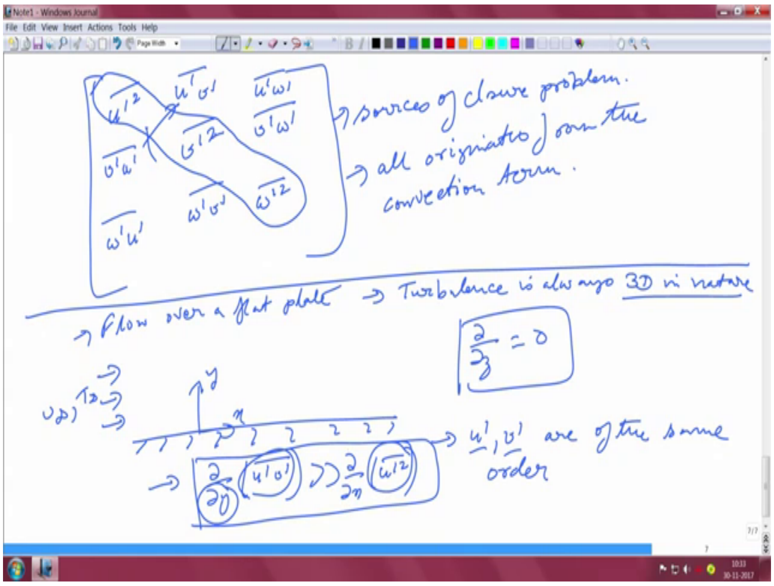

So, u prime square bar v prime square bar that will constitute the diagonal of the matrix. So, u prime v prime bar and u prime w prime bar will be the other quantities.

So, this will be v prime u prime and this will be v prime w prime. So, this particular stuff therefore, will be so, w prime u prime. So, that is the matrix. This is the diagonal element. In other words this is called the Turbulent kinetic energy. This side is basically the same; these are the same. So, it is basically you have 3 plus 3, 6 unknowns; though this is how in the matrix form you should write.

So, these are the sources of the closure problem. So, when somebody tells you about the closure problem, this comes only if you do the averaging; not due to anything else. Once you do the averaging and what are the sources of them they all have their origin from the convection term; all originates from the convection term. All has got their origin from the convection term.

So, this poses one of the key problems that how to solve for these quantities. So, there are different ways and this makes the problem really hard very situation dependent. That means, if the flow is over a flat plate versus the flow through a pipe, these and that there will be a lot of variations in our reasoning. So, there is no universal reasoning always for a turbulent flow.

So, let us take the problem of flow over a flat plate. So, as we know that turbulence is always 3D in nature. So, this is a flat plate this is u infinity this is T infinity. So, this is over a flat plate and in this case we assume that these derivative is equal to 0 though I say turbulent is 3D, statistically you can say that it is a 2 dimensional flow. So, but there are these are kind of hand waving arguments.

Also we say that u prime and v prime are of the same order and if you are there in the boundary layer region; that means, your du by dy u prime v prime bar is actually going to be much much greater than dx u prime square bar. This is from the boundary layer because once again, this is your y this is your x; the same reason that y scales as delta x scales as l is the same logic over there.

So, therefore, these terms as we say that u prime v prime; this is of the same order as this term. So, when you actually scale them this will be much larger than that. So, the same argument what we did in the boundary layer still applies over here. I will bit with a little bit of a twist and that twist we have explained what that twist is.

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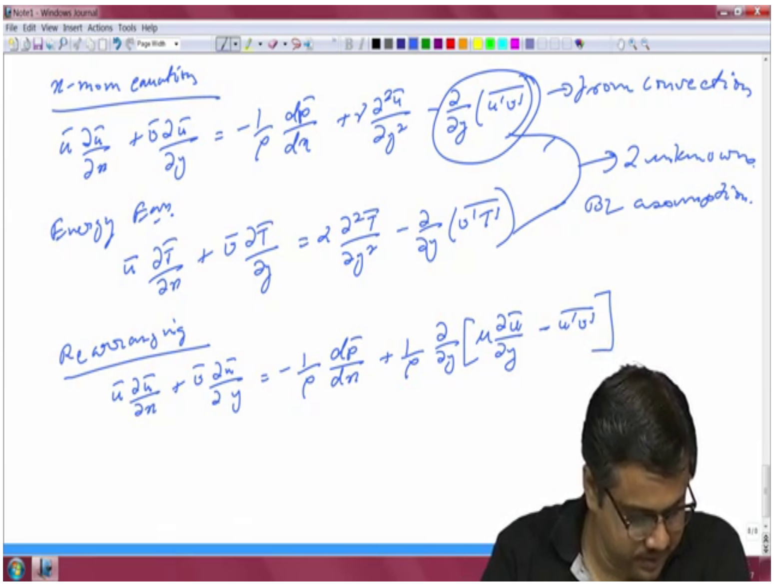

So, the x-momentum equation now, it becomes a little bit more palatable. This is p. So, that is because all the other terms kind of evaporate. Because we said it is not 3D. So, the z component goes away. So, basically the matrix instead of being by 3 by 3 becomes 2 by 2 matrix essentially. So, the energy equation similarly it becomes.

So, now basically you have 2 unknowns. We are doing it in a simplistic way actually there are 6; actually there are 9. So, we are reducing it to 2 unknowns basically depending on the boundary layer assumption. All this originates from the boundary layer assumption. Now we can do a little bit of rearranging. Once again, remember these are from convection because many students make this mistake that they think that this is coming from the viscous terms. It is not from the viscous term; viscous term is not non-linear.

So, it is coming essentially from the convective part of the Navier stokes equation or the energy equation wherever there is non-linear term you will have this issue; coming up into the picture. So, rearranging this is you u. So, one way to write it, the other way to write it is basically.

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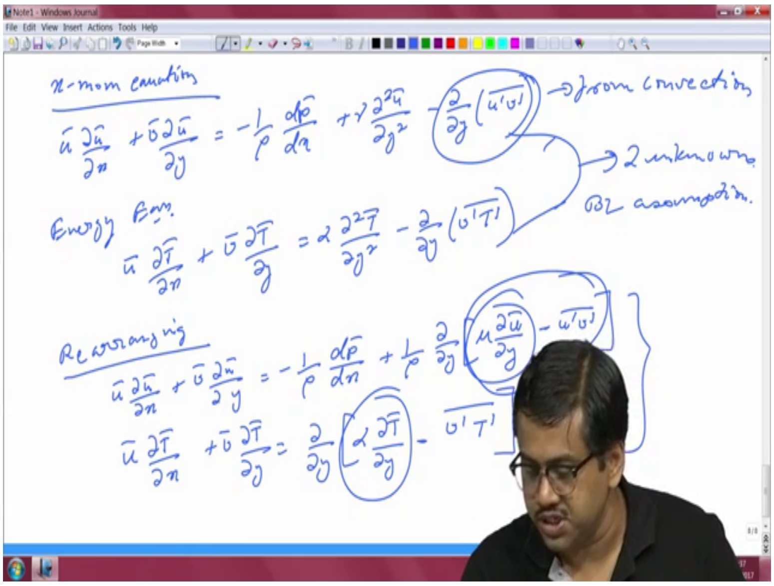

Once again, these are the 2 non-zero terms that you have in your expression. These are the 2 non-zero terms that you have in your expression.

So, but we have written it in such a way that this is now written with the diffusion terms. So, this the viscous diffusion and this is the thermal diffusion. So, you have written it in such a way that they are always in that viscous diffusion terms. Now let us explain a small concept over here.

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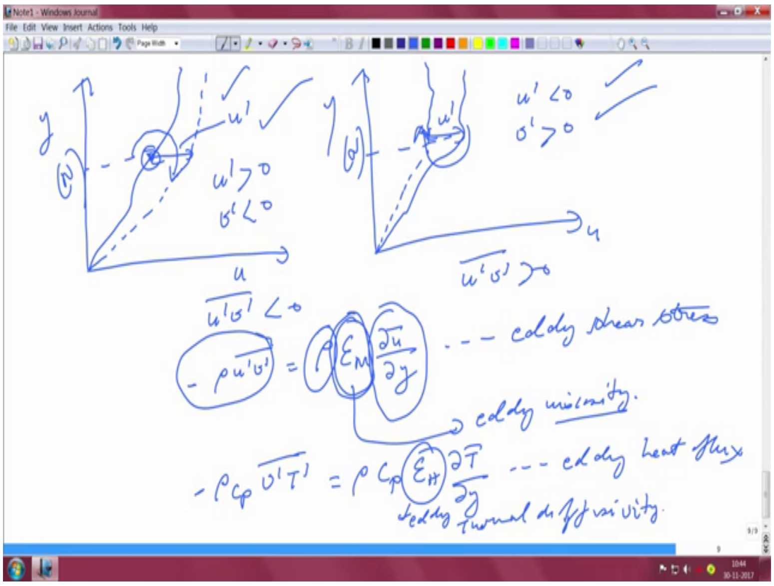

Let's do this y this is with u; say at some point this is the boundary layer profile and we choose any particular location. It is marked that location like this. So, what you have in this location is that you have a 7 bout of an eddy; which basically shifts this line, basically because you have added an additional velocity component. And it makes it go in this particular direction.

So, you suddenly have an increment of u prime because of an eddy; that is one situation. The other situation is that you have u this is your y; once again, this is your whatever your boundary layer profile. You have so, this is a positive u because it has shifted the boundary layer in the positive direction. You can also have a negative u; that means it is under in the negative direction. So, that means, the profile now becomes something like that say for example.

So, it has shifted in the inward direction to about a u prime. This is very commonly happens like you have a blob of this eddy which was not there in your laminar flow. This blob of eddy basically brings some mass, some high momentum eddy has come from some source and it has shifted; it has locally added a velocity in that particular point which is the point here, let us mark that point is n.

In this point n it has added this additional u prime. As soon as it has added that additional u prime, what will happen to this u prime? It will shift in this direction or there could be some eddy which has transported momentum back or some eddy with a lower momentum has come here.

So, it will shift the boundary layer in the opposite direction. Now as it does it, the interesting thing that happens is that when your u prime is positive; u prime is positive. If you look at the boundary layer profile, what happens is that? It shifts downwards. It shifts downwards. So, your v prime becomes actually negative because your shifted your boundary layer downward.

So, your v prime is actually negative; while on the other hand, in this case your u prime is basically a negative quantity. But it has shifted the boundary layer upward; it has become kind of more steeper. So, your v prime is therefore, negative positive. So, in other words, no matter what we do? You will find that u prime v prime here is less than 0; your u prime v prime over here is greater than 0. The correlation of the 2.

So, this is an interesting statement. So, here it is like this, here it is like that. So, this in this is rather interesting and this enables us that when you actually have u prime v prime bar, can we write it as something like this. So, u prime v prime bar has been written in terms of the same quantity rho. There is some parameter epsilon m that we have brought about and this is the mean gradient of the flow field.

So, you have written the product of 2 fluctuation quanta, fluctuating quantities or the mean of the product of 2 fluctuating quantities in terms of write some ad hoc quantity which is epsilon m and the mean flow gradient; mean flow gradient is u bar by dy. So, this particular thing is called basically the eddy viscosity.

So in other words, we have this tremendous idea that why cannot we actually because we have put it already with the viscous terms. This ingenious idea can crop up, if we can represent these fluctuating quantities in terms of the mean velocity gradient; which is actually the viscous stress is actually put in that particular way. In terms of the mean velocity gradient and some ad hoc quantity epsilon m; then, this particular term can be almost represented by like a viscous stress term.

Let us call this an eddy viscous stress or eddy viscosity or a eddy shear stress called eddy shear stress. Let us put this as eddy shear stress acing to the viscous shear stress though the origin is not really viscous; because once again u prime v prime cannot be we do not know what the nature of those terms are. So, this is a model equation. This has got no relevance to any physics. It is our model understanding that can we represent.

If you look at these diagrams, this is represented by the change in the velocity slope that is all that we have shown over here. That if u prime is positive and v prime is negative there is a change in the slope of this particular profile; if you can see here and here. So, all it does this, this u prime v prime combo is basically it shifts the velocity profile. It changes the gradient of the velocity profile and this velocity profile is what it is u versus y which is essentially the mean flow gradient that is du u bar by dy.

So, if somehow I can correlate my u prime v prime bar to this du bar by dy; then, I have now a model equation. And the ingenious part is that, this may not be exactly equal to this; let us put an ad hoc term which is epsilon m which we will call the eddy viscosity. So, this is like viscosity, but it is not viscosity. So, that is the whole point. Similarly rho cp v prime T prime bar can be written as rho cp epsilon H dT by dy, this is called the eddy heat flux.

Once again, this particular term looks like what we call the eddy thermal diffusivity. If you call that eddy viscosity, this is like an eddy thermal viscosity or thermal diffusivity. But once again, caution these are not viscous terms per say. So, if you write now in a more clean manner.

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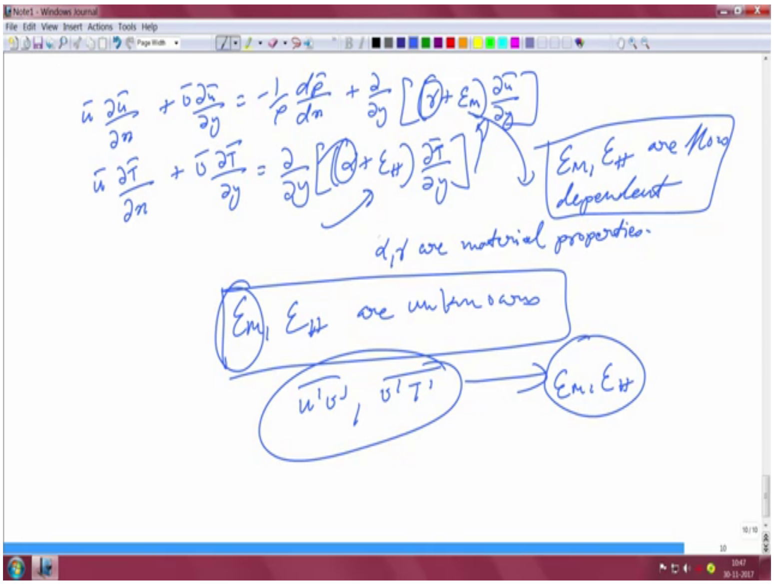

And you see what we have done. We have now packed using our model; we have packed all those quantities now inside the bracket. So, there is no u prime v prime, no v prime d prime bar anymore, all have been written in terms of the velocity gradients and the temperature gradient; the mean temperature gradient.

So, this is this solves the, I mean this kind of gets rid of the issue that we had, but because, but without using any physics we have just used some association; some hand waving arguments essentially and we have cast it in this particular form. Now of course, it as you can see over here these 2 terms unlike these 2 are not flow properties. These are not material properties; these are actually dependent on the flow.

So, epsilon M epsilon H are flow dependent. So that means, you change the flow you have to change this; whereas, your alpha and gamma are material properties. So, you it is not dependent on the flow at all these are material properties, you specify the fluid you will know these properties. This is dependent on the flow dependent, on the geometry dependent, on a whole lot of other parameters.

So, and the principle problem is that epsilon M and epsilon H are unknowns. So, really we have not got rid of the unknowns at all. What we have done is that we have changed 2 unknowns into 2 different unknowns. So, we started with u prime v prime bar and v prime t prime bar; these were the two unknowns. We have packed them somehow and we have converted them to 2 other unknowns, epsilon M and epsilon H.

So, we really have not solved the problem; what we have done essentially is that we have taken unknowns; this equation of course, looks a lot simpler, that I give it. But we have converted this to our new class of unknown that is all that we have done over here all. So, the idea is now you have to convert this particular quantity to something which is kind of meaningful; otherwise how are you going to solve the problem.

So, many of the many of the work actually spends a lot of time actually doing just exactly that. So, one of them was basically what we call the Prandtl mixing length model.

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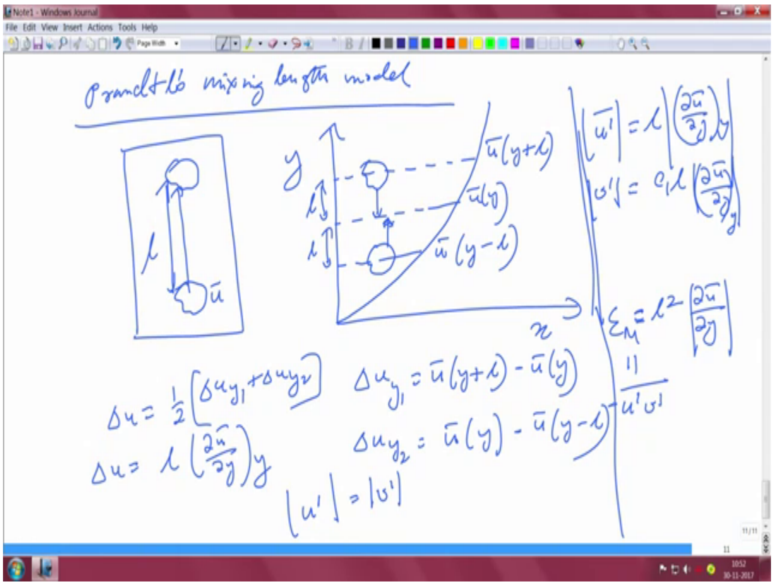

So, Prandtls mixing length model; there are numerous models. The turbulent literature is full of models actually. So, what the mixing length model actually says is that suppose you have a blob of fluid, whatever that fluid is it has got its own velocity; whatever that velocity might be. It traverses a distance l this is taken as that distance.

It can travel that particular distance without losing its initial momentum or initial velocity; before it actually mixes and gets dissipated into the whole thing. So, it is basically I am able to maintain my identity up to that particular distance. It is akin to the mean free path; that you get in your conventional statistical mechanics. It is the concept of mean free path this is the distance in which this blob of fluid is able to maintain its identity. So, that is that quantity l; we really do not know what that quantity will be.

But we can cast it in something like this. So, this is y, this is say x. So, you have a profile which looks like this. Say for example, so, this is say your u bar y. This is you say u bar y plus l and this is basically your u bar y minus l. These are all separated by distance is l; where, l is basically that Prandtl mixing length. So, if a blob of fluid comes from the upper layer; and comes down to this particular layer. So, there will be a change of velocity in this particular layer; obviously. So, it will induce a velocity in this particular layer because it is bringing higher momentum into this particular layer. So, that delta u at y will be u bar y plus l minus u bar at y; correct.

Similarly, if a blob of fluid comes from the lower layer to the top, it will bring a different kind of velocity. So, in that particular case, so, this we can call y 1. This we can call delta u y 2. Both are the same locations basically. So, u y minus u bar y minus l; so that is what is going to happen. So, your delta u is basically half of delta u y 1 plus delta u y 2; something like that.

Or in other words, this delta u is therefore, given as l d u bar by dy evaluated at y and u prime bar, the modulus is the same as your v prime bar v prime not the modulus. They are both the same and if you take the absolute quantity of this; this will be equal to l du bar by dy evaluated at y modulus of that and therefore, your v prime will be also equal to some C1 l some constant into d u bar by dy; well this is also evaluated at y.

So, what we get essentially out of doing all this. All this math is basically now we get epsilon M because this is how the terms have been cast in terms of l. So, it is becoming l square du bar by dy bar; the absolute value of this. So, in other words, your this was also equal to your u prime v prime bar basically minus of that. Because that is exactly what your model is.

So, epsilon is like this, I am sorry. So, epsilon is epsilon M is that.

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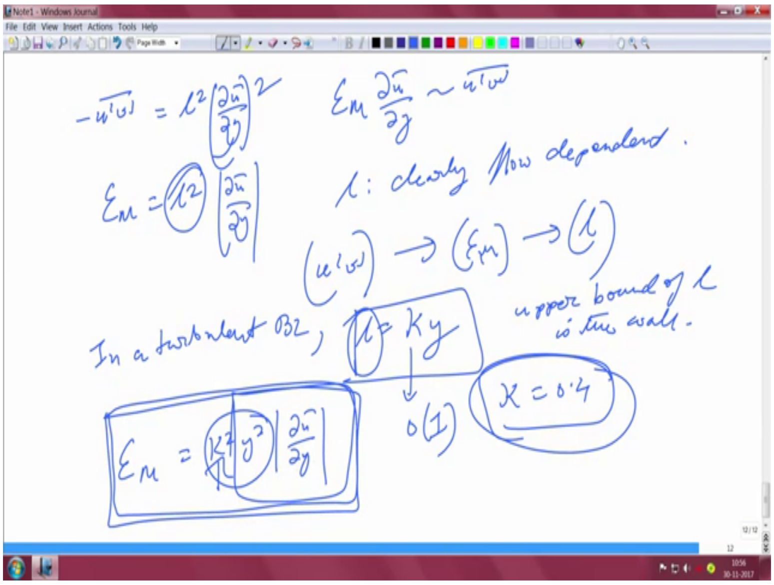

But your u prime v prime square bar is basically equal to l square du bar by dy square because it will be squared. But your epsilon M is basically l square into du bar by dy bar because there is already a du by dy bar for your epsilon. Because, the shear stress or the viscous stress is basically this. So, you multiply that; that is how you get your u prime v prime bar.

That's how you wrote initials on it. Now there is no general rule for estimating the mixing length because l is clearly flow dependent. So again, you still have this constant that is looming over there. We have converted now from u prime v prime, what we did? We converted that to epsilon M; now we have converted it to l. So, these are the transformations that has happened. So, it will vary l is clearly going to vary from 1 flow to the other.

So, the in a turbulent boundary layer. However, turbulent boundary layer; one can always say that l is actually equal to some K into y; where K, there is this is like an upper bound. So, upper bound of l must be the wall because if you hit the wall. So, therefore, people normally set it as l equal to K into y. So, this is an empirical constant which is of the order 1 essentially.

Now, for flat plate boundary layer, this K turns out to be about 0.4. This K turns out to be. So, the upper bound of l; you can write it down upper bound of l is the wall. So, this is once again ad hoc. So, we have substituted it by K into y. So, epsilon M is basically K square y square du bar by dy; this solves the first of the closure.

So, now we have quantities K is basically we say it is about 0.4. It is a constant of the order 1, the rest of the terms are basically all known. So, basically now we can solve this problem. We can solve this problem by substituting it into our basic governing equations, but it depends. It highly is dependent that what is the value of this l is that highly debatable quantity and how this l should actually vary with y that is also a debatable quantity.

But by using a simple argument that a blob of like a mean free path kind of an argument; we have been able to close at least one of the viscosities which is eddy viscosity and we have been able to show that this is what it is. So, the next last what we are going to do is basically we are going to look at the, now the problem of the turbulent boundary layer what a flat plate. And analyze that, how we can analyze this. Now that we know that there is something called a Prandtl mixing length? We have already isolated, how epsilon M can be cast in terms of known quantities.

So, using this formulation let us see that what the boundary layer profile will look like. What will be the shear stress; what will be the heat transfer and things like that. So, see you in the next class.