Convective Heat Transfer
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Lecture – 46
Reynold's Averaged Navier Stokes equation – I

In this lecture we started with Turbulent flows and we said a few things about Turbulence flows.

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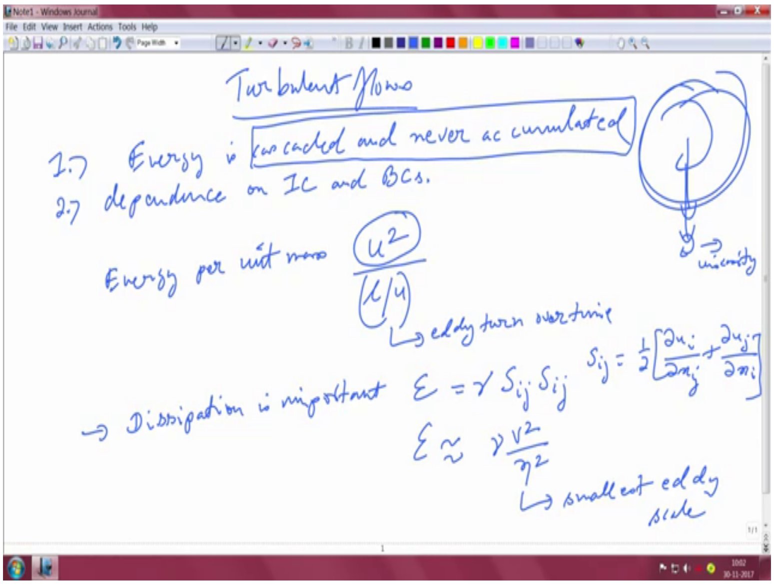

And the 1st point that we mentioned about turbulent flow is that energy. Energy is cascaded and never accumulated. This is one of the most important thing and of course, the dependence on initial conditions and boundary conditions. This I explained by giving you the example that if you take the same measurement at the same time instant even for a pipe flow, you will find that the velocity profiles will come out to be very different.

Whereas, the statistics that is the 1st moment and the 2nd moments those will match. So, no matter, how good your experiment is? Or how good your simulation is? It is always the nature of Turbulence is to amplify those minute differences. So, that you get very widely different profiles in the temporal space. So, it is, but the important part was we also said that the energy is always cascaded and it is never accumulated. That means, from the large eddies, you get the energy which is transferred to the smallest eddies;

where viscosity becomes important and viscosity actually dissipates the energy in the system.

So, the energy per unit mass; per unit mass, it can be given as u square which is the that is at the highest, that is at the largest eddy level is u square divided by l by u; where this l by u is basically called the eddy turnover time. It is basically the time that it takes for an eddy to rotate, to break or to rotate. So, that is the eddy turnover time. So, if one is the length scale and the velocity is u. So, l by u gives us the time scale which is called the eddy turnover scale. And u square is basically the energy that is carried by that eddy.

Now, so, this is the energy at the largest eddy level. Now, for the smallest eddies, we said that dissipation is important. Dissipation is important. What is Dissipation? Dissipation takes place where viscous stresses are actually more, the most important parameter. So, if Dissipation is taken as epsilon it is given as. So, this is written in the index notation; where, S ij is basically given as half.

So, this basically is basically nothing but the cross derivatives or basically ui differentiate it with respect to xj. So, the contraction happens over i and j. So, this is basically what we call the Dissipative scale or the Dissipation level. Now, this will scale from our scaling arguments, we can say that the dissipation will scale something like because it is Sij Sij. So, basically this is v by eta; where, x if you remember this is the smallest eddy scale.

So, eta is the smallest eddy scale and v is the corresponding velocity. So, in other words, we know that since, there is no energy accumulation. So, whatever energy is possessed by the larger eddies has to be pumped into the smallest eddies, where they are actually dissipated.

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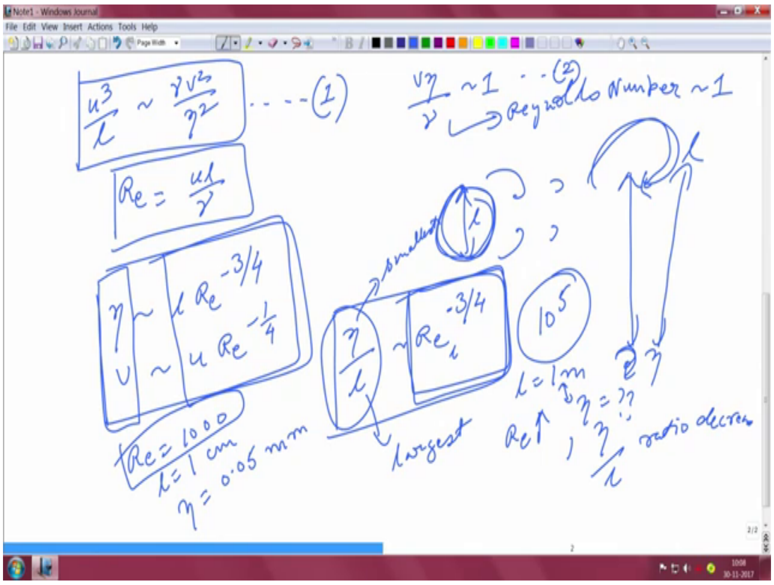

So, basically this translates to that your u cube by l has to be the same as your gamma v square by eta square. So, in other words, the energy per unit mass of the largest eddies has to match whatever is the dissipation level. And so, this is the first important relationship that we have, let us call this 1.

The other important relationship is that v eta by gamma should be of the order 1 at the smallest eddy scale because here, this means here at the Reynolds Number. This is basically nothing but the Reynolds Number.

Reynolds Number is of the order 1, when Reynolds Number is of the order 1; because it is a ratio of inertia versus the viscous stresses. When it is of the order 1; that means, your viscosity is important. When eta is of the order of l, in that particular case your Reynolds number is say 1000, 3000, whatever it is. So, in that case the inertia is much more dominant than the viscous stress.

So, these are the 2 relationships that you get. This is 2. Now, what we can do is that, we can get rid of the scales and we define our normal Reynolds number as ul by gamma. So, this is the Reynolds number based on the largest scale. In most of the cases, if you recall the example that the eddies are shed behind as sphere. Then this l will be of the order of the size of the sphere because the largest eddies are always of the size of the obstacle in this particular case.

So, if you take these 2 situations into consideration; from 1 equation, you will get your eta. It will be equal to or scale as l Reynolds number to the power of minus 3/4th. The other one will be v will be u Reynolds number to the power of minus 1/4th. So, these 2 corresponds to the smallest eddies; these basically correspond to the largest eddies as simple as that.

So, in other words, your eta by l basically scales as a Reynolds number, you can write it l if you want. It is 3 4th. In other words, the ratio of the 2 scales, this is the smallest and this is the largest. It is actually proportional to Reynolds number raised to the power of minus 1 4th or in other words, as the Reynolds number increases, the difference between these two scales actually widens.

So, if you are dealing with say for example, reynolds number equal to 100 or 300, you will have a scale difference as you go on increasing the Reynolds number; that means, this, this fraction is becoming smaller and smaller in nature. It would mean that eta will be a smaller and smaller fraction of l. So, this essentially implies that since eta is becoming a smaller and smaller fraction of one that disparity between the 2 length scales; that means, from between this and this. This is of the order l; this is of the order eta. This disparity actually goes on increasing.

So, of between the larger scale and the smallest scale, the disparity increases as you go on increasing the Reynolds number and this poses one of the most serious problems in turbulence modeling. So, as Reynolds number increases, eta and eta l ratio; eta by l ratio decreases. So, for example, if you take Reynolds number equal to 1000, say as a sample case l is of the order of 1 centimeter. Then, you will have eta which is of the order of 0.05 millimeter or in other words 50 microns.

So, for a 1000 Reynolds number case, 1 centimeter is a largest eddy; whereas, 50 microns is a smallest eddy. So, this poses a very serious problem, if you want to do say for example, Turbulence modeling. Because in that particular case, say if your Reynolds number is of the order of 10 to the power of 5 highly Turbulent flow. You can calculate that if your l is of the order of 1 meter; what will be the value of your eta? From this particular expression, it is very easy to calculate.

You will find that the disparity between these 2 scales is so huge. That means, your grid if you are trying to basically discretise this whole thing and do a CFD, of the whole problem; you will find there is one and only one important bottleneck here. And that bottleneck is that your grid size has to be very small. So, not only it has to resolve the eta scale because that is where dissipation is important.

At the same time, you have to resolve the largest scale which may be of the order of 1 meter. So, even for very simple flows, this is a very taxing process. Like for example, if you do a simple flow through a pipe or a jet that is coming out; this is a very very taxing process to begin with. And more the higher the Reynolds number and more complicated the flow is, this creates a insurmountable problem in CFD.

Sometimes, if you with the best of the supercomputers available; it can take the years, years. Actually to solve physically you know a realistic level problem say flow through a gas turbine is one of the realistic level problems. On the top of that, if you have a chemically reacting flow; that means, there is combustion and things like that, going on that happens at the molecular scale.

So, in order to resolve that, you further need to have that kind of resolution. So, the idea of turbulence is as follows, before we take on the next dive. That Navier stokes equation still governs the turbulence. Let us be clear about, that it still governs turbulence. The main problem is that the sensitivity to initial condition and boundary condition is a one of the prime as we already narrated, one of the prime factors.

Because of the sensitivity, because of the non-linearity of the Navier stokes equation any things gets amplified and turbulence is basically a manifestation of that. Now on the other side, this disparity of the length scales; that means, you have too many a whole region maybe from 50 micron all the way up to 1 centimeter, on 1 meter basically this is the range that you need to cover; that means, you not only have to have to see what is the dynamics of the large it is; you have to also see the dynamics of the smaller eddies.

So, all these things needs to be done in a computational framework because analytically it is not solvable. That is very hard to do, but people do something called Direct Numerical Simulation which is called DNS and there are other things that people do. But that is for very simplistic level problems to understand certain things. But other than that, 1 has to devise some kind of a framework through which you can address this turbulence because it is a very time consuming affair. And if you are working in industries or you are trying to solve any engineering problems, you need answers you cannot wait for years for a simulation to complete that can be long term.

But at least you need for quick design fixtures and things like that, you need to know that what is, what can be done with this Navier stokes equation; maybe a simpler version can be used to understand Turbulence. So, Navier stokes equation as such is not unclosed. So, when you hear people saying that a closure problem in turbulence is one of the main factors; that closure problem is actually created by us.

Normally the Navier stokes equation is not unclosed. It is exactly the same. It is like that when we try to extract certain statistics or we try to do some simplifications or to the Navier stokes equation; that is where we initiate certain things which you will see in due course that, what are those certain things that happens?

So, turbulence another thing that we should like to mention over here is as follows and this is also important.

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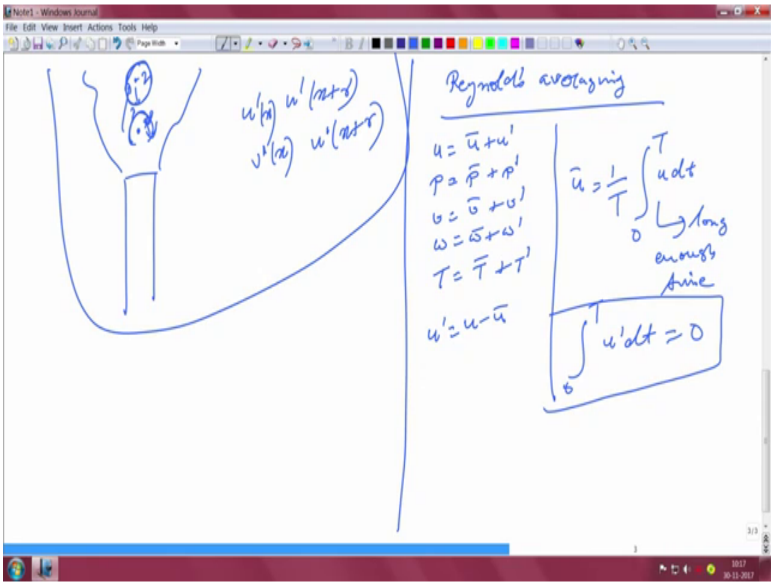

So, if you take a jet say coming out of a pipe. So, it can be the tailpipe of your engine say for example. So, the flow is turbulent and you take one point over here, another point over there separated by a certain distance. So, what do you expect to see? You expect to see that if this point 1 and 2 are basically close to each other, you will find that when there is a vortex that is being shed; when there is a eddy turnover over here that would have some effect in 2 as well if they are very close to each other.

But as 1 and 2 get separated from each other; that means, as their separation, this increases you will find that the effect slowly becomes gradually nullified. So, typically we address these problems; we will do that later, what we call the 2 point correlation kind of a technique that how if you create our disturbance here, how does it how is it correlated with a nearby point.

So, this can be done as a spatial correlation; this can be done over temporal correlation. So, all kinds of things can be done or in other words, so, you have an eddy that is being shed depending on the level of turbulence, that effect of that eddy somehow will be felt at 2 as well.

So, if you measure the velocity profile at 1 and you measure the velocity profile at 2, you will find that there will be a degree of correlation between the 2 and this correlation does not really happen with respect to 1 component of the velocity. Say for example, you are measuring u prime.

So, u prime you are measuring at 1 particular point and then, you are measuring u prime at some other points x plus r this may be x. So, you are measuring u prime at 2 particular locations. Similarly your. So, this is 1 component of the velocity disturbance has created a fluctuation in a nearby field.

Similarly, you will have like a v prime at x can actually cause a u prime at x plus r. So, basically you can have a parcel of fluid which quickly goes from 1 point to the other. It will not only cause a variation in the u prime, but also in the v prime. So, that is the whole idea.

So, all these components in turbulence is highly 3d in nature. Turbulence is highly 3d. So, it is. So, that is one of the basic concepts. So, you can have a lot of statistical techniques, some of which we will try to see given the interest of time, but will. But one important factor will be for a turbulent researcher is to find out how good the correlation is? Where the flows are basically correlated to each other?

So, but those are statistical techniques that will come a little later. But keep in mind that when you have a fluctuation in the x component of the velocity, you have do have a fluctuation in the y component of the velocity as well. So, you cannot take those terms to be equal to 0 and that is exactly paves the way for some of the closure problems that we are going to have very shortly.

So, we are going to introduce, what we call something called the Reynolds average to Navier stokes equation. But as we said the turbulence is kind of a either you know it has got some statistical origin and things like that we always say things like this. So, let us look at what is Reynolds averaged Navier stokes equation before we go to the and boundary there. So, it is called Reynolds averaging.

So, Reynolds averaging means that you have a u, which has got a u bar and you have a u prime. So, basically you are dividing the terms into 2 terms basically 1 is a mean component and we will see what the mean is. And one is a fluctuating component. Similarly, all quantities will behave like that. So, P will have a P bar plus P prime sorry. We will have a v bar plus v prime; w as I say turbulent is hardly the highly 3d and even if we have temperature T will have a T prime.

Now, here your u bar is basically summed over. This T is not the temperature. This is basically the time averaging. It is done over a long enough time period. It is done over a long time period. So, that you basically smother out everything. Similarly, your u prime; if you take an average of u prime 0 to T, u prime dt that should be equal to 0 because it is a random fluctuations. So, if you average a random fluctuating quantity over time this will be equal to 0.

So, u prime that is just the fluctuation; u prime is what u prime is basically u minus u bar. So, the average of that if you take an average of that particular quantity; this should be equal to zero. So, let us establish certain rules based on this which will be needed.

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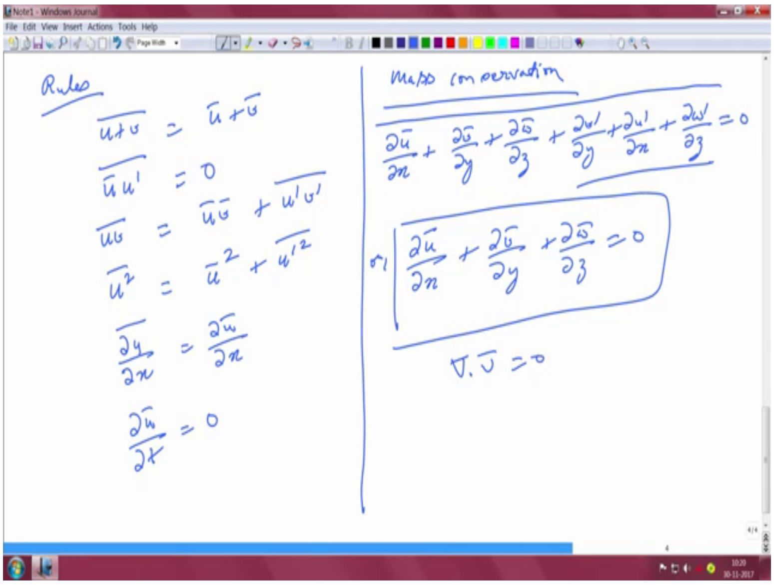

So, these are the rules; u plus v bar some of the average of the summation of 2 quantities is basically u bar plus v bar.

u bar by u prime bar; that means, it is a multiplication of u prime and u prime bar. That is equal to 0. u v bar that is equal to u bar v bar plus u prime v prime bar. This is not equal to 0. So, u prime v prime bar. Similarly, u square bar is given as u bar square plus u prime square bar. This is also not equal to 0.

So, basically the product of the 2, average of the product of the 2 fluctuations is not equal to 0. Similarly du by dx bar given by this. So, these are the rules that we lay down for the current problem. So, based on these rules, now, let us do the first stuff which is mass conservation. The mass conservation should be the first customer in the business.

So, what we do is basically you have first you split the quantities; that means, it comes out as this. Now, if you apply the rule; now over here, you see that these are all fluctuating quantities over here. So, naturally they will actually go to 0. So, effectively your mass conservation equation looks exactly the same as your mass conservation equation would look like, even if you did not do any averaging.

So, it is basically that is what we get. So, it is basically the dilatation of the averaged quantities is basically equal to 0. So, in the case of a laminar flow, you did not have those bars over there, but this exactly looks that that. So, there is absolutely no problem. So, if you apply a statistical tool, we have a basically applied a statistical averaging to the Navier stokes, I mean to the governing equations and we find that the conservation equation behaves in exactly the same way as you would expect.

I mean in a laminar because the conservation of mass equation is basically linear in nature, but things gets a little dicey.

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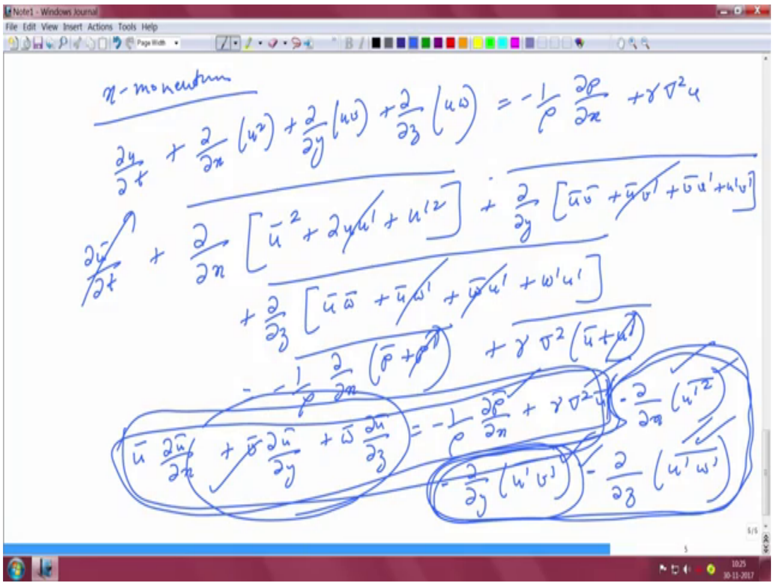

When we actually go to the x momentum equation and the y momentum will be very similar and the z momentum will be similar as well. So, what you have, if you write it in the conservative way. So, this is the equation. Now, what we do first? You split the quantities.

So, you have your du bar by dt plus and then, you take the average. So, it is u square bar plus 2u u prime plus u prime square, you take average plus d by dy u bar v bar plus u bar v prime plus u bar v prime you take the average of that plus dz u bar w bar. This is equal to minus 1 over rho d by dx P bar plus P prime. Once again, you take the average of this plus take the average of that.

Now, what will happen? Several of these terms will basically knock themselves out; these terms will all go correct. So, and this term also and all these components will go because these are all fluctuating quantities. It has got no pre factors. So, ultimately the equation will be u bar du bar dx plus v bar du bar dy plus w bar u bar dz dP bar by dx plus gamma minus, minus.

So, as you can see just pay a little bit of an attention to the last, these up to this point it looks exactly like your Navier stokes equation. Ad-vective term, pressure term, viscous term; these 3 terms that we have accumulated all of a sudden, these 3 terms that we have accumulated. These are basically has evolved because of the reason that the Ad-vective terms over here they were non-linear in nature.

Because they were non-linear in nature, out pops these 3 additional terms, these additional terms has got it is origin in the advection quantity. They are not of viscous in nature. They originate from the convection, but why we have taken it to the right hand side; we will explain in a little bit. But these terms basically we have no idea, what they are 1 is you prime square bar, 1 is u prime v prime and 1 is u prime w prime bar. So, all these correlated quantities like what we say it like how u prime is correlated to itself all.

All these correlated quantities or what is the degree of correlation, we have no idea that how these terms are related to the flow field. What is their nature? The rest of the terms we know; we know exactly that this is the Navier stokes equation. This is what happens? So, these terms are basically unclosed in nature. They are not closed and because they are not closed; there lies the problem that when you do our, apply a statistical averaging to Navier stokes equation because of the non-linearity, you do get this additional popped out terms; 3 terms here.

So, each momentum equation will have three terms essentially, but they are symmetric that is means u prime v prime is the same as v prime u prime. So, if you put them in a matrix, basically you have you have a symmetric 1. So, but the all these terms are basically unknown and we have to find a way to understand what is the nature of those terms. Otherwise, your equation simply becomes unsolvable in nature.

So, this creates the first of the closure problems. So, in the next class, what we will see that what will happen to the energy equation because the energy equation will also have similar things that will that will come up as a result of this. So, see you in the next class.