Convective Heat Transfer
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Lecture – 32
Similarity solution

So, welcome to this particular class. In the last class we did the integral analysis of the natural convection boundary layers and we showed that, what are the expressions for the nusselt number under those 2 cases. Now if you recall the 2 expressions that for a high prandtl number.

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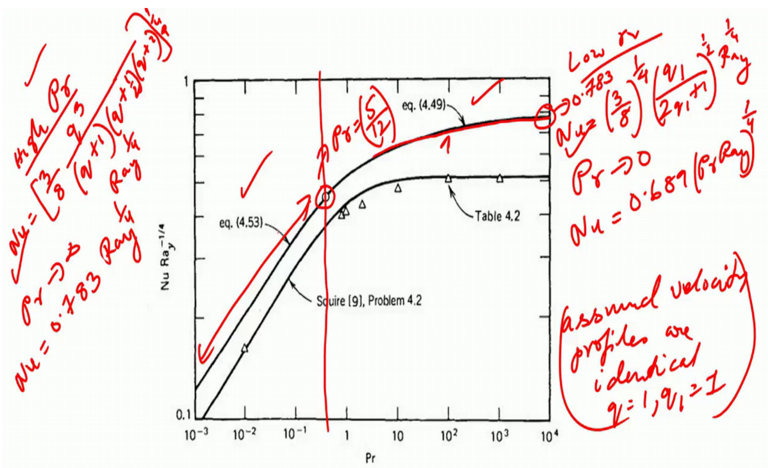

Figure 4.4 Local Nusselt number for laminar natural convection on a vertical wall: integral versus similarity results.

Your nusselt number expression was if you recall your notes, raise to the power of 1 4th into Rayleigh number to the power of 1 4th, this is raised to the power of 1 4th this particular parameter. So, that was a high prandtl number limit. So, if you look at this particular graph for example, do not consider the lower one consider the upper one this particular graph you can see that the high prandtl number regime is something like this.

So, that is the high prandtl number regime. Similarly, in the low prandtl number regime we already showed through our scaling analysis and through the integral formulation that it is 3 by 8 1 4th q 1 divided by 2 q 1 plus 1 half Rayleigh number to the power of 1 4th. So, that is the low prandtl number regime. So, that would be this portion of the graph, with the merger point close to 1. So, at about one the 2 graphs will actually merge.

So, as you can see in this particular case of course, in the limiting values we have already said that what is the limiting value; that means, when prandtl number approaches infinity, when prandtl number approaches infinity the nusselt number tends to become about 0.783 Rayleigh number y 1 4th. So, it would go this particular value will be about 0.783. On the other hand, when prandtl number goes to 0 the nusselt number becomes 0.689 into prandtl number Rayleigh number to the power of 1 4th that is a lower bound of this particular graph.

So, as you can see over here that the nusselt number expressions match at around prandtl number equal to 5 by 12. So, this point is not 1 it is about 5 by 12. This is the point where the 2 nusselt number expressions that you see here and here they basically show the same value a singular value, or very close to each other. So, they kind of merge they show a match for these 2 values.

So, the of course, here the assumed velocity profiles are identical for q 1 and q 1 1 equal to 1, that is the case. Now in this particular case you can see that the nusselt number to a certain extent depends on the analytical forms of the velocity and the temperature expressions. That it is there is some dependence, but when we choose a velocity profile or a temperature profile this always the trade off and that trade off we will see how much is that trade off, when we actually do our similarity solutions because our similarity solutions will give you that kind of a profile.

So, the similarity solution in itself has got that extra bit that it can predict the exact shape of the profiles, but this is also pretty reasonable from the 2 equations. We have been able to get the 2 expressions and we have been able to put them in a same plot and this is the point where the merger actually happens very close to one kind of and or oneish and from the 2 sides you get the 2 families of curves. So, these are the important factors that you should keep in mind before we move on to the similarity solutions now. So, let us go to the similarity solution now.

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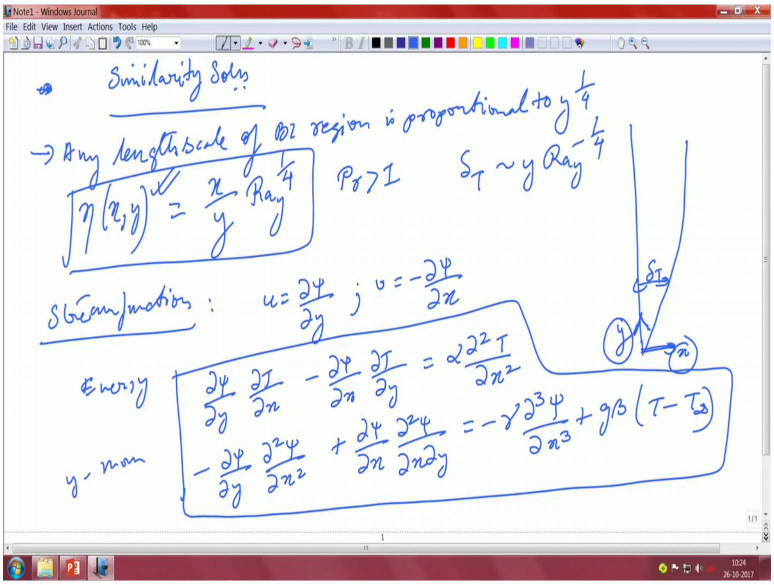

So, the similarity solution here, we will use some of the similar arguments that we have used earlier. Here of course, you see that there was a peculiar thing coming out of it that delta by h was scaling as Rayleigh number to the power of 1 4th. So, that provides like we did in the case of our flat plate boundary there remember there was delta by x was scaling as Reynolds number to the power of half. So, that was a expression that we got earlier. So, there what scaling did we do? We did the scaling with respect to that. So, here also we will use something very similar in this particular case.

So, any length scale the first point to note. Any length scale of the boundary layer region is proportional basically to y to the power of 1 4th that is what we have seen. So, the dimensionless variable can be constructed and it can be written as eta which is a function of x and y this is very standard this eta form you have seen earlier also it basically absorbs both the variations x and y and cost is in terms of one variable only. The validity of this expression will be proven that once we solve the equation and come to the final expression right it would be an o d e which would only be a function of this.

So, that was the whole idea. So, the similarity variable this particular similarity variable can be written as x by y into Rayleigh number to the power of 1 4th. This is very similar and intuitive if you have done your forced flat plate boundary layer expression right is very similar in spirit to what we have d1 over here. And let us say prandtl number is greater than 1. So, this is the most appropriate scale for the thermal boundary layer right because if you know delta T the most proportional it was proportional to something like this they this we knew.

So, we have used that very scale over here to get to this similarity variable. So, that was the boundary layer thickness. If you recall that was what was done. So, delta was a height, the delta was that particular height of the boundary layer at any particular instance where x was the direction so if you have forgotten. So, this was x that was y correct, this was x that was y.

So, this x and y were like this so what it essentially means is the delta is basically whatever is the thickness at any particular y. So, that is how the scaling has actually worked. So, this particular thing we got from the scaling and also from the integral analysis we proved it, now what we are going to do we are using this variable now proposing this variable as the similarity variable over here.

So, we also introduced the stream function if you recall you can solve this similarity solution different ways stream function is one such approach. So, u is given by d psi d y and v is given by minus d psi d x. So, the equations become. So, let us cast the energy equation. So, d psi d y d T d x minus d psi d x d T d y is equal to alpha del square T d x square.

On the other hand, the momentum equation which is basically the y momentum equation d psi d y d square psi d x square plus d psi d x d square psi d x d y equal to minus gamma d cube psi d x cube plus G beta T minus T infinity. These are the 2 expressions that we had earlier. It is just there is no similarity transformation that has happened it has been just substituted that the u and the v profiles have been just substituted over here.

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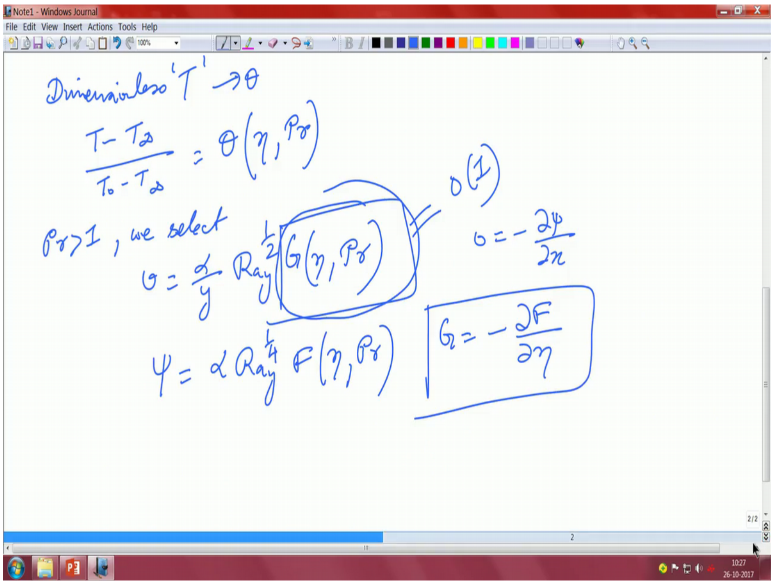

Now, we note one other thing say the dimensionless temperature, which we can call theta. In this particular case which is basically T minus T infinity divided by T not minus T infinity should be given as theta. Now this theta should be a function of definitely nita, that is obvious it should can also be a function of prandtl number give or take. It should be a function of your prandtl number as well. If you recall all the expressions that we did earlier where we had nusselt number and delta T and all these other things you saw the prandtl number dependence. So, this dependence may be actually as not a major dependence that we will also see whether this is a major dependence or not, but this dependence has to be incorporated into this particular form.

Now, for prandtl number greater than 1, we select the v velocity which is the velocity along the y direction right as alpha by y Rayleigh number to the power of half into some function which is given as a function of once again nita and prandtl number.

So, Rayleigh number to the power of half this, we already knew, alpha by y Rayleigh number to the power of half this was already there. We are just introducing this additional functions so that because we are not sure what is going to be the case and of course, we one thing that we know this will be of the order one because if you recall your v expression from your previous scaling arguments, you will find that it is alpha by y Rayleigh number to the power of half into some prandtl number coming into the picture. So, it this this factor that I have said because if our scaling is accurate then the prefactor which sits in front of it cannot be a factor which is more than 1, it should be of the order 1. So, this G that we have put over here is of the order 1 that much we can say.

So, from the definition that v is actually equal to d psi by d x. We can write psi as alpha Rayleigh number y to the power of sorry 1 4th F nita comma prandtl number where G is basically d F by d nita. It is just for convenience we have done it that way. So, this is the psi that we have.

Now, what is remaining if you look at the previous expression here everything is in terms of the T and psi. Everything is in terms of psi or it is in terms of T. So, it essentially means in this particular expression what we have done we have cast it in terms of theta. T has been cast in terms of theta and psi has been cast in terms of Rayleigh number and the function F that is all. So, essentially what we can do once we substitute all these things all the factors will actually go out all the T, all the v expressions or all the psi is will be replaced by basically Rayleigh number F G and this kind of terms. So, that is what exactly happens, let us move into the next page and show that.

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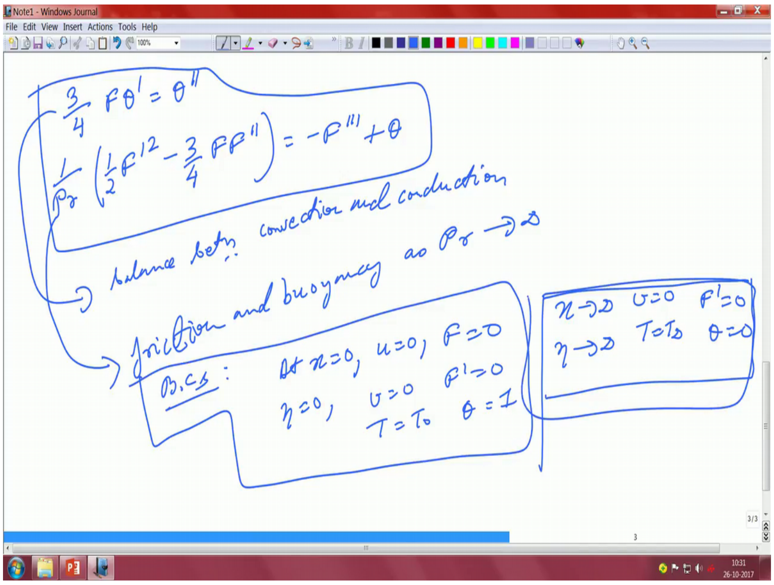

So, 3 4th F theta prime is equal to theta double prime. So, this is the expression that you get after the substitution you can do all the math yourself. Half F prime square minus 3 by 4th F into F double prime F triple prime plus theta. So, one is momentum one is energy so that is what we have got over here.

So, this this equation 4 points I mean this particular equation which in Bejans book is 4.61. This equation is a balance between convection and conduction which is obvious that is what it is supposed to be. Where the momentum equation this is the momentum equation. So, is a balance between friction and buoyancy as prandtl number goes to infinity it is a balance between friction and buoyancy, if the prandtl number goes to infinity because if you can look on the expression on that left-hand side prandtl number is there in the denominator. So, it will be 1 by infinity. So, this part will just basically vanish the left-hand side so it will be a balance between the other 2 terms.

So, this also agrees with what our scaling argument initially was. Where do we say that if the prandtl number is very high? It is a balance between friction and buoyancy, if the prandtl number is very, very low it becomes a balance between the inertia as well as the buoyancy will be always there right, if you recall those expressions.

So now what are the boundary conditions so the boundary conditions are important to solve this? So, at x equal to 0 u is equal to 0 F is equal to 0. At nita equal to 0 that would mean v equal to 0, F prime is equal to 0, T will be equal to T naught, theta will be equal to 1. Check all these things out it should be pretty self-obvious. As x goes to infinity nita goes to infinity correct. So, v goes to 0 T goes to T infinity F prime goes to 0 theta goes to 0.

So, this actually brings to the strategy; that means, we have cast now the 2-general form of the equations. And we have also cast now the boundary conditions, these 2 boundary conditions these 2 sets of boundary conditions.

Now, it is anticipated that you would just now solve this 2 expressions and there are lots of we already discussed about it there are lots of methodology. Except for here it is F and theta both so that is how you have to solve it in a very coupled kind of a fashion, but there is no ambiguity in the solution here sufficient number of boundary conditions you have the equations which are well posed the expressions are odie's in their own way. So, there is no problem as such in casting these particular forms.

So, let us look at the, what we get out of this all.

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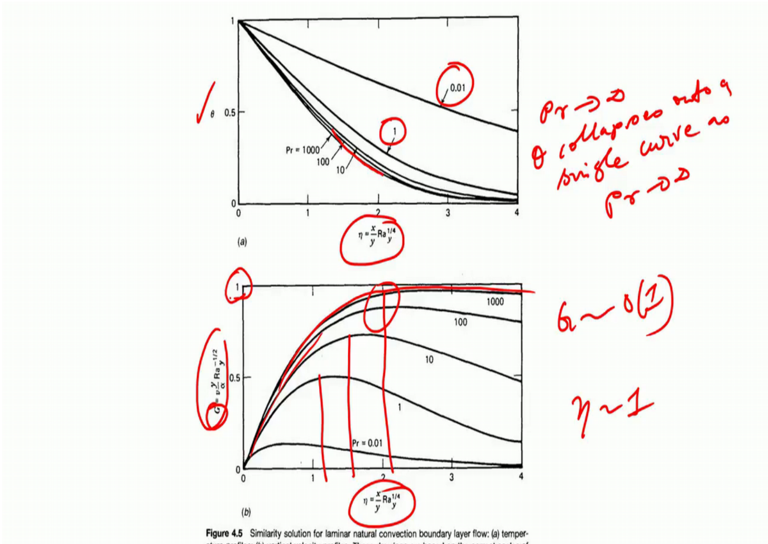

Figure 4.5 Similarity solution for laminar natural convection boundary layer flow: (a) temperature profiles; (b) vertical velocity profiles. These drawings are based on the correct scales of the δ_T-thick layer in a Pr > 1 fluid.

So, after the solution this is what we would get for a prandtl numbers, I mean you see the whole expression for all types of prandtl numbers we have exactly plotted it. So, the first one is basically theta, plotted against the similarity variable which is nita. A couple of key things that you see immediately that as prandtl number goes to infinity or becomes very, very high you can see that all these theta curves kind of merged with each other do you see that all the theta curves seems to be stacking up.

So, they tend go towards one single curve. So, they are basically moving as you can see there is a lot of difference when the prandtl number is very small. This is prandtl number 1 as you go on increasing the prandtl number from 10, 100, 1000 you see the curves are becoming very close to each other so they are stacking. So, for example, from 10 to 1000 it is a prandtl number of 2 orders change, but the graphs have not shifted at all. So, you can see what we can conclude that the theta, if you talk about theta is that, the theta basically collapses onto a single curve as my prandtl number approaches infinity.

Why does that thing happen? Now the question is that why does it is such a thing of prandtl number I mean why should this all this prandtl number should actually the all the temperature profile should actually merge in that particular way.

So, why it should not merge, why it should merge is because if you look at the corresponding nusselt number profiles, if you look at the corresponding nusselt number profiles and other things, you will find that when you actually have a high prandtl number fluid there the prandtl number dependence basically goes away.

You have already seen that that for high prandtl number fluids, really high prandtl number fluids are nusselt number dependence does not have a dependence on the prandtl number as such and that is naturally appears in this particular expression, because if you look at your national number is also given as you know it is a slope of this particular theta prime at nita equal to 0 into the corresponding Rayleigh number so that is what it is.

So, you can already see that that collapse is significant so; that means, the scaling argument was correct, it merges on to a single curve as we go on increasing the nusselt number. Now if you look at the corresponding G over here, which is basic and plotted; once again versus this. So, so this is basically the velocity expression, remember the velocity this is v y by alpha into Rayleigh number to the power of minus half. So, here you can see once again that the peak in the velocity, these are the peaks in the velocities right these are basically the peaks in the velocity they all happen when nita is of the order 1.

This is also very significant because we said that when the prandtl number is actually high. What will happen is that the peak of the velocity should happen at delta T right and delta T is basically your nita in this particular case right because nita and delta T is basically the same. So, so that is what it is over here that at nita equal to or the order 1 or y order 1, I mean just write about 1. When it is around one the portions of the velocity profile approach kind of a single graph. So, this also starts to stack up so if you draw the next profile it will be somewhere like that.

So, the dimensionless velocity peak is consistently of the order 1 that is one thing. So, this G if you recall we said that this will be of the order 1. If our scaling was correct. So, these are all characteristically coming out to be around 1 so; that means, the scaling argument was correct to begin with because all these velocity profiles now are exactly showing of the order 1 to begin with alright.

So, this is a significant finding and it also shows that our scaling argument was actually the right one. And the velocity seemed to peak around nita equal to 1 which is also predicted in that sketch that we made. And it also showed that all these velocity peaks are kind of merging towards the same profile as we go on increasing the prandtl number. So, all this things are very significant arguments over here.

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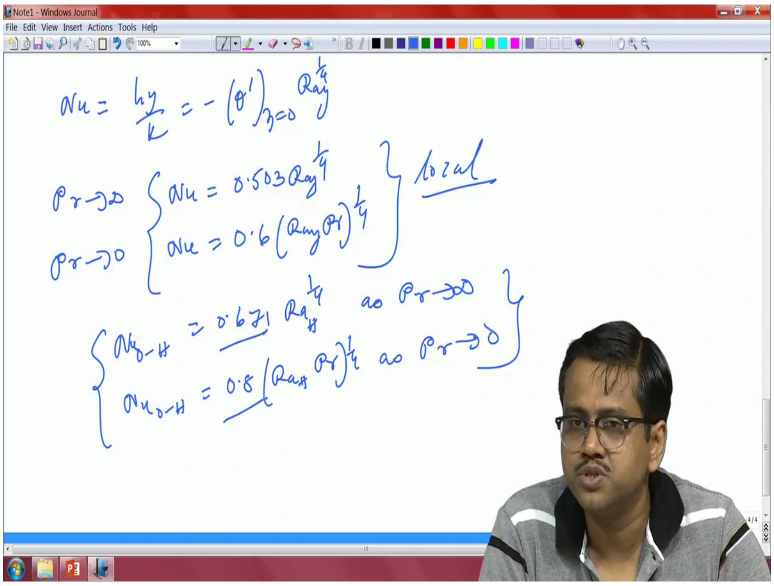

So, let us look at go back to our sheet and the local heat transfer coefficient h y by k equal to minus theta prime nita equal to 0 Ray 1 4th. So, when prandtl number goes to infinity, nusselt number becomes 0.503, Rayleigh number to the power of 1 4th, when Prandtl number goes to 0 nusselt number becomes 0.6 into Rayleigh number y prandtl number to the power of 1 4th. This is very close to what are the values that we got from our integral formulation.

The average Nusselt number these are the local these are the local nusselt numbers right the average nusselt number is when you integrate it from 0 to h; that means, along the length of the plate this becomes 0.671 into Rayleigh number sorry this will be no Rayleigh number h to the power of 1 4th as prandtl number goes to infinity and this will be 0 to h is equal to 0.8 into Rayleigh number prandtl number 1 4th as prandtl number goes to 0.

So, this is what we got earlier this is the local and it comes out to be pretty close to what has been predicted by the integral formulation. So, we can see from this exercise that how just by using the similarity transformation in a very similar way like what we did in our flat plate boundary layer forced convection part, we are able to get the similar form over here as well. So, you can also see one other thing if you look at this table now.

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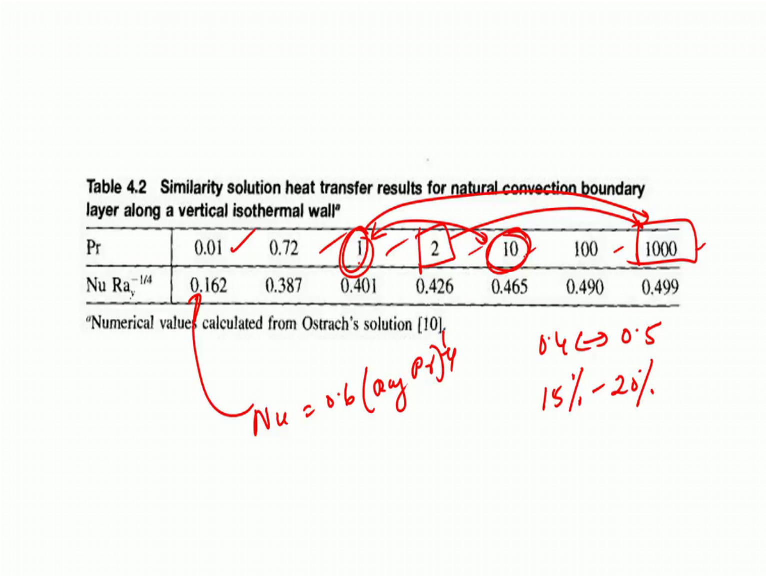

Table 4.2 Similarity solution heat transfer results for natural convection boundary layer along a vertical isothermal wall[a]

Pr	0.01	0.72	1	2	10	100	1000
$Nu\, Ra_y^{-1/4}$	0.162	0.387	0.401	0.426	0.465	0.490	0.499

[a]Numerical values calculated from Ostrach's solution [10].

So, here you can see this is the basically whatever I said for different prandtl numbers we have noted down the values of the nusselt number corresponding Nusselt number. So, as we can see that when nusselt number is of the order of infinity this comes out to be around 0.5 which is what we predicted right and when nusselt number is very small this becomes about 0.612 taking into account the nusselt number into consideration, when Nusselt so basically as you know the nusselt number is about 0.6 into Rayleigh y into prandtl to the power of 1 4th. So, this corresponds to this. So, if you put whatever is the value of nusselt number here say 0.01 raised to the power of 1 4th and then multiply it by 0.6 this is the kind of value that you are going to get.

So, as you can see, but one couple of interesting things that you can note out of this exercise is that, when we change the prandtl number from 2 all the way up to 1000 there is almost order 3 change. Let us take 1 to 1000 that is proper order 3 change. The this nusselt number values the prefactor changes from 0.4 to about 0.5 right 0.4 to about 0.5. So, that would roughly translate to not as very significant change about 15 percent.

So, we get about a 15 percent of that order kind of a change 15 to 20 percent change, when the prandtl number varies by 3 orders. So, and if you observe now between 1 and 10 this change is even miniscule alright. So, one is for example, for air it is about 0.72. So, it is roughly of the order 1 right and on the other hand for water it is about 5 to 6, 6 is so, there is not much of a difference.

So, this would mean that the nusselt number is a much remember when you started this exercise we said it that our F or our, the function, which depends on nita as well as prandtl number. We actually said that prandtl number dependence can be a weak dependence and it is actually shown in this calculation that indeed that prandtl number dependence is pretty weak because we are changing the a number quite a bit the result is changing by a very small amount.

As long as you keep your Rayleigh number to be the same if your Rayleigh number changes there is a huge change because it is you can see it is Rayleigh number to the power of minus 1 4th. So, it is a huge change. So, it is a weak dependence on prandtl number this also satisfies one other requirement that for example, if you are trying to for example, carrying out experiments with water is much easier than carrying with air because air being gaseous and all those things.

So, people can get away they can conduct experiments with water and still they can use the same value for air because the change is not more than like about 10 to 15 percent. So, you know the exact error is not that much and you can live with that error. So, you can do the work in a much you know benign system like water which is easy to handle or vice versa depending on which kind of problem you are looking at and still you can get away with it.

So, this is the most interesting part that can that has come out of this particular exercise. That you can actually have you know this kind of behaviour where we have clearly established Rayleigh number dependence is much stronger than the corresponding prandtl number dependence. See you in the next class with the next feature which will be on uniform heat flux.