Convective Heat Transfer
Prof. Saptarshi Basu
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

Lecture – 31
Integral solution

So, welcome to this particular lecture. Now in the last lecture, if you remember we basically solve the 2 scaling extremes and we showed that what are the relevant numbers, non dimensional numbers as well as what are the corresponding boundary layer thicknesses. Mainly it was delta T, delta and delta V. These are the three types of length scales that we actually derived. Now, to compile all of them, you can look at table 4.1 of Adrian Bejan.

(Refer Slide Time: 00:43)

Table 4.1 Summary of flow and heat transfer scales in a natural convection boundary layer along a vertical wall

Prandtl Number Range	Thermal Boundary Layer Thickness	Wall Jet Velocity Profile: Distance from Wall to Velocity Peak	Wall Jet Velocity Profile: Thickness of Wall Jet	Wall Jet Velocity Profile: Velocity Scale	Nusselt Number $Nu = \frac{hH}{k}$
$Pr > 1$	$H\,Ra_H^{-1/4}$	$H\,Ra_H^{-1/4}$	$Pr^{1/2}(H\,Ra_H^{-1/4})$	$\frac{\alpha}{H} Ra_H^{1/2}$	$Ra_H^{1/4}$
$Pr < 1$	$Pr^{-1/4}(H\,Ra_H^{-1/4})$	$Pr^{1/4}(H\,Ra_H^{-1/4})$	$Pr^{-1/4}(H\,Ra_H^{-1/4})$	$\frac{\alpha}{H}(Pr\,Ra_H)^{1/2}$	$(Pr\,Ra_H)^{1/4}$

So, here you can see that, when the Prandtl number is greater than one, the thermal boundary layer thickness which is your delta T is basically H into a Rayleigh number to the power of minus one fourth standard, when Prandtl number is less than one, you just add the Prandtl number to the power of minus one fourth factor, this is already what we did. So, this is a compilation. So, this is basically your delta T.

Then, there are three things regarding the corresponding velocity scale that is introduced. The corresponding wall jet thickness, as well as the distance from the wall to the velocity peak. Three parameters which are important. So, the velocity scales are very straight forward they are always introduced by the delta T. Remember one thing, even if you forget everything else. It is the delta T. It is a buoyancy term, which basically introduces whatever is that relevant velocity scale. So, that is what it is. It is always the buoyancy term, regardless of whether you are mapping it, with inertia or with friction that is what we have done in the 2 cases. It is always determined by the buoyancy term. So, the velocity scales comes out to be like this 2 marks that we said. So, this is V. Now the distance thickness of the wall jet is basically the thickness of the delta; that means, whatever is the delta, that is delta V or delta that you are dealing with, that is what it is. So, that is what it is. So, that has been kind of put, forward over here and the distance from the wall to the velocity peak; that means, where the velocity peak will kind of happen, that is given by these 2 terms, over here. So, all you can see all are H into R a H to the power of minus one fourth.

So, basically if you talk in terms of cloning, these are all cloned kind of quantities with a stretching factor appropriately added depending on the situation of the problem, because ideally if your Prandtl number is equal to one all of this thing just vanishes. You get a what we call a royal I mean everything is the same there is no limit is; obviously, it is just that Prandtl number is equal to one So, it is just because this in takes into account the differential diffusivity and the difference between momentum diffusivity and the thermal diffusivity into the picture. That is all that it has done. The Prandtl number is a quantity which essentially takes care of that momentum diffusivity versus the thermal diffusivity, it is because of that we need the stretching factors. We need the stretching factors essentially because of that, of course, these are at the 2 extremes that we have pointed things out.

And as I said earlier there are three numbers B o H and G r H. So, why where B o H is basically R a H into Prandtl. This is basically R a H divided by Prandtl. So, these are the factors, that are responsible. These are the three most important quantities. Look at their applicability. So, next time when you look at a problem, look at the applicability that which number is the most important one and not just really, it depends on what problem you are actually dealing with. For example, in liquid metals grash of number will be very important and things like that, but normally in the case of a natural convection problems you know what we do, we basically use this interchangeably, we use grash of Rayleigh and all these things that are very mixed match kind of a fashion. We do not partition it like this, sometimes it may be very difficult to partition it in the first place, but normally as you can see, these numbers are all pretty much close to each other with the Prandtl number factor added here and there.

So, in essence they designate the basically the scales of the problem; that means, what are the different scales that are actually responsible for each of the problem. So, of course, you can see that why your another thing to note in normals natural circulation problems, you will find that usually this Rayleigh number and grash of number values are very high. You will find 10 to the power of 4, 10 to the power of 5 and things like that. That is a very normal practice, even before you move on to the turbulent regime, the reason because is that everything is scaled as one fourth. So, everything is kind of, there is a factor of scaling that is always in built into the system. So, it is not just a Rayleigh number per say it is a one fourth of the Rayleigh number that actually matters. So, these are some of the subtle things that you should kind of remember and these are the things that can come in handy, that it is very different from the natural convection from the forced convection problem.

(Refer Slide Time: 05:51)

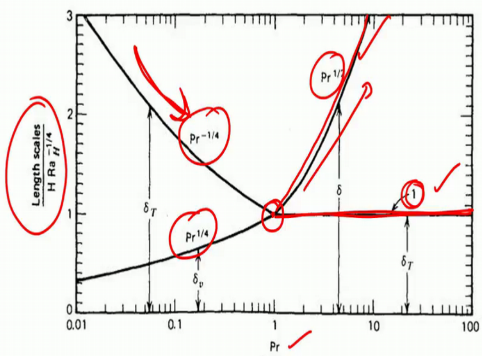


Figure 4.3 Length scales (thicknesses) of natural convection boundary layers.

So, let us look at the Length scales put in a kind of a formal graph over here. So, what we have done is that basically what professor Adrian Bejan has done is, basically you have divided by the length scales divided by this factor H into Rayleigh number to the power of minus one fourth, because that we saw was universal regardless, wherever you put it is the same. So, it is kind of a regime map per say, this plotted with Prandtl number and here you can map out the different quantities,. So, for example, when your Prandtl number is less than 1, you have the Prandtl number to the power of minus one fourth graph, you have that particular line of graph which basically gives you a delta T.

So, you can basically map out that you now belong to this family it is like a family tree. So, you belong to this particular family. Similarly you can map out for Prandtl number greater than one; you can map out that what will be this. What will be the value of your delta That kind of a thing and similar thing goes and you can see that here it becomes flat that means, when for Prandtl number one up to about hundred, you will find that this particular delta T is of the order one, why that particular thing happens, it is Prandtl number greater than one situation.

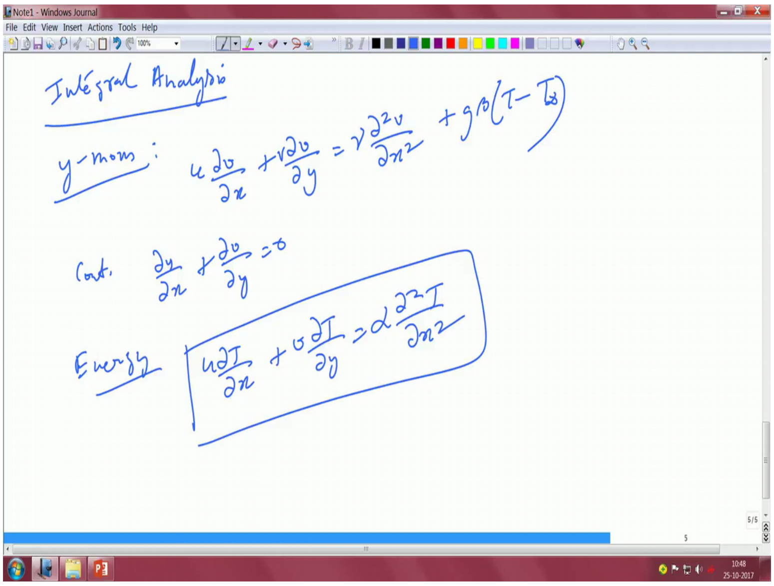
So, remember the delta T in that particular case is independent of Prandtl number. You remember that it is H R a H to the power of minus one fourth. So, naturally this value will be one. So, on and beyond it is basically independent of Prandtl number correct. So, thats why you get your delta T; however, the velocity boundary layer is not the same way. So, for Prandtl number greater than one you have this particular kind of a dependence Prandtl number to the power of half dependence on your velocity that will be evident if, you guys get confused, is basically if you look at this particular thing which is the basically the thickness of the wall jet it will be very apparent it is Prandtl number half factor is there here that factor is not present all. So, this is the delta T, this is your delta or delta V whatever you call it.

So, this has got a Prandtl number half, this has got no Prandtl number, now if you go and look at this particular these 2 lines of the plot, you will find here it is one flat because there is no Prandtl number, no matter what is the Prandtl number value it is of the order one.Here of course, the velocity shows that Prandtl number half dependence. So, with Prandtl number that will go up, which is logical also as you increase the Prandtl number your velocity will be is going to be felt at larger and larger distances into the reservoir, now let us look at the other situation which is the second one in that particular case you see it is a Prandtl number minus one fourth, the thickness of the wall jet is very small and this is the corresponding Prandtl number value on the other side. So, if you look at then this particular plot and this particular plot you will get the meaning. Both actually has got that Prandtl number dependence now and your velocity boundary layer is different from your temperature boundary layer and this is; obviously, going to be smaller than that.

So, as you go on decreasing the Prandtl number, our velocity boundary layer becomes thinner and thinner and thinner, which is kind of obvious because you are I mean restricting it very closer and closer to the wall. Whereas, in the other case you will see the reverse trend that will start to happen. So, you can see there are basically a family of four curves that you can see four families with Prandtl number equal to one is being the point, the focal point where everything kind of falls on the money. So, there is absolutely it is basically as if you are kind of diverging from that particular point as you go in either directions Prandtl number greater than one or Prandtl number less than one.

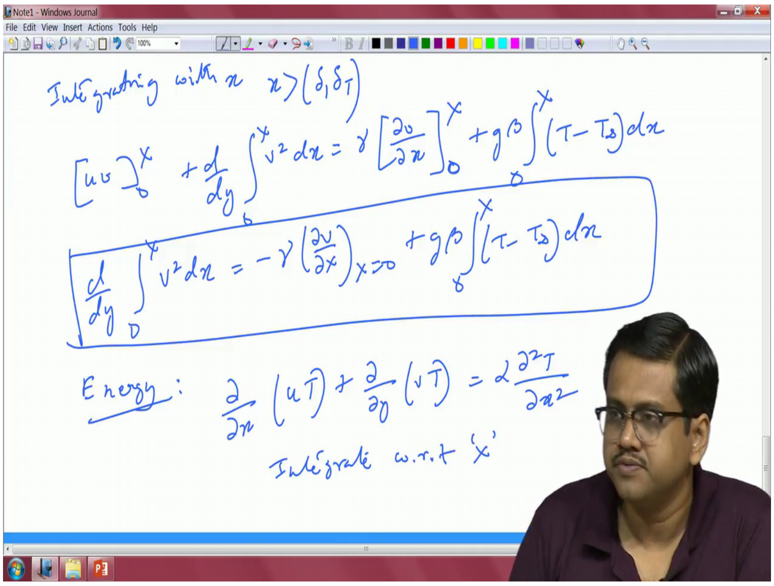
So, based on this now, let us look at so far we have done a lot of scaling. So, now, it is a time to do a little bit of math which is basically we will shift to the integral analysis.

(Refer Slide Time: 10:35)



So, Integral Analysis and by now, you are most familiar with what Integral Analysis is. So, I will write the y momentum equation because that is the only equation that is kind of as important. Continuity, energy, these things are kind of very common. So, now, what is the tradition to do in an Integral Analysis, you basically integrate out one of the dimensions in this particular dimension it is a x dimension previously, it was a y dimension, but now your plate is actually rotated. So, your x points out a way perpendicular to the plate. So, naturally you are going to do the x dimension, x integration first; that means, integrate out with respect to x.

(Refer Slide Time: 12:05)

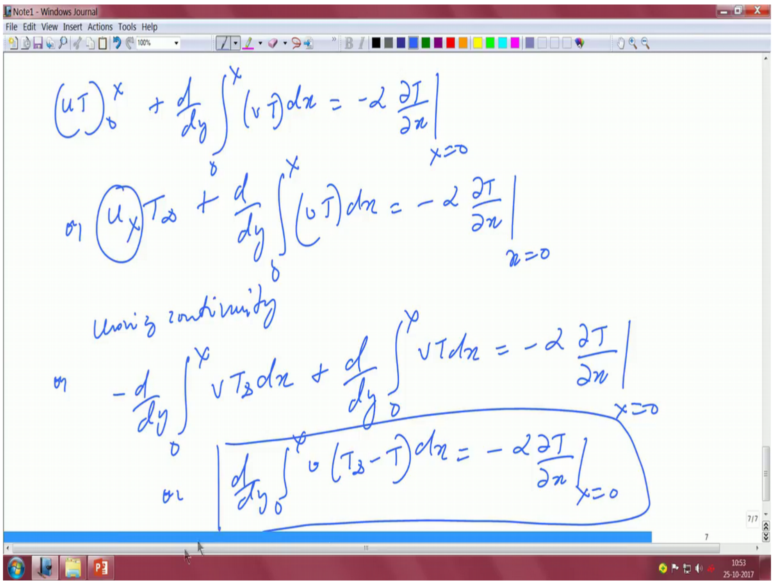


So, integrating with x, where x is greater than delta or delta T ; that means, it is greater than both actually and we apply this to the y momentum equation first. So, what will happen? Ii am not going over the Lebanese and other kind of rules. So, you can read about it once you consult your forced convection notes.

So, now getting read of a few terms, why is the first term disappearing that is because of the simple reason at the wall and at x you should be at actually equal to 0 because we have taken x which is beyond all of this limit and v also should be equal to 0 and the wall of course, u and v both are equal to 0 in the far field of course, you have v to be equal to 0.

So, anyway the first term goes away because of that particular reason, second term stays and you have this particular term, now here of course, we are evaluating at x equal to 0, but because at x equal to x, which is far away from the wall, you naturally would not have any shear stress coming into the picture because it is a consent fluid, there is no provision of having any shear. So, this is the y momentum equation. Similarly let us look at the energy equation now. This is the energy equation. You once again integrate with respect to x all you integrate with respect to x. So, you get u T, 0 to x it is a little bit boring, but I mean this is a then necessary thing that we have to do this equal to 0.

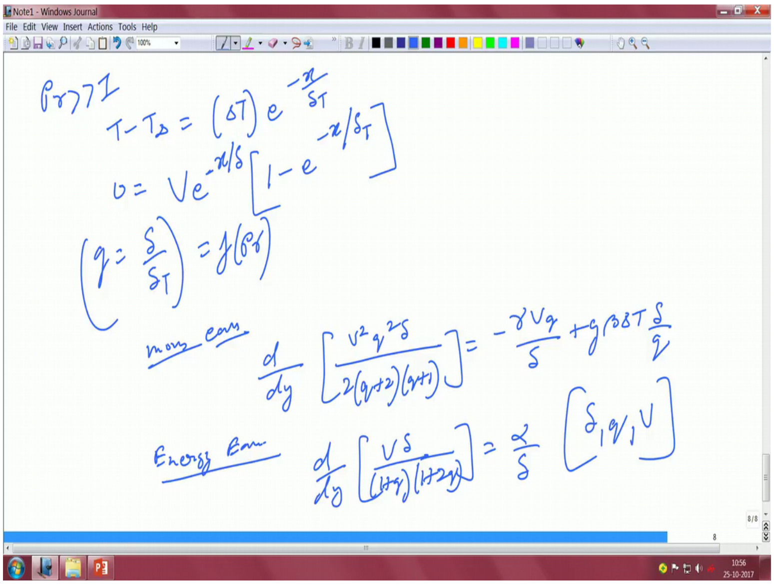
(Refer Slide Time: 14:46)



So, that is once again the same logic for field you do not have a temperature gradient anymore at x equal to x. So, here of course, the situation is at x T infinity and d by dy integrate it from 0 to x, vT dx minus alpha dT by dx evaluated at x equal to 0 all if you substitute this particular guy from the continuity thing, any x that you are actually evaluating it. So, you get using continuity you get minus d by dy 0 to x VT infinity d x plus d by y or in other words d by dy 0 to x v T infinity minus T d x minus d T by d x is equal to 0.

So, that is what you get now what is the next step that you usually do in such cases, you basically choose a profile for temperature and velocity and you already got a feel that these profiles looks very exponentialish to you, velocity for example, goes up and then it comes down. So, it should have a multiplicative effect somehow, if you when you choose the temperature profile or the velocity profile you should have that thing in mind temperature profile is kind of shows that monotonic kind of decay, the delta T from a delta T perspective. So, let us say if the Prandtl number is much greater than one that is one of the limit T minus T infinity if I write it like this x by delta T, will this be reasonable?

(Refer Slide Time: 17:25)

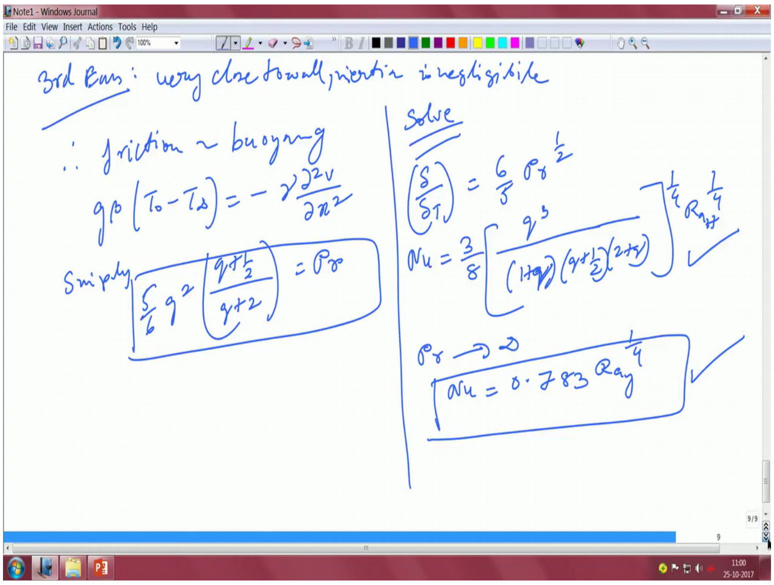


I think it would be reasonable because it shows that exponential decay of the quantity and it looks pretty exponential decay, when we actually did the scaling, but velocity is a little bit different it is a little bit of a different beast than that. So, for Prandtl number much greater than one, the temperature profile is given by this I think we can buy into this particular argument.

Whereas the velocity profile, as I said velocity is a 2 parameter model. So, you can have something like that. So, it is one minus that and it is exponential of x to the power x by delta. So, it is like a 2 parameter model that we are proposing over here in addition we define one more quantity which is q which is delta by delta T, which is as we have proved by now; it is a function of Prandtl number or some sort. So, what will happen? Now you substitute these 2 things into those 2 equations momentum and energy and from the momentum you are going to get. So, from momentum this math you can do the energy equation; however, there is one small catch over here.

The catch is that, what are the unknowns in this particular equation. In this particular equation that we have done, there is a delta which is of course, an unknown there is q which is of course, an unknown because we do not know what function of it is in terms of Prandtl number and then there is of course, that v the maximum velocity we know that delta T, we do not know what is the velocity that is going to be incorporated. So, based on these we need a third equation.

(Refer Slide Time: 20:30)



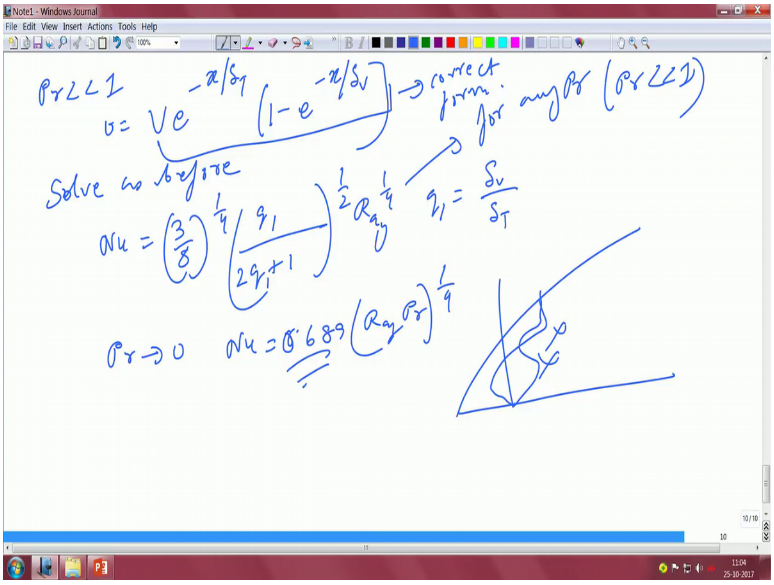
These equations have got three unknowns, 2 equations in post problem you cannot solve it. So, you essentially need one more parameter over here, one more equation and that equation is usually derived the third equation let us call that the third equation.

So, once again very close to the wall this is the same logic that we did earlier very close to the wall inertia is negligible. So, it is basically friction that is balancing the buoyancy all. So, you have g beta T, not minus T infinity is given by gamma all that is equation all. So, simplifying if you now once again back substitute the same thing 5 by 6 q square q plus half and q plus 2 go to Prandtl number the third equation.

So, basically now what you are going to do you are going to solve all these three equations in a coupled manner and basically find out what is the solution not very trivial it requires about 2 classes to work it out the full thing, but I mean you can do the solution as a practice, but the whole point of this is not to solve the equation, but basically to show that delta by delta T is given by 6 by 5 Prandtl number to the power of half, which agrees with the scaling argument that we formulated earlier. We said it is going to be Prandtl number to the power of half.

So, 6 by 5 is just the factor that sit in front. Nusselt number is given by 3 by 8 q cube divided by 1 plus q, q plus half and 2 plus q that is the Nusselt number, raised to the power of one fourth, Rayleigh number to the power of one fourth, this is q this is not number, Now for Prandtl number as it goes to infinity because it is a high Prandtl number, fluid situation the Nusselt number basically becomes 0.783, Rayleigh number to the power of one fourth, which almost agrees with the order that we calculated, earlier we said it is going to be Rayleigh number to the power of one fourth and in the limiting case of Prandtl number going to infinity it comes out to be about 0.8 of Rayleigh number to the power of one fourth. So, this is an extraordinarily powerful thing that we did in the scaling, we provided the correct scaling and that is validated over here including the ratio of the 2 boundary layers. So, this is the most generic form for any Prandtl number. This is the form when the limiting case when Prandtl number goes to infinity for the low Prandtl number.

(Refer Slide Time: 23:27)



That is Prandtl number much less than one here your u v is x by delta T one minus exponential x by that is the v profile. Once again look that we have changed it a little bit the inside has changed now to delta v previously it was delta T that is, because the peak is now appearing within this, previously it was appearing within the delta T. now it is appearing within delta v. So, it takes into account the correct estimation and what about the temperature? The temperature profile should remain the same. There is no reason to change the temperature profile because it is still delta T whether delta T is big or small that does not matter.

So, what we do is that, once again you solve as before. So, you get your Nusselt number to be 3 by 8 to the power of one fourth, q1 2q one plus one half Rayleigh number to the power of one fourth where q1 is delta v by delta T, in the limiting case when Prandtl number goes to 0 almost close to 0 this becomes 0.689 in to Rayleigh number y into Prandtl number to the power of one fourth, once again the scaling works that is because we are getting a number which is of the order one correct.

So, this particular thing if you look at this particular problem, now you will see that we have established this is the most generic for any Prandtl number of course, given that Prandtl number is much less than one not for any, this velocity profile is very important takes into account the correct form. Correct form is very important because when I said earlier that you need a velocity profile, which is remember in your flat plate boundary layer we said, if you recall the situation that this is your boundary layer profile, you have to match a few boundary conditions. So, what prevents us from having this, you can have that or what prevents us from having this. So, long as you match the gradients somehow and match the boundary conditions you are good, but these profiles are not correct because they do not have the correct form. Correct form is very important because you cannot have kings and other things appearing in the velocity boundary layer profile, which is unrealistic what will lead to that there is no physics.

So, similarly here the correct form is very important velocity should go up and then it should come down, it should not do it should not show a multi modal peak. For example, you might argue why not a multi modal peak what is the why there should be a multi modal peak in the first place what actually generates a multi modal peak. We are assuring that a 2 ends the velocity should be 0. So, there it must have a peak somewhere that peak in a certain case is within that delta v that peak in some other cases within delta T, because in that case delta is more than your delta v and the peak cannot be actually in the delta region, which is beyond delta T, because there is no driver. So, why should the peak actually shoot up there. It should shoot up within delta T and it should decay within delta similarly in the other case because the driver is still delta T, the peak happens within that small layer of delta v because your momentum diffusivity it is a low Prandtl number fluid situation. So, the momentum is more arrested or in that particular region.

So, based on these 2 conditions the correct velocity profile descriptions are given by that though this is kind of not exact because we have taken exponential, it could be other functions, which is similar to that and you can also show that, if you use other functions you will get very similar results as well. So, it is not like that your this number at this 0.689, this might change a little bit.

Of course, we will do in the next class the similarity solution and we will try to show in through the similarity solution that, what is the exact values of this particular profiles and how these profiles actually evolve, because this is still done through the integral arguments. So, see you in the next class where we will start the similarity solutions for this and try to see that how the profiles look like.

Thank you.