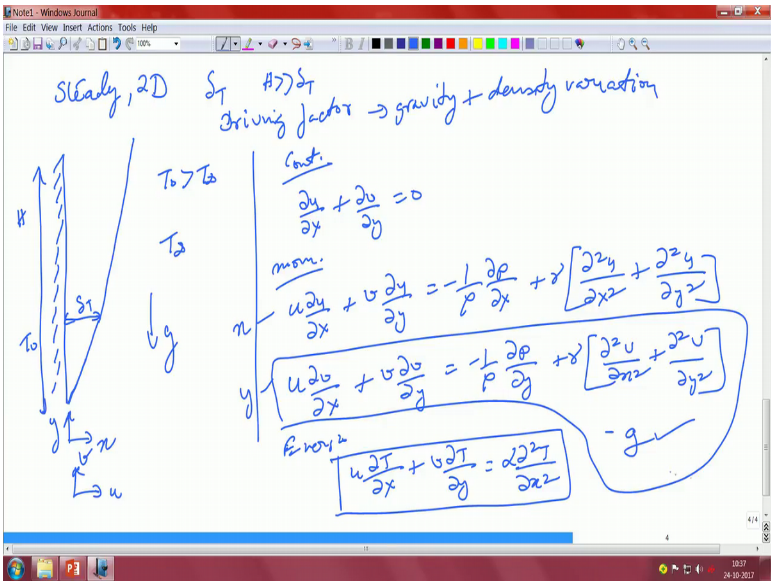
Convective Heat Transfer
Prof. Saptarshi Basu
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

Lecture - 29
Scaling analysis – I

So, in the last class we did a lot of analysis and we saw that what will be the requirement for natural convection. So, in this particular class we are going to start it off with some of the formulations. So, we write down the governing equations first and from the governing equations let us see that how we can basically through scaling and other arguments establish the relevant parameters that are responsible for driving this natural convection.

(Refer Slide Time: 00:48)



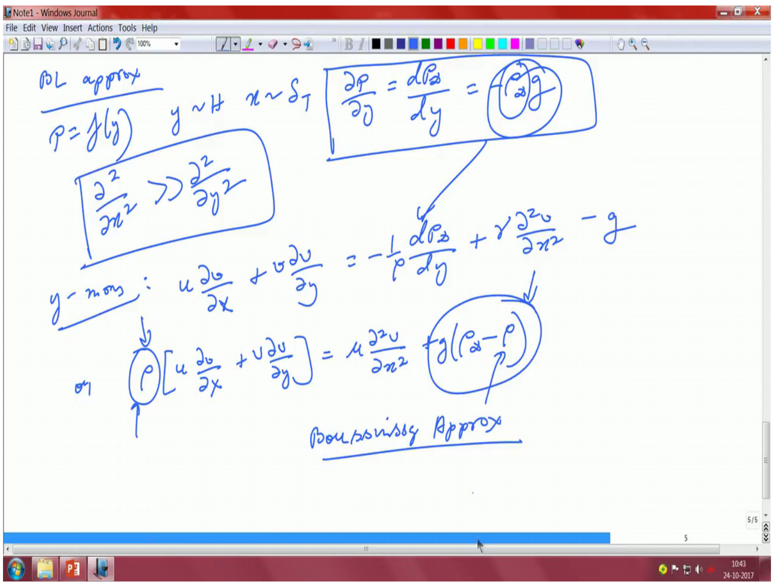
So, our equation sets will be steady 2D equations our delta T which is basically if you consider this as your wall which is at a temperature T naught, and delta T is basically the layer, so let us assume this for the time being. So, this is your delta t. So, H being the height. So, H is much much greater than delta T. So, here the driving factor as we mentioned earlier factor is gravity plus the density variation. The gravity is an important parameter over here the outside fluid is at a temperature T infinity T naught is greater than T infinity, I mean these are the cases under this considerations we will solve.

So, the first the continuity, the continuity equation is once again if I this is x, this is y, this is u this is v. So, dux by dy plus dv by dy is equal to 0. First continuity equation then comes the momentum equations the momentum equations of course, there are two momentum equations now v du dy its exactly the same. So, there is not much to describe a square u by dx square plus d square, square u dy square ok. So, this is in the momentum in the x direction no gravity comes into the picture gravity is acting downwards.

On the other hand got it minus g, g is the additional term that you have right. So, up to this of course, we have just and then of course, the energy equation, energy equation is du dT by dx plus v dT by dy alpha d square T by dx square right. So, x momentum, y momentum continuity these are the equations that you have this is a part of the y momentum at g.

So, this is the only additional term that you see over here which you are not have not seen the same in the case of forced convection because force convection is forced. So, gravity is not gravity, gravity is too small for those things. Of course, forcing can be important gravity can be important in certain situations in forced convection also let us not make a sweeping comment about that, but normally most of the problems that we dealt with we did not consider that to be the case. So, we do the boundary layer approximation right.

(Refer Slide Time: 04:27)



So, the boundary layer approximation, where we say that P is a function of y right this is the same thing that we did earlier that P is not a function of x its only a function of y. Remember our flat plate boundary layer there of course, the axes were a little bit different.

So, but based on that you get what I am talking about y scales as H that is the height x scales as delta T right and dp by dy is equal to dp infinity by dy is equal to minus rho infinity into g. So, it is essentially hydrostatic in your region got it. So, P is only a function of y that it has to be right because P is not a function of x that this we already proved in our flat plate boundary layer problem alright. Now, in this particular case; obviously, obviously because it is gravity assistant this particular pressure drop is nothing, but the given by the hydrostatic head. That rho infinity g is nothing, but the hydrostatic head over here.

And also d square by dx square is much much greater than d square by dy square right because y being the direction of the flow and x being the boundary layer direction, or the depth of the boundary layer direction. So, naturally this parameter will be also satisfied alright. So, therefore, the y momentum equation which is the most important equation here in this case anyways is basically given as d dv by dx plus v dv dy all to minus 1 over rho dp infinity by dy plus gamma d squared v dx square minus g my minus g.

Now, you are going to substitute this back here that is what we are going to do. So, and we are going to take the rho to the left hand side. So, what will happen is that u dv dx plus v dv dy equal to mu d square v dx square plus g rho infinity minus rho alright rho infinity is basically the density of the fluid at the infinity right in the reservoir section right. So, this is the equation this is the y momentum equation.

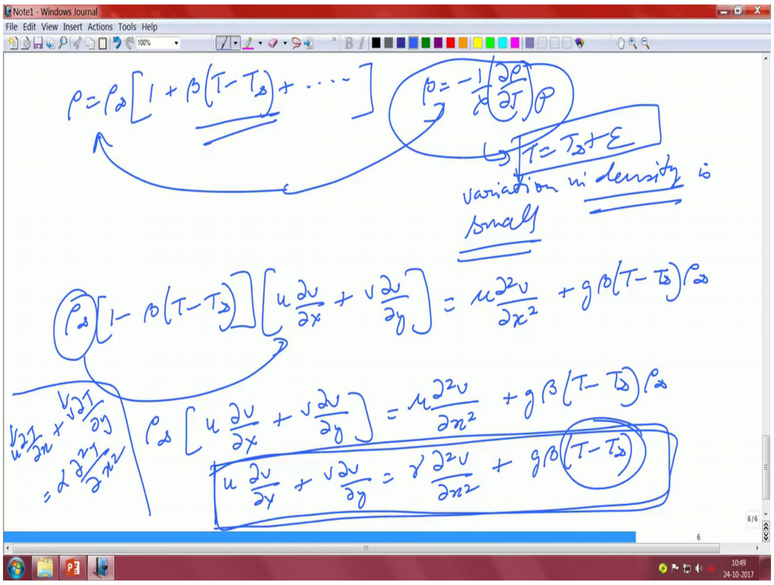
Now, here as we can see two interesting things comes out. So, there is a rho here there is a rho there right. So, ideally for a system like this where the rho is actually changing right you have to incorporate the change everywhere right so that means, this rho now we have to cast you in terms of rho infinity and see what kind of a variation we get or cast it in terms of temperature right because that was the argument that if the temperature is driving the density then we have to cast density in terms of temperature right. Now, and that we will use that beta that we already have right that gives you the relationship between density change and temperature change.

Now, interesting point to note over here is that here you have actually a density rho here you once again have a density difference which is basically like the buoyancic type of term that you have right. Now, in this particular situation looking at this you would ideally say let us substitute whatever is the density change here as well as whatever is the density change here right and this will make the problem a little bit more complicated than what we anticipate alright. Because density on the left hand side is associated with a convective term right density on the right hand side is associated with buoyancy and the body force term alright, it is associated with the body force term.

So, we make here an assumption which is going to be valid for small temperature differences only and that approximation is called the Boussinesq approximation. So, what does Boussinesq approximation entail? We will come to that and in a little bit.Boussinesq approximation actually tells you or the spirit of the Boussinesq let us put at put it as a spirit of the of the Boussinesq approximation. The spirit of the Boussinesq approximation is that the density on the left hand side; that means, associated with the convective terms let us take that to be the same density as the reservoir fluid right. Let us take that to be constant point number 1.

Point number 2 is that the density on the right hand side let us now express that density in terms of temperature. So, you see that we are differentiating the problem we are stating that this is still an incompressible type of a problem; that means, most part of the Navier Stokes equation is actually behaving with the constant density term right; however, in the body force term when density is present we are taking its variation into consideration. So, that is the thing that Boussinesq approximately did. So, what happens is that now let us see that what the density will be.

(Refer Slide Time: 10:18)



So, density is given by rho. So, if you take it to be expansion. So, just take a first order expansion in this particular case. So, beta T minus T infinity plus there are a lot of higher order terms right this essentially tells you that beta is basically a linear that that slope if you recall that beta is nothing, but 1 over rho d rho by dT at constant P right. What we have done using this expression and this expression combining these two this actually tells you that beta is basically a linear slope; that means, you take rho minus rho infinity divided by T minus T infinity you get beta right.

So, it is a like a linear interpolation of the whole thing alright. You are not taking into account any second order curvature effects right square terms that we have over here. So, this can only happen, this can only work when the density or the temperature difference is small; that means, T is basically T infinity plus some epsilon. So, it is small. So, the variation in density radiation in density is small, T equal to T infinity minus epsilon variation in the density is actually small understood. So, this can only happen when you actually take only the first order term. So, right the leading order term alright.

So, if you now substitute, if you substitute the whole messy looking thing now in the governing equation what you would normally expect to get is something like this alright, 1 minus beta T minus T infinity that will be the first term u dv dx plus v dv dy right. This will be the total form if we write it like this mu d square v dx square plus g beta T minus T infinity into rho infinity correct. But however, what we are going to do we are going to do only the dominant term. So, we are going to take this term only associated with this as we told earlier. So, we are going to take it as rho infinity u dv dx plus v dv dy is equal to mu d square v dx square plus g beta T minus T infinity into rho infinity.

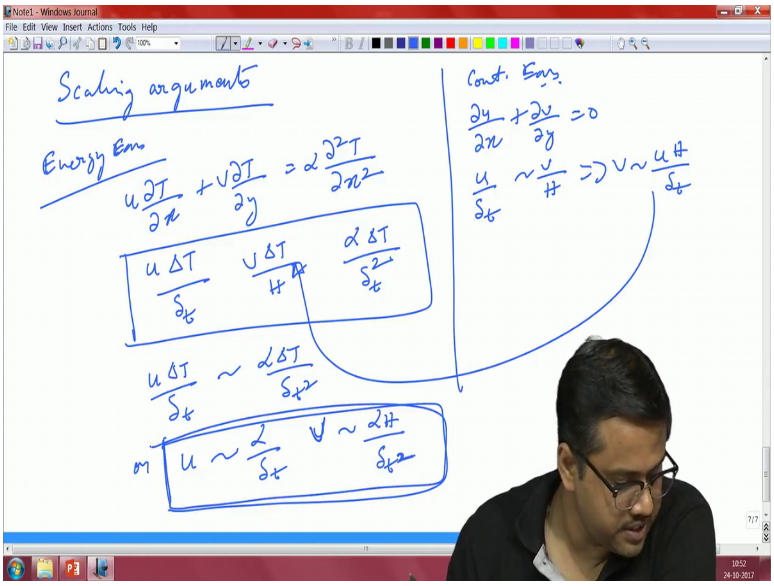
Now, if you divide out with the density term now so, what you will get is, dv by dx into v dv by dy this becomes your kinematic viscosity, kinematic viscosity is still given by the rho infinity. This is given by gB T minus T infinity that is what you should get alright. So, that is the expression, that is the expression that you have right. So, the momentum entire momentum equation kind of sandwiches and becomes exactly like this right. So, what are the key things that we have done? We have taken that in the convective term there is no variation in density, density is a constant.

In the on the other hand in the body force term what we have done is that we have decomposed the density variation we have used beta alright which is the volumetric expansion coefficient. We have done it in such a way that for small changes in density; that means, for small changes in temperature this can be given by a linear interpolation. So, that is exactly what this has been alright and now we see that the momentum and the energy equations are intricately coupled to each other because this has got minus T infinity over here alright and if you write the energy question in this particular case which is du dT by dx plus v dT by dy is equal to alpha T squared by d x square that will be what it will be alright.

So, that expression you see that there is velocity of course, that is already given that they would have the velocity terms. Now, you have the temperature term also appearing in the momentum equation which was not the case in the case of your forced convection right. So, this complicates things a little bit; that means, the velocity now is also dependent on the temperature to a certain extent right and the temperature is also dependent on the velocity that is obvious because the heat transfer has got both velocity as well as temperature right. So, this complicates the issue a little bit. But we are trying to attempt something like a scaling argument as we have been following the we follow the trend like scaling, similarity, integral in whatever ways we can provide more insights into the problem right.

So, that brings up to the next part of the argument now that we know what the governing equations actually look like with Boussinesq approximation. If the temperature variation is very large say 100, 200 degree Celsius for example, this will not be this simple you have to solve the full equation right. You have to solve the full equation, you have to take into account second order effects that comes into the picture and you have to do a full fledged coupled solution of the whole thing. So, that is also possible in plasmas and other things. You can actually need that kind of an equation.

(Refer Slide Time: 16:10)

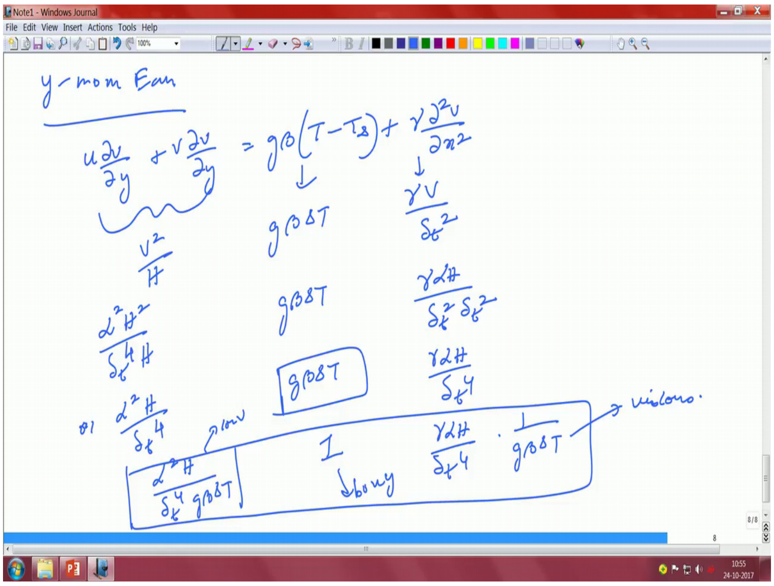


So, let us look at the scaling arguments. So, let us target the energy equation first because this is normally what we do. Now of course, delta T is given by that delta T alright, x scales as T as delta T that we have already shown earlier, v scales as delta T by H and this will be like delta T by delta T right. These are the 4 expressions that you, 3 expressions that you get, got it. So, that is the first thing. The continuity equation therefore, all to 0 alright.

So, this is given by delta T u delta T scales as v by H right. So, this leads to v scaling as u H by delta T correct. Now, if you back substitute this particular expression over here now alright, if you back substitute that expression over here you will find both of them will scale as u over delta T alright that is usually the case. So, this will scale as alpha delta T by delta T square or in other words you will get u scaling as alpha by delta T that is a first 1 and v scales as alpha H by delta T square right. Remember your u is the velocity in the x direction that is a nominal velocity v is the velocity in the y direction which is the principal component of the velocity alright. So, you can see that this is the logic, this is the logic.

And you can also see that why u and v, why v will be higher than u because H by delta P is basically a much much larger number, much much larger than 1, alright. So, these are the two expressions that you are getting as of now, got it. From the scaling argument v is proportional to alpha H by delta T square.

(Refer Slide Time: 19:15)



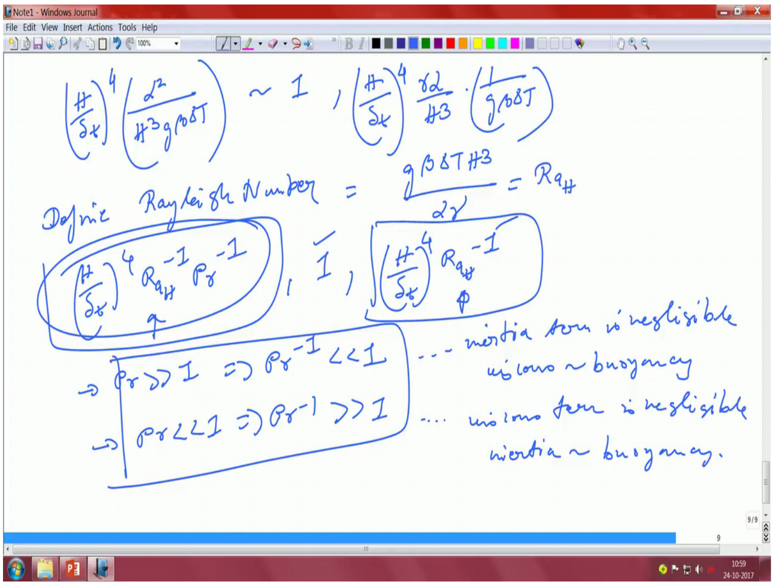
Now, from the y momentum equation let us attack the y momentum equation now. So, y momentum equation let us write it once again to be on the safe side. Now, these two will be the same you know that from your old logic right as we did earlier. So, this will be like v square by H alright, that would be the logic this will be g beta into delta T right that would be the scaling natural scaling alright, I need not say much about it. This is v divided by delta T square. These are the 3 expressions right. This leads to this, this leads to that.

Now, if you substitute for v which we have already done in the previous page, this one, right that is alpha H by delta T square alright. So, let us put alpha square H squared by delta T square divided by H this is still g beta into delta T this is gamma alpha H by delta T square into delta T squared right. So, that will be the expression right. So, or we get alpha square H by delta T square delta T square. So, sorry this will be delta T to the power of 4, because this is g beta into delta T this is gamma alpha H by delta T to the power of 4.

So, what we do, divide everything out by this particular guy. So, if we divide everything out by this particular guy we will get alpha square H by delta T to the power of 4, g beta into delta T this will be a 1, order 1, then gamma alpha H by delta T to the power of 4 into 1 over g beta, delta T got it, set of equations that you get right. Up to this it should be very clear that why we have got it the way it is, alright.

So, you just substituted and, but you should know the nature of this term this term is basically convective, this is basically the buoyancy term alright this term is basically the viscous term right. So, you should know that what are the origins of each of the term that will come in handy a little later. So, now, that we can do some manipulations and you we can take this like this, this is alpha square H cube g beta delta T scales as 1, 4 gamma alpha H cube. So, that is already something that we have.

(Refer Slide Time: 22:21)



Now, these terms are looking very awkward. So, let us define something which can make it a little bit more palatable. So, define something called a Rayleigh number. What is the significance of Rayleigh number, we will see in a little bit it is g beta, delta T this is the definition gamma alpha right. So, that is the Rayleigh number and we are defining it in terms of H, H is the length scale in this particular number right, Rayleigh number H. So, now, therefore, it becomes H over delta T 4 Rayleigh number H minus 1, the other one is a Prandtl number alright. So, that is and then you have 1 and then you have H over delta T raised to the power of 4 Rayleigh number minus 1.

So, these two expressions look almost very similar except that there is a Prandtl number inverse. So, if Prandtl number is equal to 1, then of course, all the expressions are kind of same order. But if Prandtl number is different very high very low in the two extremes that we are concerned with we would have a very different take on the Rayleigh number.

Now, if Prandtl number is much much greater than 1, this implies that Prandtl number inverse is much much less than 1 right. If Prandtl number is much much greater than 1; that means, Prandtl number inverse is much much less than 1, then what will happen is that the inertia term. Inertia term is which one? This one right, the inertia term is negligible is not that so. So, that would mean that it will be a basically balance between viscous and buoyancy; that means, between this term and this term alright. However, on the other side if Prandtl number is much much less than 1, implies Prandtl number inverse is much much greater than 1 right. In that particular case viscous term becomes very negligible right viscous term is negligible which would mean that inertia is the same as the buoyancy, alright. So, then naturally this proves that we have to solve this equation in two ways, in the two limits in the two extreme conditions, one is for Prandtl number much much greater than 1, one is for Prandtl number much much lesser than 1 right; because depending on the terms that we see over here that they are very similar except they are differentiated by the Prandtl number right.

So, in the next class what we are going to do is that we are going to take each of these Prandtl number limits alright and try to work out the problem and see that; what are the key features that come out of each of those Prandtl number situations right and once we are done with that then we can move to more formal analysis using integral and subsequently similarity transformations.

Thank you.