Compliant Mechanisms: Principles and Design
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Lecture - 08
Types of Elastic Pairs (Flexures)

We discussed various uses of Elastic Pairs; now let us actually try to model their stiffness.

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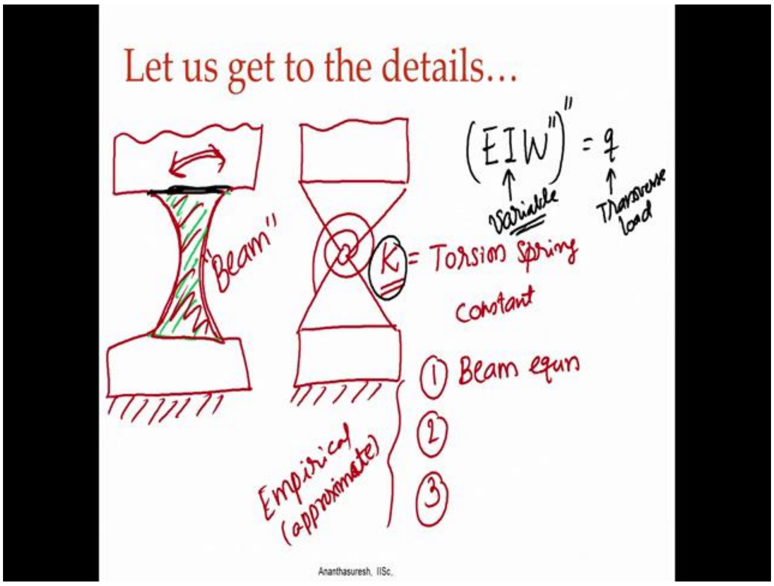

So, let us look at the details of these Flexures. Let us begin with Circular Notch Flexure that is if I have this Circular Notches like this; there is again that a relatively rigid or there a little part here and we are saying this is fixed and this can be a very big one that is going to rotate. When we do that what we want to know is if I were to module this that is this entire portion is what we call Circular Notch Hinge with an equivalent. So, if I were to draw again like I had drawn earlier we have this rigid body here and there is a frame.

Now, I would like to model by putting a hinge there because that is a effect of rotation under Torsion Spring. I want to know what this Kappa is; this is the Torsion Spring. It is a model constant. This has been done by various people over the last I guess more than half of the century; starting from 1960s to now; even now papers are published on how to get this Kappa as accurately as possible of course, we have Finite Element Analysis to analyze this, but that may not be suitable when you try to design.

Because you have many Elastic Pairs in a complicated Precision Complaint Mechanism and if you know a simple formula for this Kappa; Torsion Spring Constant then one can use it to design these mechanisms. So, there are three methods; actually one can say many more methods; the three methods.

The first one is to use the Beam Equation; you can think of this one. So, what I have hatched here this whole thing is like a Beam of variable cross section. Normally we take a Beam to be uniform cross section, but we have variable cross section Beam and it is deforming when I say that this is getting some rotation; it is because this Beam so called Beam it may not satisfy all of Beam theories assumptions, but it can still be modeled as a Beam.

These are all approximate as or I would even call Empirical based on whatever measurements that you do and lot of approximations that one do; they are approximate formulas using Beam Equation and you basically integrate this Beam Equation with variable cross sections. So, here if we recall the Beam Equation this will be Young's Modulus is E, second Moment of area I and if I say W is the transfer displacement of this Beam double prime that is d square w by d x square and then double prime and you can put the q that is the Transverse Load or Moment.

If you were to take integration twice; you can write it in terms of the Bending Moment. This is the Transverse Load. Now this I is the one that is variable it is a function of x. So, we can take that here and integrate and try to see how much rotation you get that this point and based on that you basically come up the equivalent Torsion Spring constant that is exactly what these two people did Paros and Weisbord in 1965.

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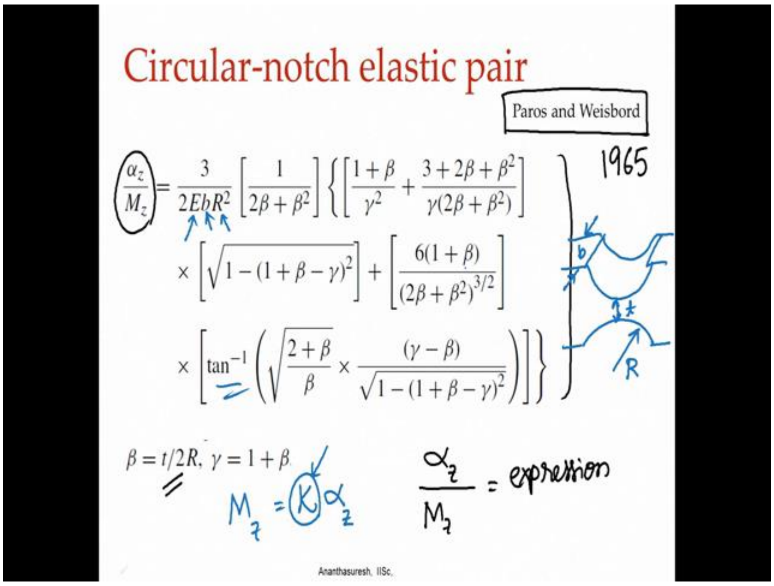

When they do that; they get or they got an expression that looks like this; what is rotation for alpha z M z; so what there are actually showing here is alpha z that is the rotation about the z axis.

So, that is z axis here is going to be out of plane. So, that one alpha z they are relating to M z equal to some large expression that you see here; if you look for formidable, but idea is simple; they are taking the Beam Equation, simply integrating for the variable cross section with lot of approximations. Even after that it is this big where beta is t by 2 R and gamma is 1 plus beta. What is t and R? Let me show it here by drawing this Flexure.

So, when I have a Flexure like this of course, R is the radius. So, this is the radius of the Flexure and this distance is t and of course, this is the rigid portion that is on both sides this is t and R which is in beta and gamma depends on beta. So, everything depends on t by R, but there also you have b which is out of plane. So, it is going to be this here and this is b that you see in the formula E is of course, Young's Modulus and R; I said radius of that notch and then you have beta, gamma various to trigonometric quantities and square roots and 3 over 2s and all that; it is a long formula, but obtained by integrating the Beam Equation.

So, what is actually getting here; this expression inverse of that is Kappa that we want because the Kappa that we want should be relating M z to Kappa here and the angle in this particular case it is shown with alpha z rotation about z axis. So, this one is 1 over this expression; the way is written; very long and while it is easy to put numbers in to it and get this Kappa value in terms of design what thickness t you should have here, what R, what b. It is not that easy here I think b is one of that comes out in R, but then this ratio t by R is everywhere in non-linear fashion.

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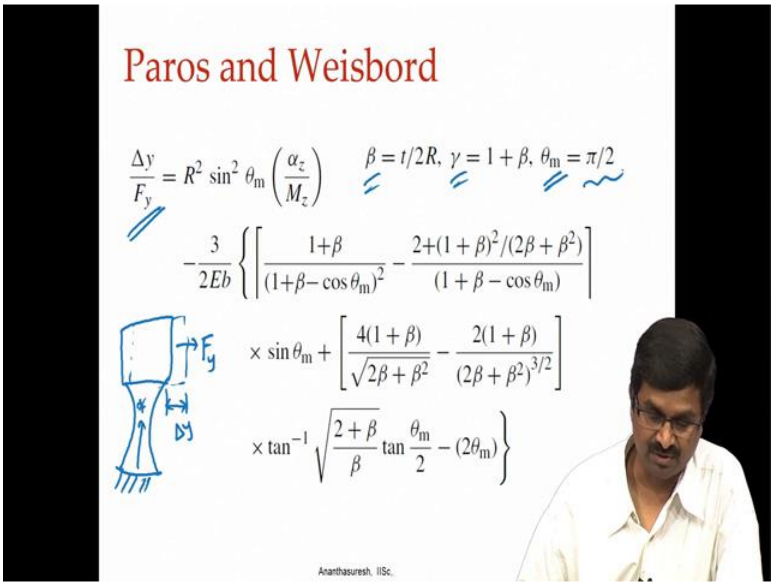

This is the long version of Paros and Weisbord formula; they got not only this rotation stiffness about the z axis out of plane axis they also got transverse thing here that delta y meaning that the same Flexure. So, let me draw it again this is a base it is fixed here and there is a rigid body right now when apply not moment or torque here; when apply a transverse force them calling it F y meaning that this will be x here. So, that becomes y; F y; how much does it move? It is going to move a little bit that is your delta y that also has a long expressions.

Now, we have beta again what you had earlier gamma, but also some other theta m is here for circular Flexures that is taken as pi by 2; lot of Empirical assumptions that are made in this to get this.

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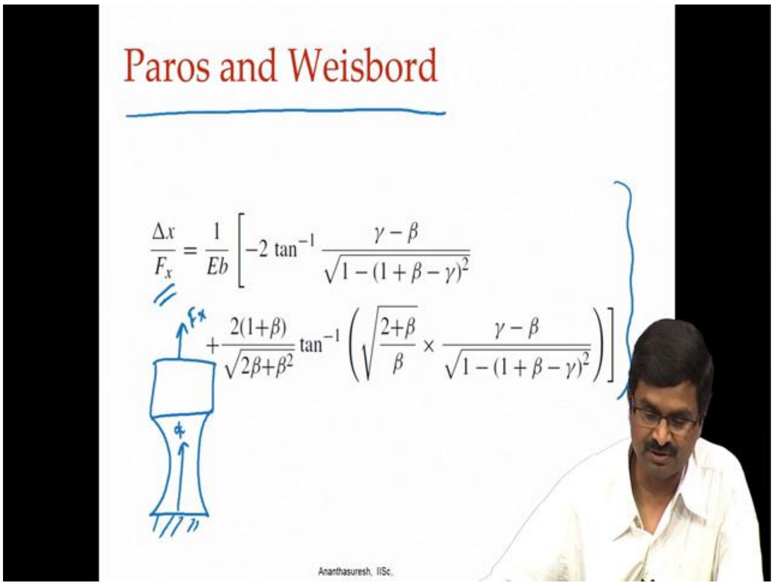

Likewise you can also get the axial that is again if I were to draw when this one fixed and I have rigid body to apply force if F x; here how much does it move? For that also a rather long formula because now it is a bar with variable area cross section; so of area cross section along the x is varying; further also you can get. So, note that just because this formula are long let us not thing that is complicated it is simple with lot of assumptions, but the work of Paros Weisbord 1965; whatever they had done was very widely used; even till today people in industry use it a lot because there were the first ones to do this and they were also very clever they had simplified versions

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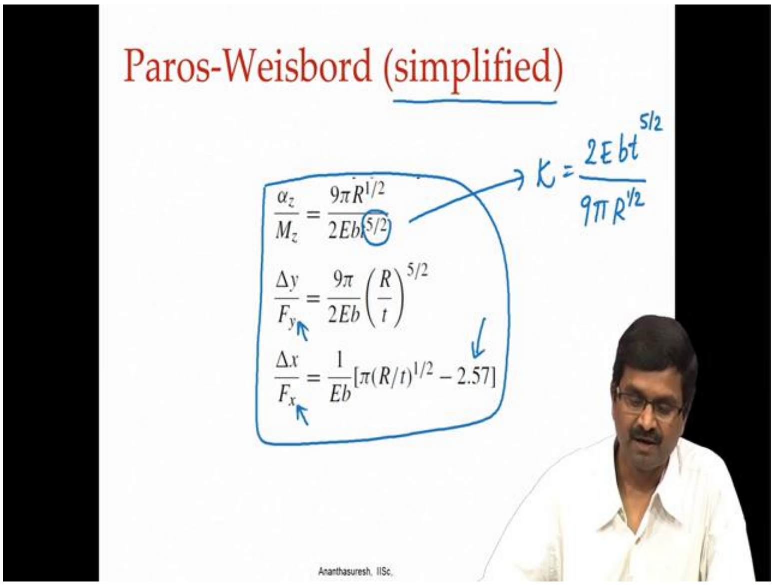

So, whatever I shown long expressions now we have the same things in a simplified form; you see you might ask where is this 2.57 coming, why is this 5 by 2? They are all approximation; the long formula for some ranges the simplified them so that you have. Now if you have a thing like this from here from this formula if I want to write Kappa I can said it should be inverse of this expression; I can write it as 2 E b t raise to 5 by 2 divided by 9 pi and then R raise to half; square root of R.

Now, we can see how Kappa changes if I were to change my radius; increase and decrease at what rate Kappa would change or this t you see this 5 over 2 I have a design inside now. That is how people love such formulae because they are not accurate in the sense that you may still incur quite a bit of error sometimes 10 percent, 20 percent also, but the Range of Motion that if we no ahead of time what error it is you can take in that into account while designing this Compact Complaint Mechanism; the Elastic Pairs; similarly for F y force F x force that is simplified formulae.

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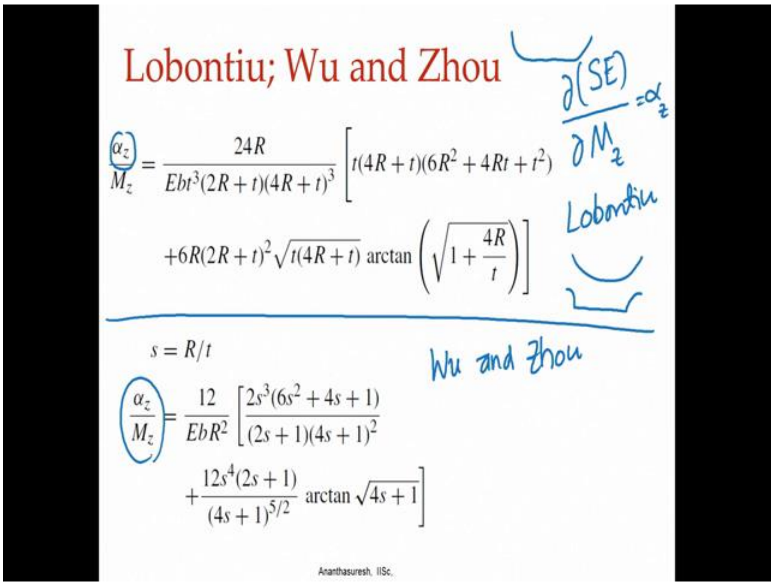

So these are very widely used, but then people are not happy that Paros-Weisbord Formula is still hold, but people want to improve there are a lot of people. So, this top formula here by a person named Lobontiu. He used a different method Paros Weisbord used basically variable cross section Beams and integrated them with lot of approximations. Lobontiu used Energy Method; so called Castigliano's Theorems that is he would right strain energy, take partial derivative of strain energy with respect to let us say if I want to get this angle alpha z; you have take partial element with respect to M z corresponding actuation if it is angle there is a moment.

So, you take strain energy, partial derivative of the strain energy with respect to this M z will give you alpha z corresponding rotation and that is what he has done; he has taken a lot of trouble to derive this expressions; lately; this is more recent not 1965 Paros and Weisbord.

So, now he has at his disposal mathematica maple; all these symbolic manipulation things to do and that is what he does and he took advantage of this modern tools symbolic algebra software. So, that now it not only for Circular Flexures he also have taken a Elliptical or it can be Corner Filleted something like this and that can be inside and I am talking about when I say I am only drawing one curve right there is other curve and you can have this or you can also have things like this and you can have Parabolic, Hyperbolic; he us taken every possible curve and at an entire book on these approximate formula by doing lot of manipulations.

So, is basically using Energy Method to derive this. This one by other set of people Wu and Zhou; they used different technique called Conformal Mapping Technique. So, you have uniform Beam; we know how to do that very easily now when you have a cross section that is varying you map these to conformally and you can use appropriate mathematical tools to derive expressions.

So, he also has it is alpha z by M z or 1 over Kappa. So, now, we have Paros and Weisbord, Lobontiu's formulae as well as Wu and Zhou and there are of few others.

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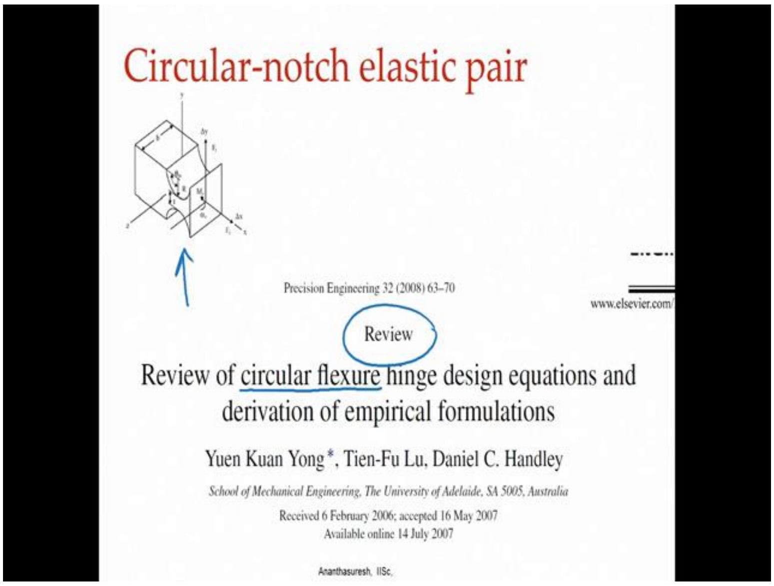

So, the lot of approximate formula and when there are so many somebody would write review paper and that what has been done. There is review paper comparing not for all this Elliptical, Hyperbolic all that Lobontiu did in entire book just looking at Circular Flexure Hinge what we talked about such as this.

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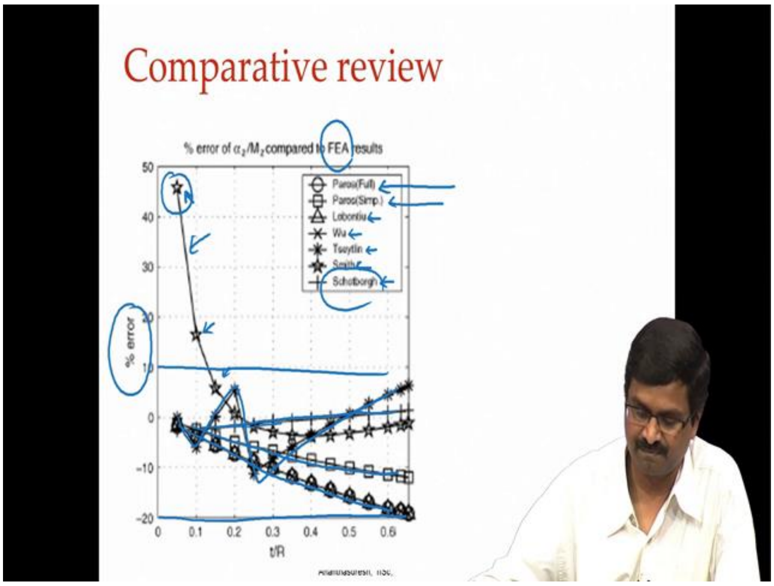

How do these things compare and it is a very nice review and they have taken a lot effort to compare all of these things with Finite Element Analysis results as well F E A is Finite Element Analysis results.

So, they are showing here several names first one is Paros in Weisbord full and other is simplified and then this Lobontiu and then Wu and somebody called Tsoylin and Smith will here this name again in this lecture and Schoborgh. IF you see let us look at Smith one right that is this stars; it is way off from what we are plotting here is error; Finite Element Analysis taken as the gold standard here; and that is perfectly all right. Today we have Finite Element Analysis software that is very very accurate by if you take proper care that (Refer Time: 14:29) theory has developed very well and implementation also reach a stage where you can trust it. So, if it compare with Finite Element Analysis how do some of them have 50 percent error you see this is going to 45 percent error and some of them are accurate. Let us see which of them is very accurate that the one is the pluses somebody called Schoborgh; that is seems to be closest to the zero error and if you look at Paros and Weisbord the circle and the square; circle is going like this full one and square is like this.

So, simplified when is more accurate then the full one. So, that probably at that time they did not know, but now when you do it you get these and something are going top (Refer Time: 15:15) we also this (Refer Time: 15:17) the error is going up and down. So, people have made various approximation to do it, but you can see apart from these the first one which has large error; most of them at 10 percent to another side you know some of these have 20 percent. So, plus or minus 20 percent error is what they give because they give a simple Torsion Spring constant view.

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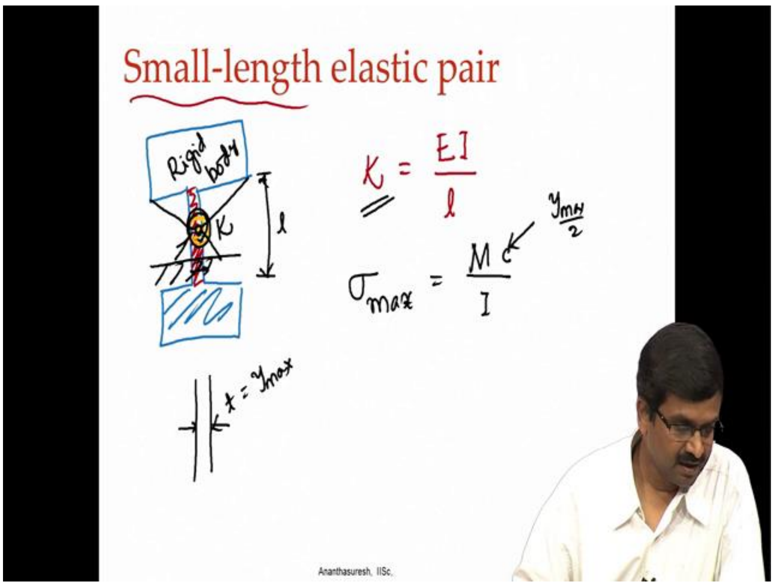

There is another thing this Small Length Flexure. So, when you take now like this is fixed and now focus is on this little thing; so this is what is Small Length Flexure; it has small length for that one we can write using just the simple Beam theory this Kappa can be written as E I by l. So, when you have this thing replaced with let us say a pin joint; I replace this I will use a different color; replace with the pin joint they will be torsion this will become the frame now for this rigid body; this is the rigid body or rigid segment there is a Torsion Spring the Kappa is given by E Young's Modulus, I second order area and the l is the length of this Flexure.

This has been done by several books in this field, but one other thing that you can do here you can also get the Maximum Stress when you are designing oh likewise Paros and Weisbord and Lobontiu and all others have given not only formula for Torsion Spring, but also for Maximum Stress there which is very useful otherwise you have to do Finite Element Analysis; we have rigid segments and narrow reigns. So, when you take machine you have to have very small elements or take a lot of effort to use different mesh density in different portions of the mechanism. So, if you want to get accurate estimate of stresses it will take a lot of effort, but these people not only have gotten Torsion Spring constants, but also have gotten expression for stress.

For example in this case if you want to get Maximum Stress this just a Beam there is no variable cross that is why one of the simplest cases here; we have a usual M c by I this is moment and second Moment of area, this is basically y max by 2; y max in this case is basically this if I have a small length of Flexure this is t and this y max going to be that or t by 2. So that is M y by I; that you remember that is what you can have. Here is one other interesting comparison paper this actually uses dimensionless design graphs.

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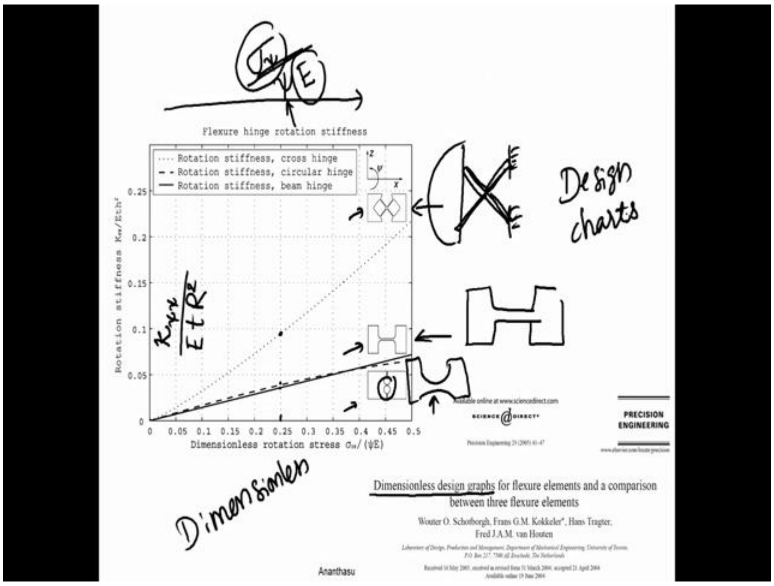

So, this is dimensionless which is nice because we can make them big and small and whatever material what they are showing in the x axis. So, on the x axis here they are showing sigma divided by they are giving a sign psi meaning rotation about z axis divided by that angle psi and the E Young's Modulus.

So, stress divided by that angle and Young's Modulus on the y axis they are giving rotational stiffness where what they are calling Kappa psi psi divided by Young's Modulus that (Refer Time: 19:22) that we have in that (Refer Time: 19:23) Flexures and then h square that must be R square radius square something in that I think they are showing what they are comparing is this Cross Flexure that we had seen which has strips like this. So, basically Small Length Flexures arranged like this where this is attach to one rigid body whereas, this is attached to a frame this one and then this Small Length Flexure where you have a rigid body and a joint like this.

And here is the circle or notch you know just like that between 2 bodies; at circular notch and it is comparing them non dimensional base. So, if we say that you have a certain material that we have a chosen then you know this sigma (Refer Time: 20:16) and E if you want to know certain angle of rotation you go there 0.25 whatever then you can see which one of them will be stiffest; there are three curves here. So, for each one of them this doted one is for the Cross Hinge that is this one and then dashed one is for the Circular Hinge, solid line is for the what a called Beam Hinges Small Length Flexure we can do this sometimes this design charts are very useful for this type of work rather the long formula we have these design charts.

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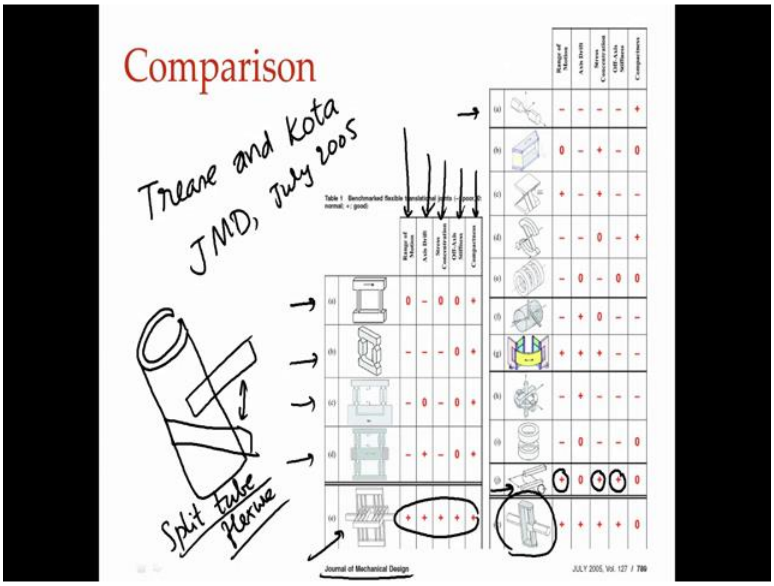

And there is another comparison this one is by Trease and Kota in relatively recent paper in 2000.

So, here they are comparing various kinds of these Elastic Pairs; some of these Small Length Flexures here; you can see the images and this is Parallel Motion and Folded and Symmetric whatever I shown earlier and a few new things as well that are there including this the dimensional one and there is something called a Bendix Pair that is very interesting one that I will show and there is a Split Tube Flexures iF you take a tube let us say we take a tube and have a little split on it some were here then if you attach at one end and then other end tangentially at 90 degrees apart between these two you can rotate and that gives you a very nice a Flexure equivalent which called you Split Tube Flexure there are number of other configurations actually here what they doing is there comparing the Range of Motion; the first one is the Range of Motion and other is Axis Drift, Stress Concentration, Off-Axis Stiffness and Compactness some time things very big.

So, for example, some of these things are very bulky. So, they have used five criteria put them 0 minus plus; 0 means neutral, minus is it is not good and plus is it is better than the others. So, again let us look at the parameter that they have taken to compare. One is the

Range of Motion; Range of Motion is limited for most of these unless use these Flexures even then it is not a whole lot, other thing that have taken is Axis Drift. That is a very important for this Flexures when they rotate their axis is not going to remain the same that I did not mention this a very important thing and modeling that is quite difficult also all these formulae do not capture anything that is why the errors in them is very high; sometimes 20 percent as we saw; Axis Drift and the Stress Concentration because that being heavily stressed in narrow portion of the material of the Compliant Mechanism Stress Concentration is very important; then there is Off-Axis Stiffness in one axis is very flexible what about this Split Tube Flexure was developed because it is Off-Axis Stiffness is quite good and Compactness how compact it is.

So, if I look at the Split Tube Flexure which is over here you see plus sign this Cross Axis Stiffness is very good, its Stress Concentration is also very low that is plus, Range of Motion is quit high you can go 90 degrees or more and other things are neutral. So, some of these that have only pluses of course, this paper was focused on this particular Flexure; it has all pluses and 1 0. It is not compact here they have put even though it is not that compare they put all pluses; it is subject if comparison of various joints. This paper is worth looking at; this is by Trease and Kota Journal of Mechanical Design July 2005.

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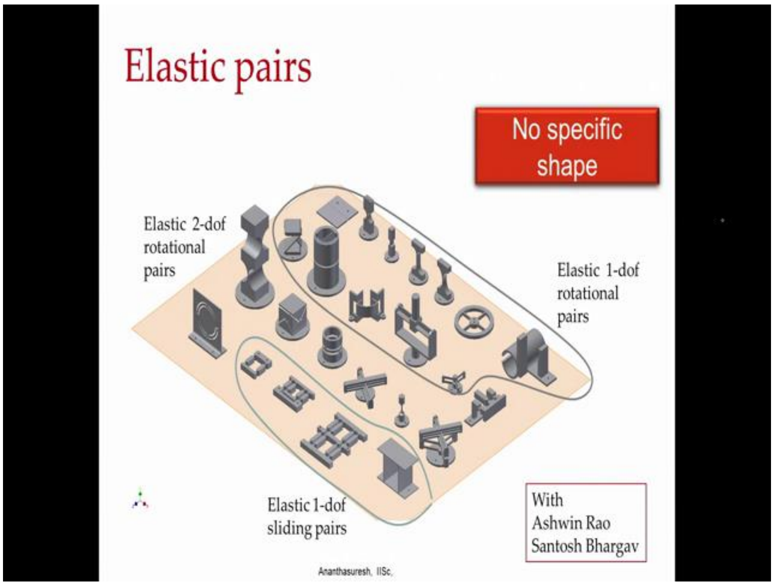

And these are all the different things that we have made in solid model and 3 D print and you can compare lots and lots of Elastic Pairs are there which we had already emphasized that they have no specific shape one can come up lot are deferent shapes and get these things.

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And this is a Bendix Joint that is been commercially available for a long time, a very nice joint. So, it has basically plates I think you can see a plate over here let me change the color; there is a plate over here and there is a plate over here; 2 plates cross there when apply them movie you will see how it works.

So, you can see the plates moving and this is the side view; how this plates move; they are not touching one another; you get this rotation it looks very compact actually. They are are sold one can one can by them in metal and plastic and lot of different thing. So, very successful elastic bend, Range of Motion is not that bad actually in this.

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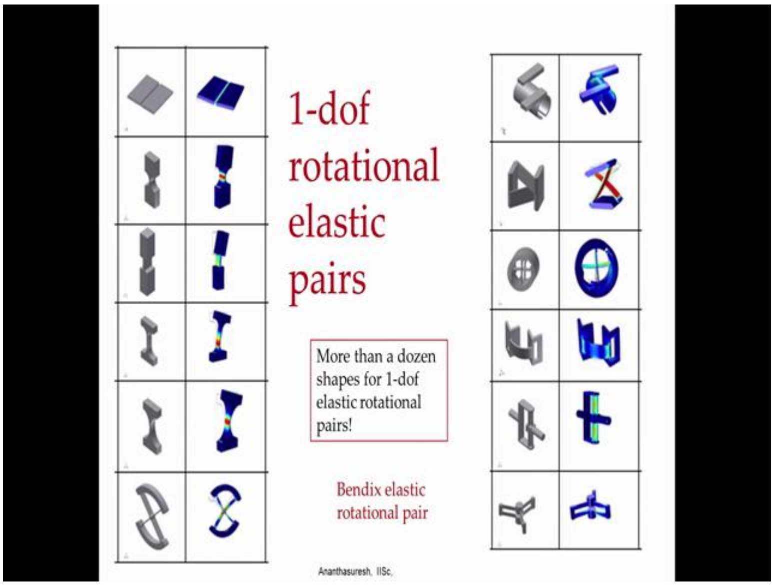

And you also have Finite Element Analysis of all of these different Flexures and like I said people have compared them quite a bit.

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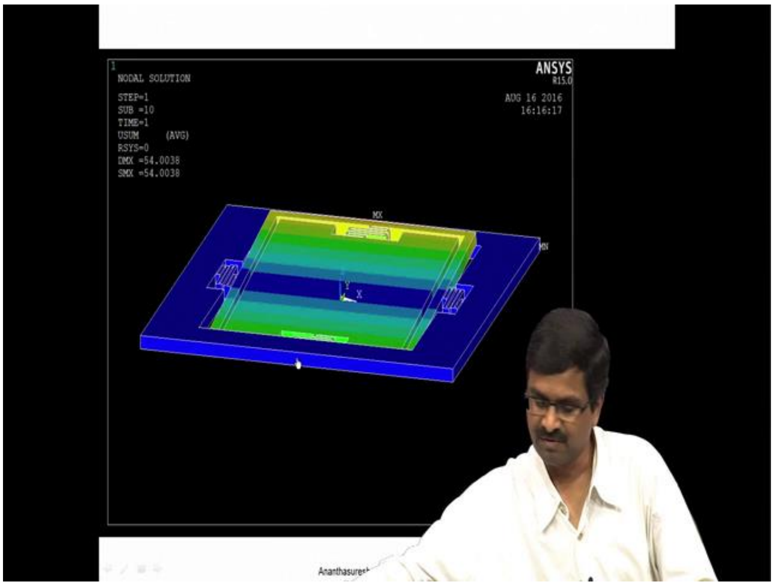

And here is one interesting one where we are moving away from Simple Flexures. Now we can have these Beams; they are still small compared to the trust of the structure. This is actually was used by a company called Agere which is now from Bell Labs. They had use this in optical cross connects as they called for fiber optic communication to move this optical beams and fiber optic one place to another fiber there more like post office for several signals coming through as single fiber. In fact, you can have bunch of fibers they had this mirrors that I have a video of this and a real prototype to show you how this Elastic Flexures Rotational Hinges were obtained at bending of the Beams

So, here it just another view of that, but you can look at the animation here simulation finite element simulation. So, basically it is a plate that rotates now we can see tilting of the plate about one axis and here you can see the tilting about the other axis. So, let us look at a real device that is there now; so we have this over here, we can rotate like this and if I am hold it like they may we can see. I can rotate this plate like that or I can also rotate the other way around if I mean if I hold it like this I can also rotate this way and that way. I can rotate about this axis like that, I can rotate about this axis and in fact, we have made it in a way that you can see these beams easily, but you do not have to remove all this material.

This material can be there and they can be tiny slots I mean if you are using Silicon Edging; you cannot see; you will see only a plate, but a center one can rotate about 2 axis such Flexures also can be done and people have used a number of designs like this very ingenious designs that they have. All of these things have one goal which is to have this Multi-Axis Stiffness here. So, Multi-Axis Stiffness that thing I am not getting a color let me see.

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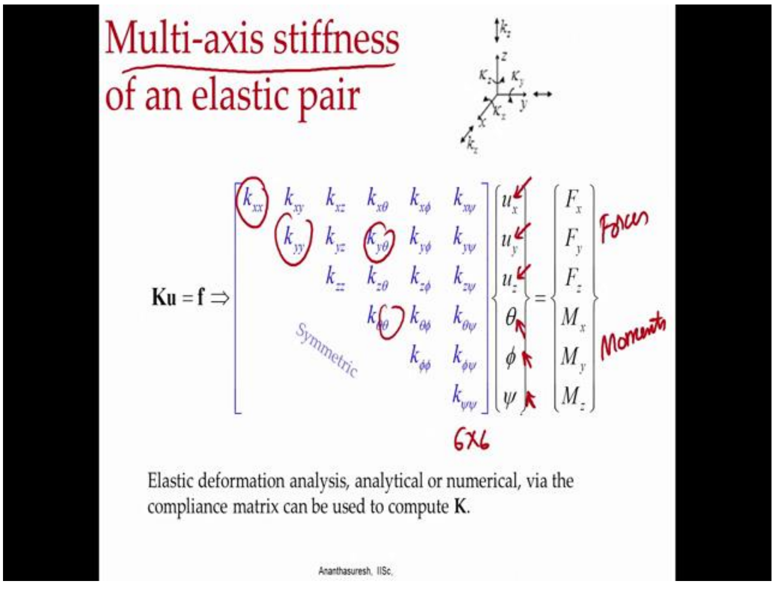

So, we have this Multi-Axis Stiffness; this 6 by 6 matrix that is we have translations about x axis, y axis and z axis and also rotations and corresponding Forces and Moments. So, between 2 relatively rigid bodies these Flexures are giving this multi axis; all these things differ as we had noted in the first lecture itself how these values you know cross values, diagonal values how do they differ; how do you construct this Stiffness Matrix or it is inverse the Compliance Matrix; how do you do that to do this.

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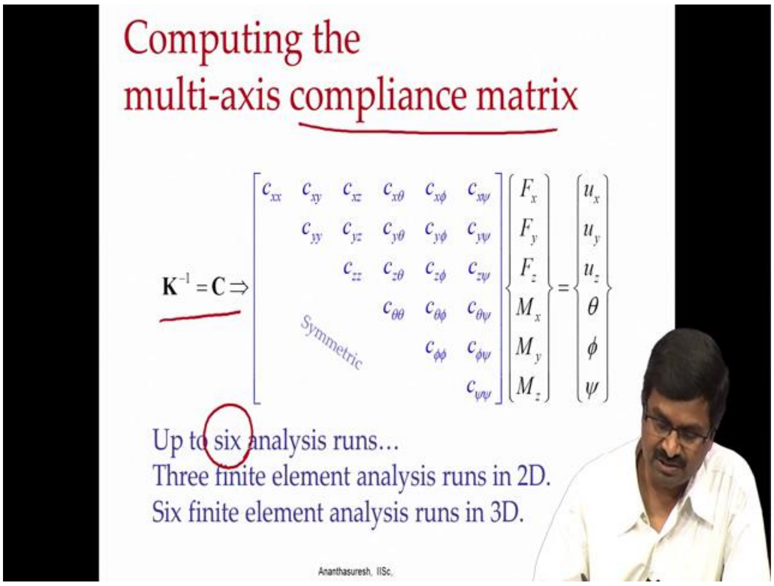

We have to do 6 Finite Element Analysis runs; today we have finite elements we do not have to go for all these approximate formula because there of approximate, but they are useful in design let us not forget that, but if you want one that you have chosen everything is fine now you want to be very accurate you can construct these things instantaneous that is you take a particular Flexure and try to do this.

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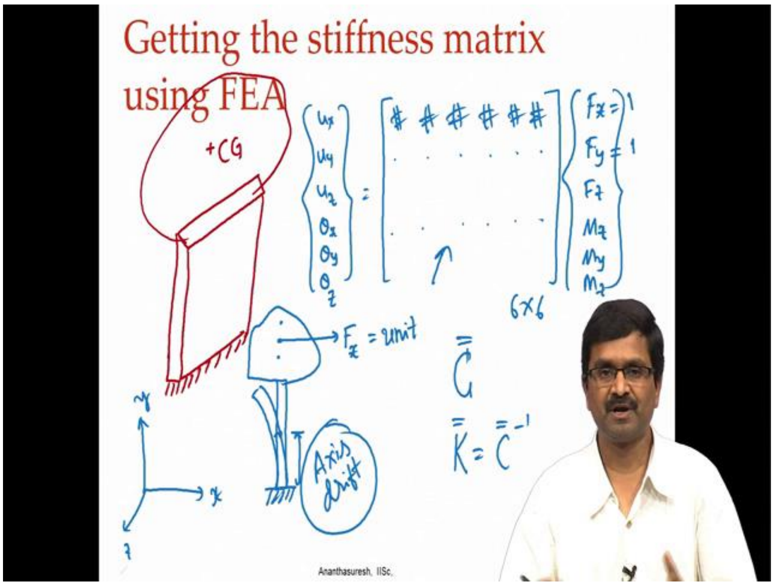

IF you want to do that you have to let us say we take a Small Length Flexures. So, let us say that I have a small beam like this actually at thin plate; let us say that we fix this here and to this one I attach a rigid body something and let us a there is a center of gravity for this and at that point I apply let us choose a coordinate system; let us call this x y and z. So, let me just showing the cross section here it is fixed over there. So, x y z; now, there is a rigid body or this at the C G of that let us I apply F x unit magnitude over there then I get displacements of this thing based on what this Flexures Stiffness is; I will get displacements in the x direction, y direction, z direction rotation about x, rotation about y, rotation about z all of those I record in a matrix 1, 2, 3, 4, 5, 6.

So are we not getting Stiffness Matrix now actually we get a Compliance Matrix meaning that if I put F x here and then F y, F z, M z sorry M x and M y and M z what we want are the displacements that is U x, U y, U z and then theta x rotation about x axis rotation y to y axis rotation about z axis; when I apply unit force in x direction one Finite Element Analysis I get all these numbers; I do not have any other forces now I applied only F x equal to 1. So, whatever displacements I get from the Finite Element Analysis that will be U x, U y, U z theta x theta y theta z.

Then I switch to this and make that unit and get the next row and likewise F z and M x and when I say M x I have to apply let say x axis is this way I have to choose 2 points close by to central gravity, apply a force above and then another one below, create a rotation about x axis or moment and record all these things; I will get this 6 by 6 matrix.

This will be our C matrix it is relating U to forces; now if I take in inverse of this C matrix then I get the Stiffness Matrix we can construct that by doing 6 Finite Element Analysis in the present configuration; let us not forget that in this configuration after I rotate it by some amount I have to redo this matrix because this matrix values in this keep changing. In fact, axis initially maybe infect not may be it will be fixed here.

But after it rotates a little bit it will go like this; it is distance from here to here will actually change because it is rotating. So, axis is going to drift. So, you have to note down where the axis is center of rotation is and what this values are each time into Finite Element Analysis you can get that and we have actually done that I think supplement will put those values for one of this small in Flexures and compare with what is out there.

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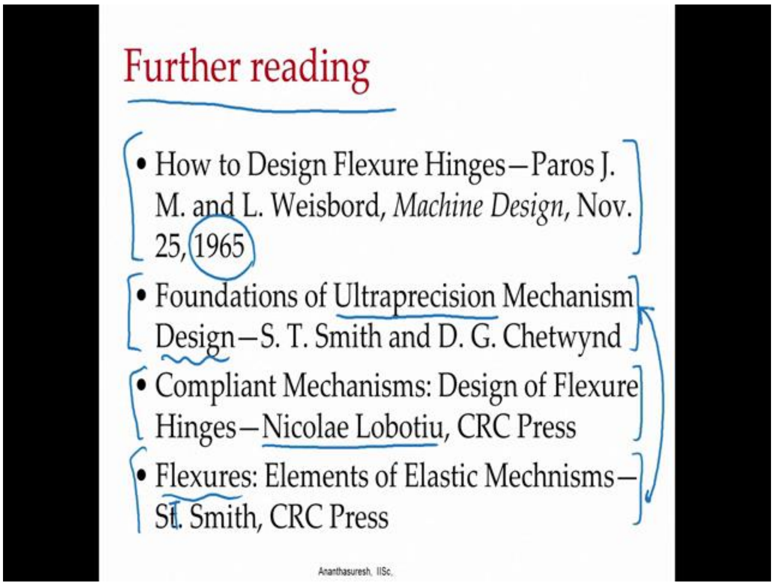

You can look at those values you can do it yourself you have Finite Element Analysis software with you and to conclude before that let us actually look at some reading that you can do. This classic paper of Paros and Weisbord I will strongly recommend till you to read how these people had used just simple integration of the Beam Equation of variable cross section and give formula that are still used 1965 to now 2016.

We could still use that and now this book by Smith and Chetwynd is also quite interesting because they talk about lot of this Precession Mechanism application and focus on the design with all this simple formula how do you design how they arrange this element some are them are inspired by a rigid body linkages.

But you can design components very effectively for small Range of Motion and this Nicolae Lobotiu had done this not just for circular cross section or circular notches, but Elliptical, Hyperbolic and so forth (Refer Time: 34:46) taking work to derive all this formulae using Energy Method and this Flexures by ST. Smith what called Flexures: Elements of Elastic Mechanisms it also in interesting book which as some overlap with this one, but has very useful information about this conformal method as well. So, what we are discussed in these two lectures is how Elastic Flexures can be used in applications and how we can get the stiffness. If you have Finite Element Analysis at your disposal you can get them very accurately including the Axis Drift and Axis Shifting not the center moving, but axis itself may be changing all of that you can capture while you do for every configuration.

What we will discuss next in the next couple of lectures is to use Finite Element Analysis not only to analyze Complaint Mechanism with Elastic Pairs or these Flexures, but also Elastic Segments; then we go towards Distributed Complaint Mechanisms.

Thank you.