Compliant Mechanisms: Principles and Design

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Lecture- 56
Analysis of bistable arches

Hello, we are at the 10th week and this week the first 3 lectures or going to focus on bistable compliant mechanisms, in the last lecture we discussed bistable phenomenon in general in elastic systems and compliant mechanism being in elastic system it does apply to compliant mechanism, we looked at the basic phenomenon of bistability saying that, it has two stable points and there is a unstable point which is sandwiched between them so; that means, that there is a minimum and a minimum between 2 minima there is a maximum, that is what will looked at in terms of the energy landscape and understood bistable phenomenon. And then we look at a number of applications number of consumer products and really big impacting applications as well we discuss in a last lecture and we ended with bistable arches by looking at couple of video clips were we see how beams that have 2 stable states without a prestress in them that is the important point that you with a preload or prestress.

You can always get bistability rather easily. So, we saw a leaf having bistable behavior that is because, it has built in residual stress surface stress, but even without prestress preload you can get bistability in the case of arches and also some shells that we will see in the next lectures. But in this lecture let just analyze the bistability of arches which are planar arches basically, curved beams which are design to deform in the plane of the arch itself.

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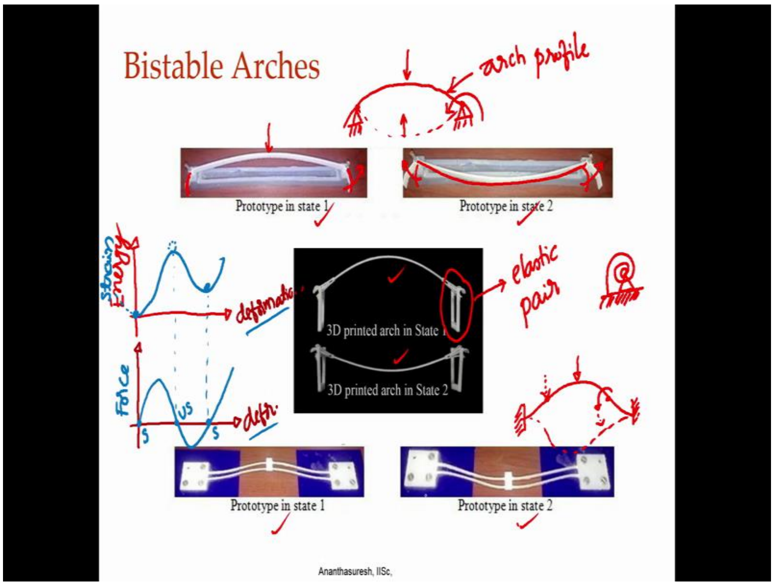

So, let us look at this analysis of bistable arches in this lecture. So, what are bistable arches here are an example. So, here we have 2 pin joints basically, the idea here is that 2 pin joints which are fixed and there is an arch connecting these two that can be of many different shapes that will be what we call arch profile. In this particular case, it is very close to a sign curve between these two have sign and there also this flaps here will come to what; that means, and you can see when we apply force here this one becomes a shape like this which is shown here, which is also something like a sign and it flaps the toward there have know some shape here, they have also moved a little bit when it comes here what was originally here they have moved that way. So, what originally here that is this and this have moved apart like this.

So, here we can apply force here and then apply the force back to switch it back to original state or when we have this flaps we can also apply a movement here and then switch them, that is I can move it like this get then state it will actually switch in that sense they are bimodal bistable arches, but will come that in next lecture. Next now understand, the arches and how we can understand or analyze there bistable non-linear behavior because, the force this placement characteristics of the bistable structures is non-linear. So, it is another one were instead of having hinges we have a inelastic pair that is an elastic pair; that means, that it can be modeled as a hinge with torsional spring.

So, if this is hinge now also included torsional spring to it. So, that there is a movement resisting movement as something turns that hinge. Here is another one that does not have hinge is a tall. So, here it is just fixed. So, this one is more like this where it is built in like a cantilever support and both sides now we can apply the force here, we can apply somewhere else also, but the way if it would deformed to another state would pretty much remain the same and will see the effect it, when we apply force verses here verses there and somewhere else may be we can apply a movement there, because mutually it is turning this way. So, we can also apply a movement, but you can the 2 stable states. So, this is state 1 state 2 for that, state 1 state 2 for this, state 1 and state for this these are the arches. So, in all of these again let us recall that we talked about the energy landscape less energy it starts from minimum goes to a maximum comes to another minimum.

So, this is actually minimum few or that way, this is a stable point there is this unstable point there. Now for this one if we also plot the force curve that is important to see this is the deformation. So, this is deformation variable displacement variable it is same, some variable there and let us see how the force look like which we did not discuss there. So, we can see that the derivative of the energy. So, this can be the strain energy when you also add actuation then the work potential comes strain energy plus work potential means a potential energy right now it strain energy.

So, now what will plot is the force the strain energy derivate with respect to the deformation variable that is this will give you the force. So, here the slope is 0. So, the force will be 0 and same thing over here the forces is 0 again because, the slope is 0 the gradient of strain energy is the force likewise over here. How do you join these 3 points and in relation to the energy one, here we have past to gradient decreasing. So, the force from here will go like this, reaches the maximum then decreases and comes back and goes like this. This is the force displacement behavior bistable because, this is stable this is stable this is unstable this is u s unstable. So, this is stable and stable then unstable here because, this corresponds a maximum this corresponds a minimum this corresponds to a minimum as well. This is a non-linear behavior and that is what this arches are undergoing and that is what will analyze today our starting point is actually post buckling analysis.

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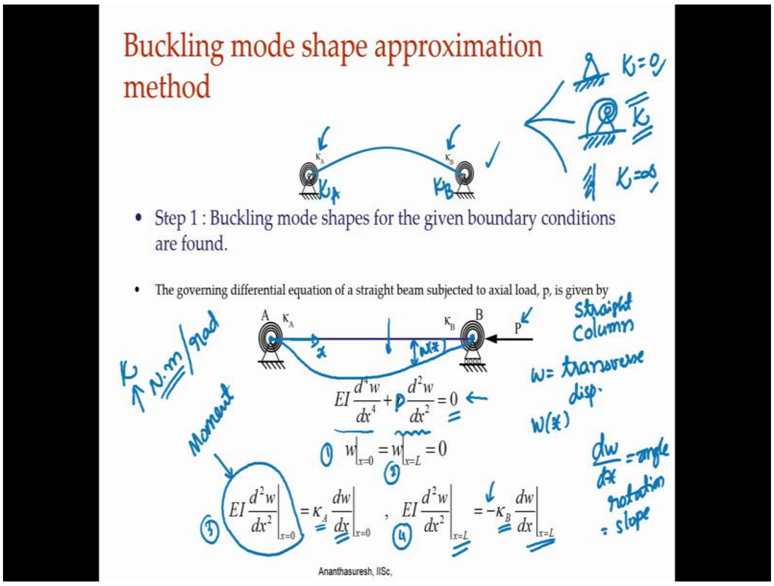

So, if I have a column and such as the one what is shown, here it is a column we call it a column whenever there is a compressive axial load that what it has, but a straight beam or a straight column. So, we have a straight column, on that is our starting point for analyzing things like this. In the previous slide, we saw 3 versions there was only a pin joint that the first one and then we had one where there was a pin joint and then a additional torsional spring, there was another one which is completely fixed. So, all of those can be modeled if we use this torsion spring constant which is this case will have a kappa here kappa is equal to 0 here kappa is actually in a way infinity that is there is no rotation allow in this point here, rotation is freely allowed in this case, that is why it is 0 and here is infinite mean rotation is not allow that is the third one we saw in the previous slide this intermediate.

So, we can model if we can model with intentionally it is given as kappa a here and kappa b here. So, that we can even consider cases were one end the different from the other end. So, in order to analyze such situations we take a straight beam with hinges and torsion spring. So, that we can analyze all 3 cases and anything between these actually more like an intermediate case that takes the entire a spectrum between one end the other end.

So, this is our starting point for the buckling analysis, and the governing equation for that is shown here. So, this is familiar right now, there is no transverse force right, when we do this analysis. So, that is why that q that E I forth derivative of w, w is the transverse displacement of the straight beam transverse meaning in the perpendicular direction. So, it is gone to then something like that a table point here, is that w which is a function of x, x is this way x equal to 0 x equal to l. So, the w is transverse displacement of the beam by the way w is a function of x fourth derivative times E I, e is hence modulus i is a second movement of area for the cross section.

So, i for a rectangular section is B d q, but by 12 were B is the breadth of the cross section, d is the depth of the cross section we have that for usual bending, but now we have an additional term which is P times d square w by d x square this is second derivative, and this P is same as the axial compressive force that is over there. The derivation of that we can either do using variation calculation approach there is an another NPTEL course that discus as this or you can force balance as were, but that have to look up the buckling analysis to see what were this equation comes from we have e i forth derivative of the w plus P times second derivative of W that is equal to 0, and then we also have the boundary conditions that W in this case, at x equal to 0 as well as x equal to L is equal to 0 because, we have this hinges there is no transverse displacement there. And then we also have the other set of boundary conditions, because this is a fourth degree differential equation.

So, we need to have 4 boundary conditions we have 1 and 2 here now we need too more this is 3 this is 4 what is that this one says that at x equal to 0 E I second derivative of w which is basically movement. So, this is movement one more derivative this shear force this is movement that related to kappa a times the slope that point w by d x that is what is here, there is a torsion spring when it turned by in angle that angle is d w by d x, because slope of the transverse placement d w by d x is our angle of rotation at that point that is a slope of the beam.

So, that times a kappa torsion spring constant will be equal to the movement because, the units of kappa you see units of kappa will be Newton meter per radian multiply by radian we get back Newton meter it is movement likewise you have E I d square by d x square evaluated x equal to l is equal to kappa at the other end kappa b and then slope over there minus sign because of the sign convention that we have taken here. So, that is just our sign convention, because of the way things turned over here with regard to the slope.

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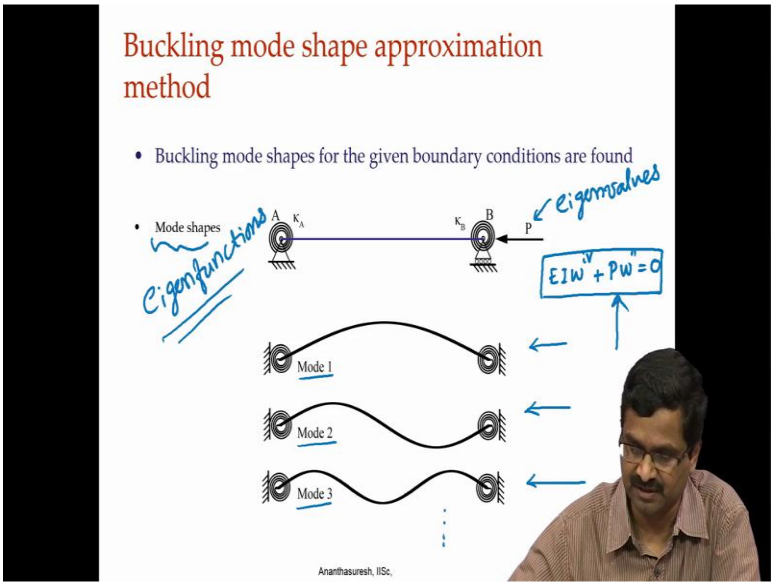

So, with these with this differential equation and the boundary conditions, we can solve for the moved shapes of that because, that is like Eigen value problem that we have here that the homogeneous equation. So, we have if we write it E I and then fourth derivative were write it like i v plus P times W double prime is equal to 0, this is like a stomlivi problem or Eigen value problem we can get infinite solutions for it and each solution is called a mode shape, this is not the mode shape that we use in dynamic analysis or other free vibrate analysis is the buckling problem that also has a different Eigen equation and for that will have Eigen functions which are mode shapes.

So, mode shapes are Eigen functions the corresponding Eigen values are going to be the critical buckling modes just like the mode shapes and Eigen values which are like natural frequencies when you free vibration here, the Eigen values going to be the critical buckling load which we call oiler buckling loads. So, here we have that the first mode shape looks like this, second mode shape is asymmetric looks like that, third mode shape looks like this, and you can get more and more fourth fifth how many hour you have in your discretized model we can get if you have n degrees of freedom we can get n mode shapes, but you are doing analysis here, you will get actually infinite you can go keep on mode 1 mode 2 mode 3 and up to infinity. So, many solutions are permitted by this Eigen value problem.

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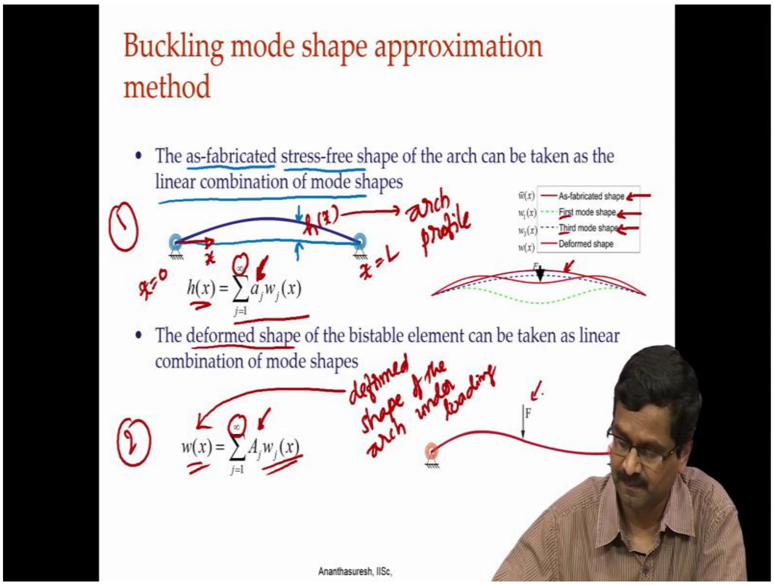

So, here we take, we showed 3 mode shapes for the conditions of kappa a kappa b given here. Once we have this we can do 2 things with buckling mode shapes; the first one is that the as-fabricated stress-free shape that is a key word as-fabricated stress-free there are 2 key words. So, when you take an arch you fabricate it in that way. So, there if your material bulk material that you have taken to machine this arch that if does not have in a residual stress we are not going to introduce any more, to this as-fabricated from that we can use c n c milling machine and then cut out. In fact, the one that we showed here, the beginning this was done by taking a sheet and cutting out the shape as fabricated was like this, and so was this was actually 3 d printed and so was this there 3 d printed. In fact, it is later 3 d printed.

So, as-fabricated that is stress-free in that state are as that is shown, and you that and that shape that is a arch profile can be taken as a linear combination of the mode shapes which is saw in the previous slide that there is mode 1 mode 2 and mode 3 and so for. It turns out that these buckling mode shapes when you adapt them for the arch profile that is to make an arch, they are likely to be are most often there bistable those arch profiles are bistable not all of them, but they have the inherent properties. So, you can take that as a linear combination that is if I take let us I do the straight line here and then say what is this that is the height, that is height h that the height h, h of x that is shown here, once again x is that way x equal to 0 x equal to L that is span between 2 pin joints here torsion spring added spring joint.

So, there that height that is a arch profile this h indicates the arch profile can be put as a linear combination of this mode shapes, this w here w j if as a w 1 w 3 there all the buckling mode shapes which is shown here. So, there is that arch here which is now a combination of several of these buckling mode shapes, that is something that we can do to reduce the problem to a discreet set of variables a j's; j equal to 1 to infinity if you take infinite mode shapes we have to take all of them, but you can choose to do only with 3 or 4 5 even a 1 if we want. So, that becomes approximate solution to the differential equation we had to analyze this.

So, the arch profile itself it can be taken as a few you can take 1 itself that we bistable by itself you can take 2 3 and so forth. Another thing we can do is, that deformed shape of the arch that is different that can also be taken as a linear combination of the same buckling mode shapes we have small a j is there now you have A J is, this is the deformed shape of the arch under loading.

So, when you apply a load how it deforms that can also be captured using the combination both remember, one way arch profile in a linear combination second the deformation is also a linear combination then you can take instead of infinite we take finite and get on to the analysis. And it is indicated the as-fabricated shape is that color that is to one green first mode shape third mode shape second is skip because, here we are interested in symmetric arch profile second were is asymmetric. So we go to first and third and not the second one. So, fourth one will also the asymmetric and so forth, deformed shape that we have here is a linear combination of these three and again we have reduced the differential equation problem to a discreet problem to the Eigen functions by taking only F finite small a is that define the arch profile capital A is the define the deformed profile under loading such as this here.

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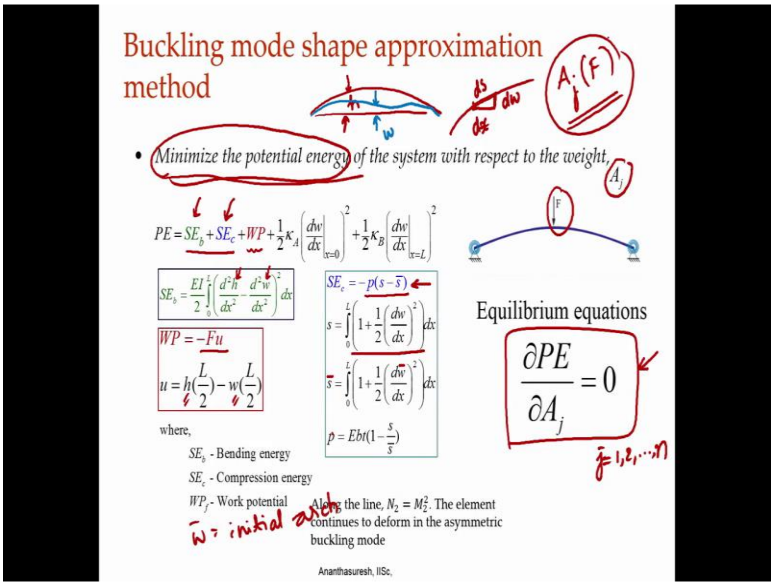

So, when you do that as you apply force is going to deformed as it is shown in this video clip, and what we are determining here are the A j's as a function of the force. So, this A j each A j that we have is the function of force, when the force is 0 and as force increases to a these increase. So, if we have 3 mode shape there will be a 1 capital A 1 capital A 2 capital A 3 each of them very differently and the deformed profile will also be changing. How do we do that we do that by minimize in the potential energy now, we are going from a strain energy to potential energy. So, potential energy is a combination of strain energy and then second one which is the work potential WP.

So, energy has 2 components there is a bending component and there is a c which is the axial compression component, and then this term takes care of the spring which is also strain energy at the end. So, this also strain energy due to kappa A that is spring and then this is strain energy due to kappa B will write this potential energy there is work potential which is due to the loading that we have there. So, let us look at each term.

So, SE b is bending strain energy so. In fact, we should write that it is bending strain energy SE c is compressive strain energy and this is the work potential energy WP. So, expression for bending strain energy is shown here, EI by 2 o to L d square h by d x square minus d square w by d x square. So, basically h indicates arch profile second derivative these deformed one that is, if I have the beam like this let us say the beam like that from this line, this would be our h as we said already now if this were to deformed to let us say the different color let us say this deforms to something like this, then this height is w, that is say deformed height that is height is change.

Now, that is the deformation that is what we have here. So, we have bending strain energy and then work potential is negative of the work done by the external force here the F is the force how much work is it doing. So, force times displacement u here h L by 2 minus w L by 2 the difference because, if the midpoint if you take that is by the force being applied, were it is verses were it is after a deforms were it is after deforms w h is the initial height or the arch profile we get u that is the work potential. And then we have the bending strain energy all ready written here and now let us look at compressive strain energy that is s and s bar. So, let us say this is s and this bar is missing bar, s is again it is an approximation up to a first order for the arch here to take this 1 plus half d w by d x square the weight comes is that if I take a there is a arch like this, if I take a point like this, simply Pythagoras theorem.

So, this will be d w that height and this will be d x. So, this one that we need if I call that d s that is going to be square root of d x square plus d w square, now you divide by d x and takes d x out and then you get a square root of 1 plus d w by d x square d x that integral, that square root of 1 plus d w by d x square is approximated as 1 plus half d w by d x is square. that is a first order tell Taylor series approximation, and similarly; for the changed one, if you take that is initial when w bar initial length is s bar this is the change one let us say w bar here is a initial arch. Then you can see how the compression the p the force p times that change in the axial that is the arc length of the arch clearly an arch like this, if it has to buckle somewhere it is really compressed stores a lot of strain energy that is a requirement for bistable phenomenon and then it becomes longer again compare to original.

So, that what done by that axial force in changing this given by this, once you let potential energy in order to find these A j is we have to use this equation, this potential energy is minimized that is the problem here, minimize potential energy of this whole system. So, necessary conditions dictate that the, derivative of potential energy with respect to each A j. So, j can be any number you take 1 2 you can go of infinity what will do want few things are enough some n. So many equations you get and you have to solve.

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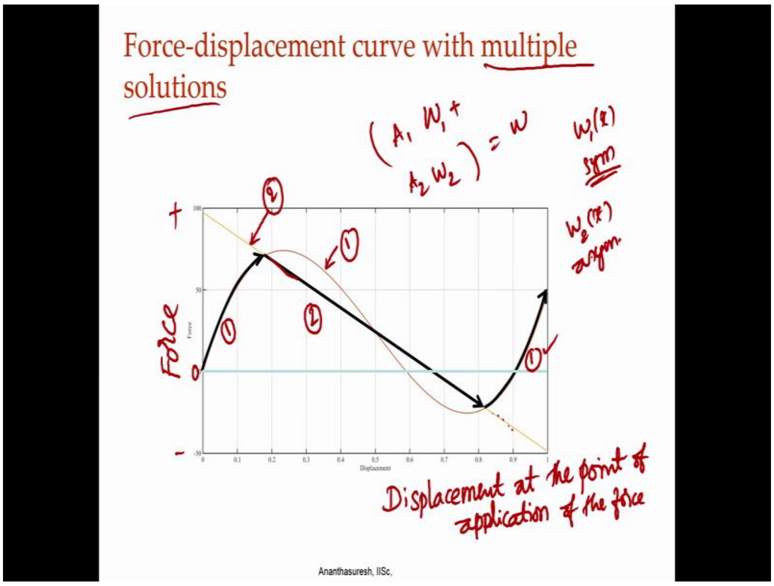

Along the line you have how many our mode shapes you take; you can do that, when you do that these, what you get. So, what we have here is the force what we have here is the displacement in the case of an arch this is the displacement and the point of application of the force displacement at the point of application of the force application of the force that again is a linear combination of our A j is and the buckling mode shape. So, here you get multiple solutions more than one solution that happens here that can happened with non-linear equations here equations indeed non-linear when you do this thing over there, the resulting equations in terms of this A j is non-linear which allow multiple solutions sometime there is no solution which is there is no equilibrium, but when there are that can be more.

So, 2 solutions are shown here, this is solution 1 and there is a solution 2 which one does the arch take when you deformed, turns out that the arch would take what is highlighted now there are 2 solutions one you looks more like a (Refer Time: 28:19) either there is a straight line the arch actually takes in this particular case what is shown in black lines, it follows 1 for sometime switch is to 2 and then goes back to 1. So, it goes as it is shown. What is it do that, because of again the stability from here to here up to this point this is a higher force and after this point the second buckling mode should be the lower force which has to that, then comes here after this is higher this is negative we see 0 is here, this is negative this is positive so here this is larger.

So, it prefers that and goes he takes that list path of resistance if you not exactly that, but it prefers the lower force and follows that shape. So, there is inherent asymmetry that goes because, the second one here will be asymmetric is a first one symmetric, second is asymmetric that is we have mode shapes W 1 and W 2 and then will have A 1 then A 2 combination of these 2 is over w w 1 of x is first one symmetric w 2 of x is second one that is asymmetric.

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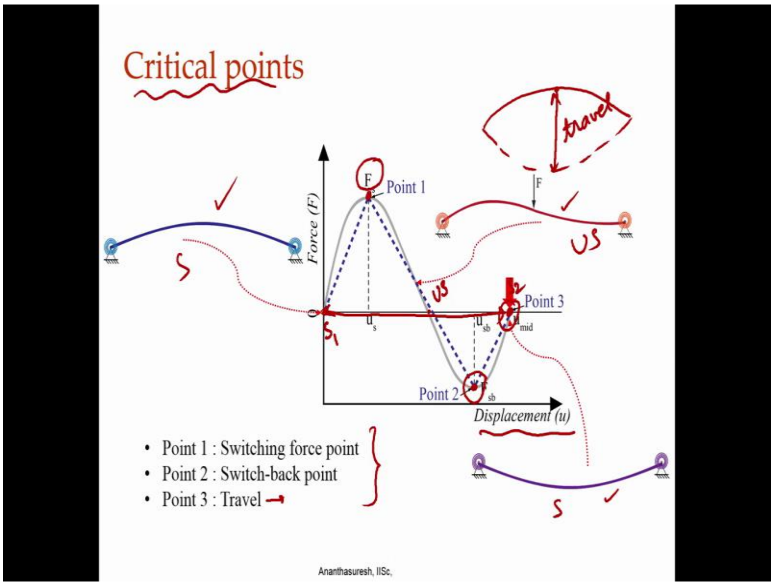

Here, there are some critical points we can solve those equations, but it is not always possible when you put this torsion springs and more complicated boundary conditions for pen-pen fixed-fixed you can easily do those solutions, but other once when you kappa it becomes difficult what could be done in that case is to look at some critical points.

The critical point here when the force becomes a maximum the other critical points the force becomes a minimum, the third critical point were the force reaches the second state the force becomes 0 to get to the second stable state. This is the first stable state and this is the second stable state between force and displacement. In between there is another one, we can called a forth critical point, but we do not needed that is the unstable state the u s between 2 stable states. So, here a switching force point switch back force point and then the travel that is between, when you have an arch and that is goes to the other state here is these where applying the force how much is it that is the travel, that is the distance travel that will be from here to here that is a displacement to the other thing that we need. And these critical points can be found numerically more easily for this complicated situations than the solving for the A j is completely.

So, we have this thing showed for is s 1 the arch is like this and in between un stable is over here, and third stable state here this is force were as, this is un stable this is stable this is stable over that 3 states that we look at, in between of our critical points at different critical points whether force is maximum switching force switch back force and second state first state any way we know that is our starting point which is stress-free. So the 3, that is a first one is second one and is the third one.

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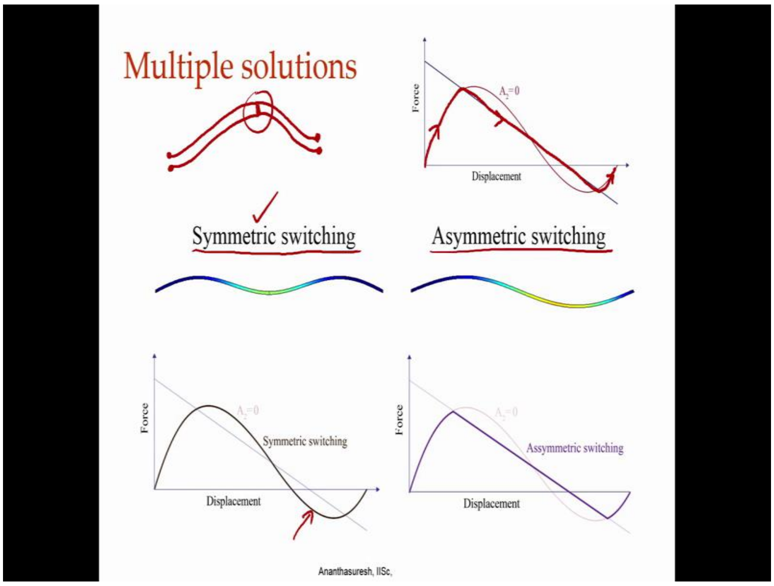

And this multiple solutions we can find showing that it will actually take this one is what we said it goes like this, goes like this and goes like that, but if you want you can switch symmetrically also. But you have to constrain that asymmetric not a load it will take this part, which is shown in the black line here or if you take asymmetric you can see if you do not restricted it goes there becomes asymmetric and then goes back to symmetric.

So, after this it again goes back to symmetric. Initially, it is a symmetric little part there it is asymmetric and the asymmetric for the most part then goes back to symmetric. The symmetric switching, asymmetric switching here we do something special to prevent asymmetric mode shape taking a chance. So, that is done in various ways in the case of thing we saw there were two arches like this which are again. So, this was another prototype you had this was actually cosine curve if you have another cosine curve each of them is fixed here, fixed here, fixed here, fixed here and there is in attachment that remedial if you do that does not prevent rotation of that one. So, asymmetric mode does not happen due to main symmetric, but even there asymmetric the can still happen, but more or less one can prevent it depends on various parameter that takes place there.

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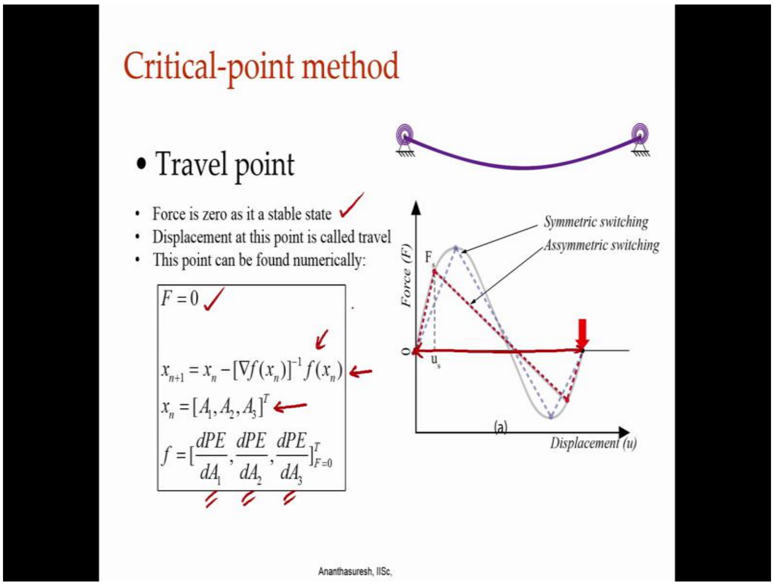

So, the travel point force is 0 at a stable state we know derived this point we have the travel that we already discuss this can also be found numerical that is from here to hear how much ever travel is. So, in both cases we can find force equal to 0; the other hand of course, this is a trivial solution were A j is are 0 other case; A j is are not where you can use some numerical solution were this the variable for as A 1 A 2 A 3, if you take three buckling mode shapes and equations that we get by taking derivative of potential energy with respect to each of them partial derivative A 1 A 2 A 3 equal to 0, and you can update them by taking this gradient in this manner and then numerically solve the problem when do that you can find this critical point, switching back switching points were it is a maximum thing.

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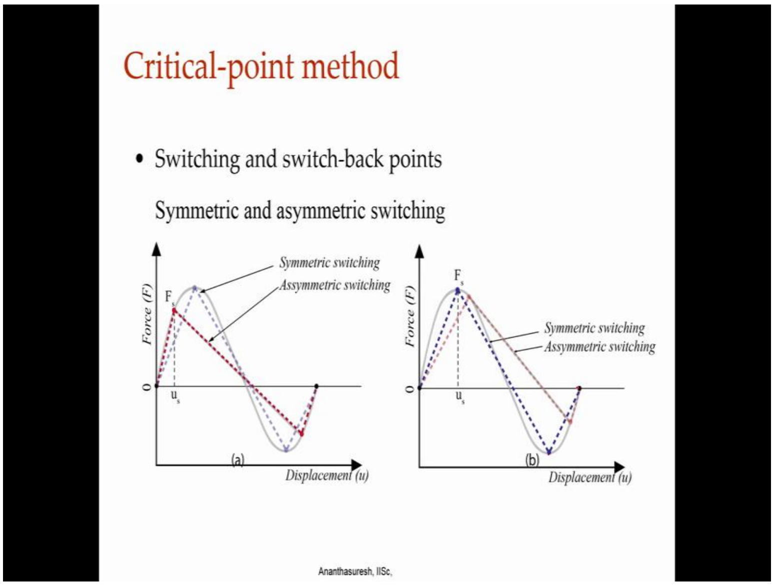

And here is the minimum and then will also get the travel point all can be done numerically.

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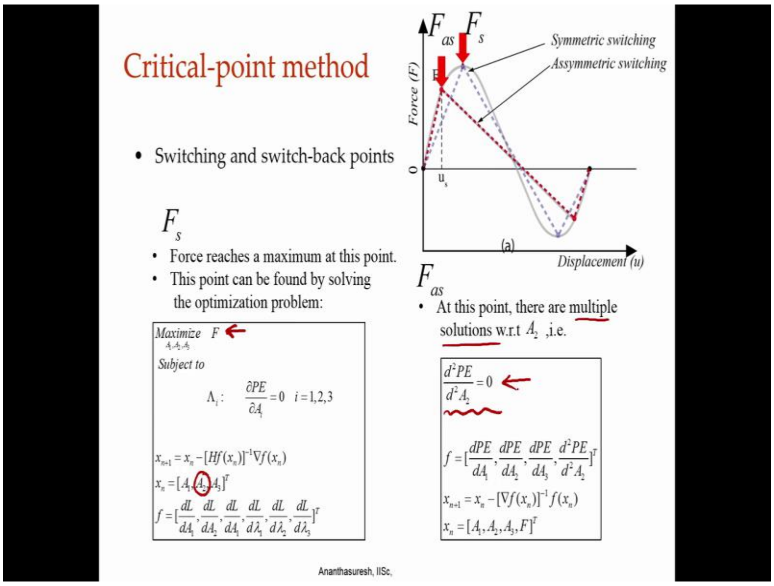

So, these the maximize if you do you get this critical point, which is shown right now big red arrow and then symmetric or asymmetric depending on what you allow that is; A 2 is included that becomes asymmetric and A 2 is not there becomes symmetric you get two different points. And the other way, that when it actually goes to the 0, where it exactly? There are multiple solutions for A 2; we can find them by equal in the second derivative to look at this stability. So, whether it is just stable or not we can also include that, I can derivative called an energy which is a stability check for it.

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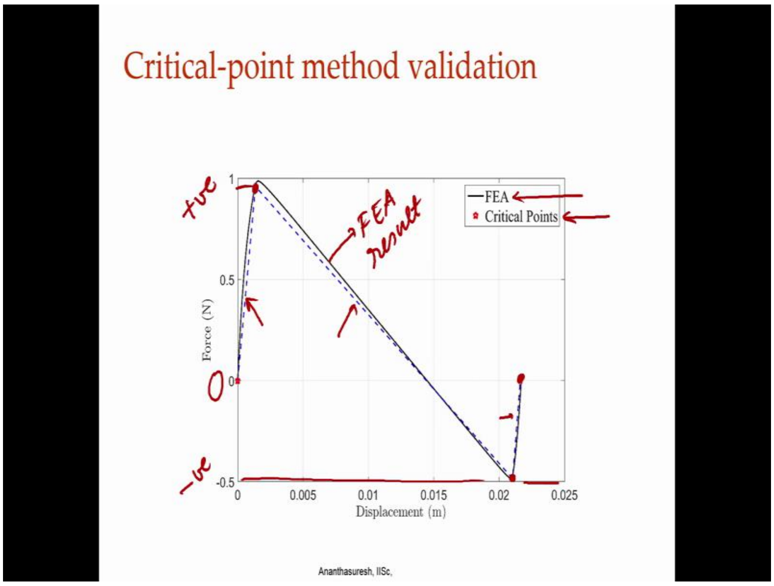

So, we can find those points as well and it is a comparison of the finite element and critical points, critical points this joining does not actually makes sense we only know that, there is this point and this point and this point. We just happens to joins we can see how it varies and this is the finite element analysis result and it is pretty good due to solution, if you do it right we do get the points in between is just imagination it goes like that, our interest is the switching force how much force is there to switch from first state to the second state and how much force is needed to switch back here it is 0 in this case, force at apply in the opposite direction that is why these negative and this side it is positive.

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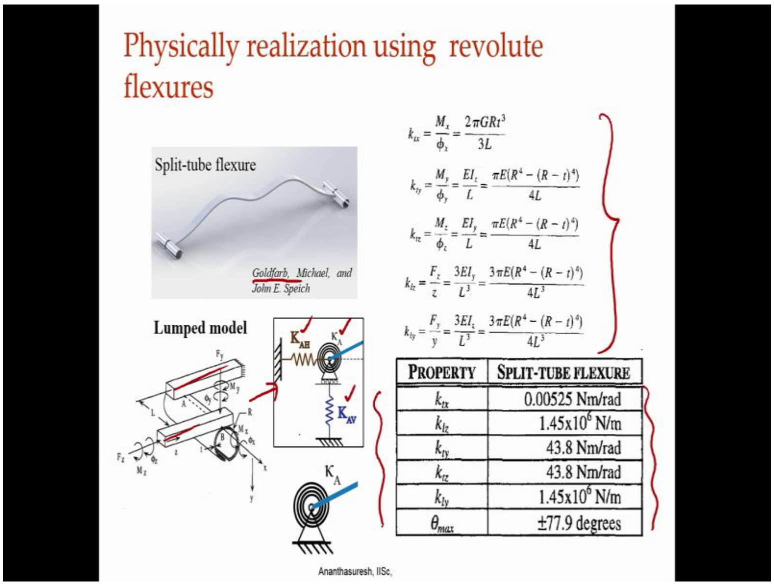

And we can take a lumped model in this case split tube flexure is taken which you had talked about earlier we talk mode elastic pairs and we can physically realize this split tube flexure is basically, a tube that has a small split all along. So, the tube goes like this. So, this one if you join two things there, you get a torsion join there boundary conditions for at and it is stiffness in various directions, are given the multi axial stiffness. So, that can be modeled as a transnational spring the axial direction and translation spring the transverse direction and then rotational stiffens all that correspond to this, we can use a lumped model get all those parameters that are needed and use them in our analysis.

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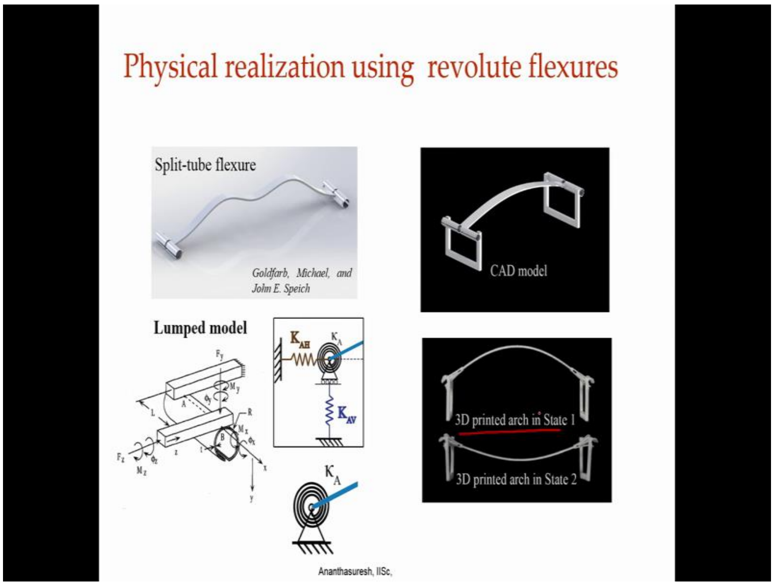

So, that we can actually synthesize these things real 3 d printed 1, wean analyze using this arch analyze we just discussed we are to the mode shapes and buckling mode shapes and look at that.

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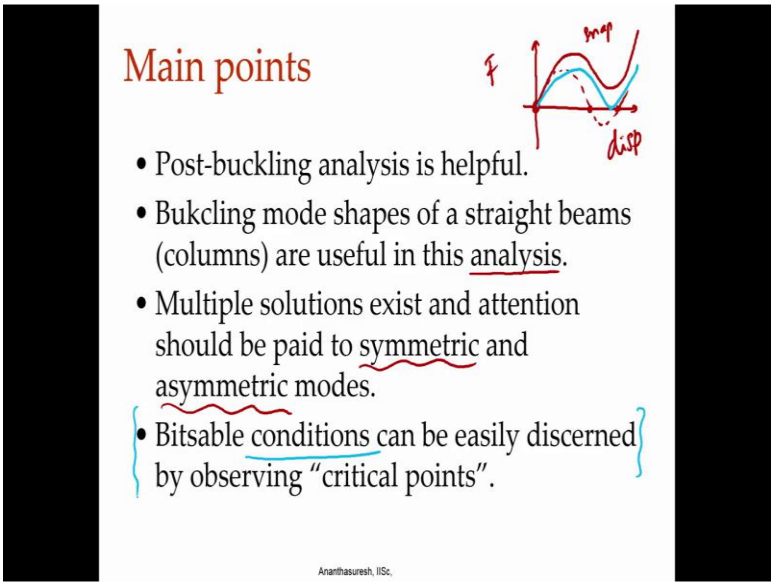

So, what we have discussed, is the post buckling analysis of a straight beam gives us buckling mode shapes which can be used as it is in the as-fabricated shape and without stress, we can make them as it is and then can strain them, constraining we can do in multiple ways we can do like a fixed connection or a pin joint without any torsion spring or a pin joint with torsion spring we torsion spring constant changes what you do is, we have to solve the Eigen equation that we have get the buckling mode shapes.

Once you get them there useful in analysis as well as the in synthesis we saw both because, we can express the arch profile as a linear combination of the buckling mode shapes and we can express the deform shape of the arch also as a linear combination of the buckling mode shapes small a is for arch profile big as that lower case for arch and then upper case as for A j that we had for the deformation and then we can use the principle of minimum potential energy to formulate the equations and solve them. Solving them is not easy that is why we have explained a critical point method, we get the critical points and then try to get the enough information for as to analyze approximately only have few points and critical points we can imagine shape like that, but in general it will be a smooth shape.

We also discuss how we can include the symmetric, asymmetric once physically as symmetric be prevented by constriction there that we have were restriction of the middle for a cosine arch that we talked about, but we can analyze both and we can also use them to synthesize rather, bistable conditions for bi for a given arch in terms of it is length and cross section and so for, can also be discerned that is we can understand when we get the bistability when we do not. For that let us look at one thing here, if I have this is the force and this is the displacement if I have force displacement like this, this is simply called as a snap through this is not bistable, because one force is 0 after that force is not 0 for yet to be 0, this not be external force required which could actually be like this, then we have 0 and 0.

Sometimes what happens is, this will be just touching, that is; I will just use a different color it could be that it just touches like this, that will be the limiting case such conditions, bistable conditions can be discerned from this critical points that we discussed. So, in the next lecture we take this analysis and then show how we can actually solve some applications were we use this arches to real applications and how this analysis helps us in actually synthesis are designing.

Thank you.