**Compliant Mechanisms: Principles and Design**
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Lecture – 50
Stiffness and Compliance Ellipsoids

Hello, we are talking about design methods for Compliant Mechanisms. We have so far, discussed five different design methods, today we are going to discuss in this lecture and the one that is after this a design method, which is a 6th one in our series of design method that we are discussing, this one is a new method that is inspired by electrical circuit synthesis where you would take building blocks and make a circuit.

And in this case a compliant mechanism. So, this a building block based synthesis where we will discuss some theory.

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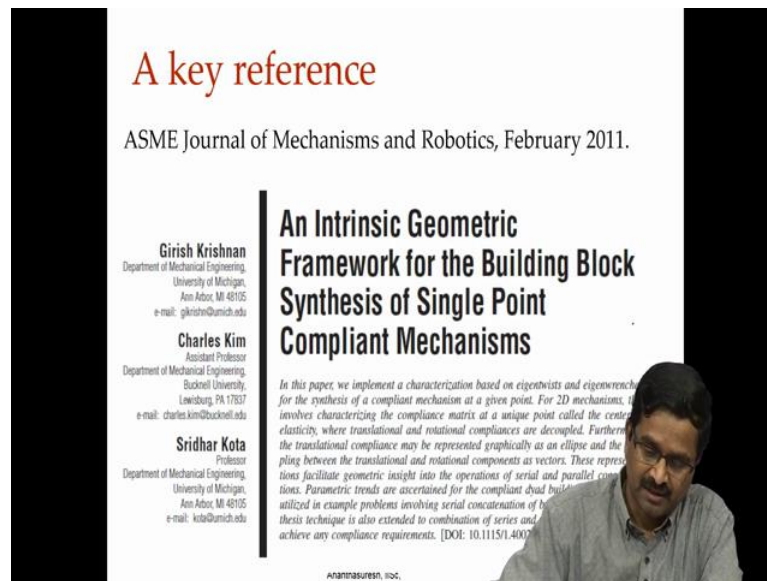
And the theory pertains to what are called these stiffness and compliance ellipsoids. So, these are the key words for today stiffness and compliance ellipsoids, which describe the building block with which when we combine different building blocks to make a compliant mechanism there is a sort of a systematic way that one can design or it is also partly intuitive as we will see, but the theory is very good and that is what we will discuss first and then talk about the implementation.

This is a key reference for what we are going to discuss, this is from ASME Journal of Mechanisms and Robotics in 2011, sort of recent work of Girish Krishnan, Charles Kim, and Sridhar Kota.

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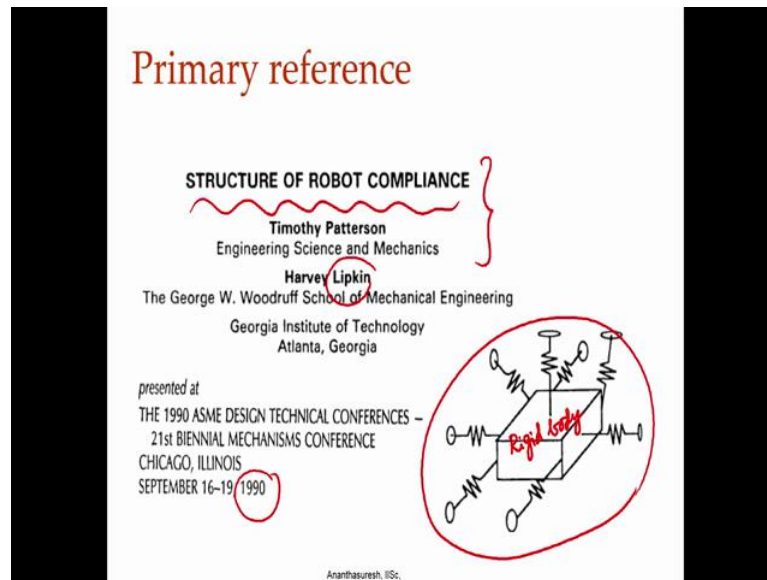
The roots of this work can be traced to a conference paper by professor Harvey Lipkin and Timothy Patterson, tim patterson which is in 1990, conference paper where the title is the Structure Of Robot Compliance, most of the robots they are made of rigid bodies, but when they move at high speeds or when they are carry large loads they do deform a little bit. So, robotics people also have to worry about compliance or flexibility there the flexibility of course, is not intended at least until recently now people talk about compliant robotics, which sometimes require soft robotics.

So, but this notion of flexibility modeling exists in the robotics literature this is one of those papers where they talk about a rigid body. So, what we see here is a rigid body, which this paper considers where it is suspended by bunch of springs, here only translation springs shown here, but you can also have a costional or rotational springs when we have such a thing we can represent the stiffness of this rigid body suspended with elastic elements that is springs in a non or invariant way, non variant or invariant way using eigen analysis and that is what this paper does and that is the basis of the discussion that we are going to have today

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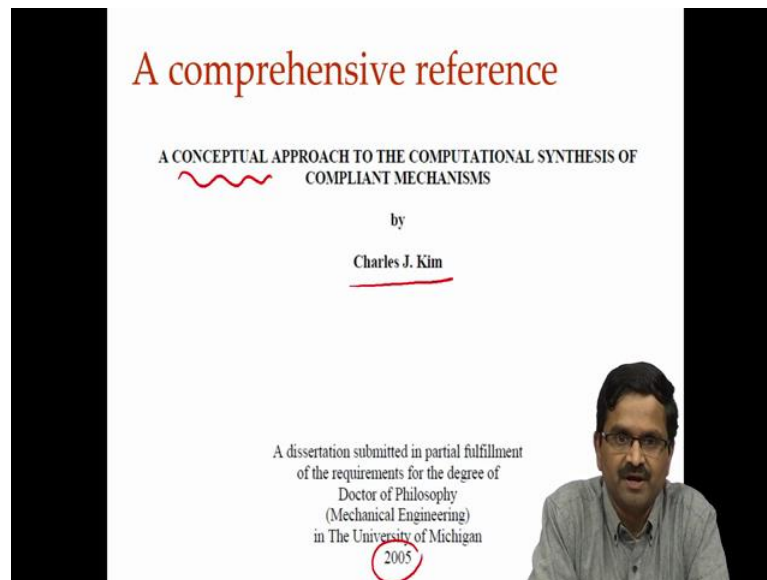
A comprehensive reference is a PhD. thesis at the University of Michigan in 2005, by Charles Kim. Where it says a conceptual approach to the computational synthesis of compliant mechanisms, any design method we take are computational, but here it is a conceptual approach meaning that this will give you a way of arriving at concepts for compliant mechanisms using this building block approach.

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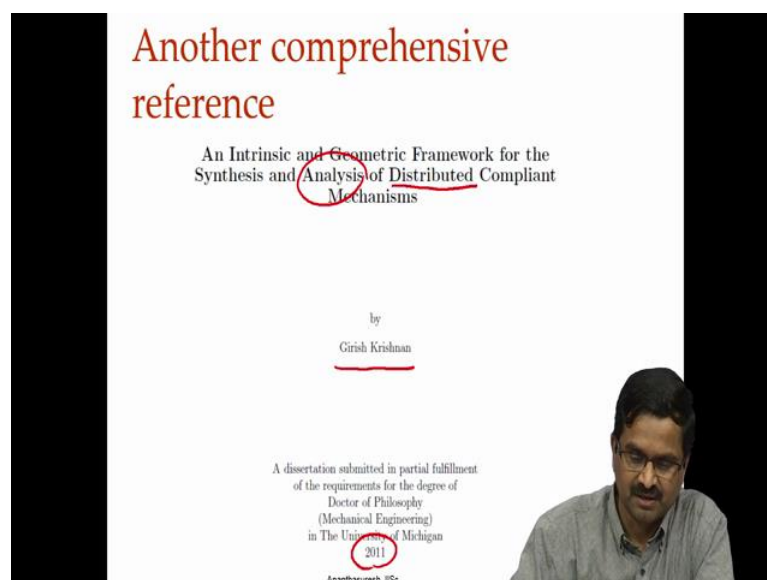
And there is another comprehensive reference; this is another PhD. thesis 2011, by Girish Krishnan. Who is now at university (Refer Time: 04:21) and the title of his thesis

as you can see here, is Intrinsic and Geometric Framework or the Synthesis of Synthesis and Analysis of Distributed Compliant Mechanisms. We have already talked about distributed compliance versus discreet compliance, when we say distributed compliance we mean that we have slender beams rather than elastic flexures.

So, here when we have beams how do you compose them because it is synthesis of course, it talks about analysis also what we will discuss in todays lecture as well as the next lecture, here you have this beam segments like building blocks and you have to put them together to make up a compliant mechanism that is what these two speech thesis look at this design technique.

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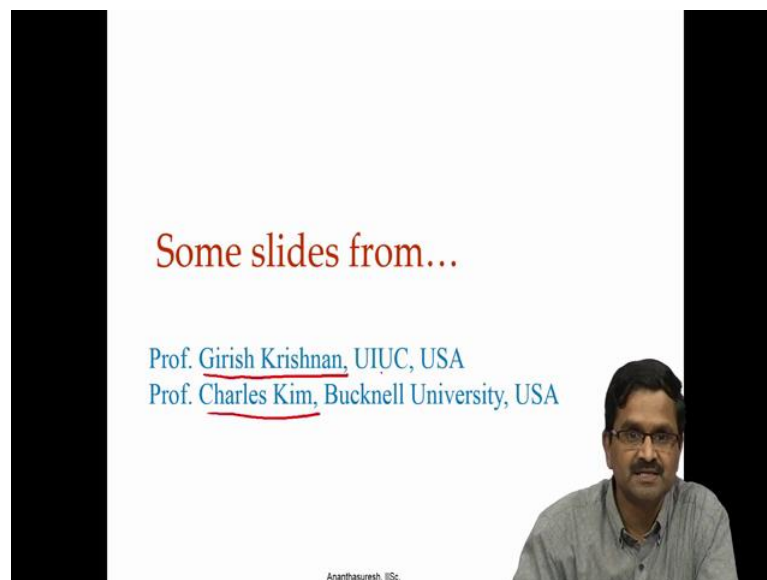
The slides that I am going to use are given to me by Prof. Girish Krishnan, whose speech thesis we just referred and Charles Kim, he who is at Bucknell University in USA, and Girish Krishnan is at UIUC in the US. So, I am using their slides is their work.
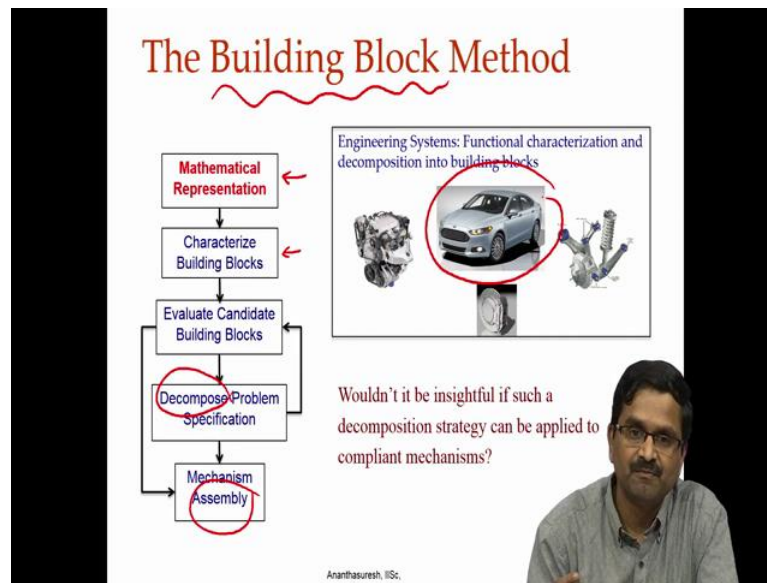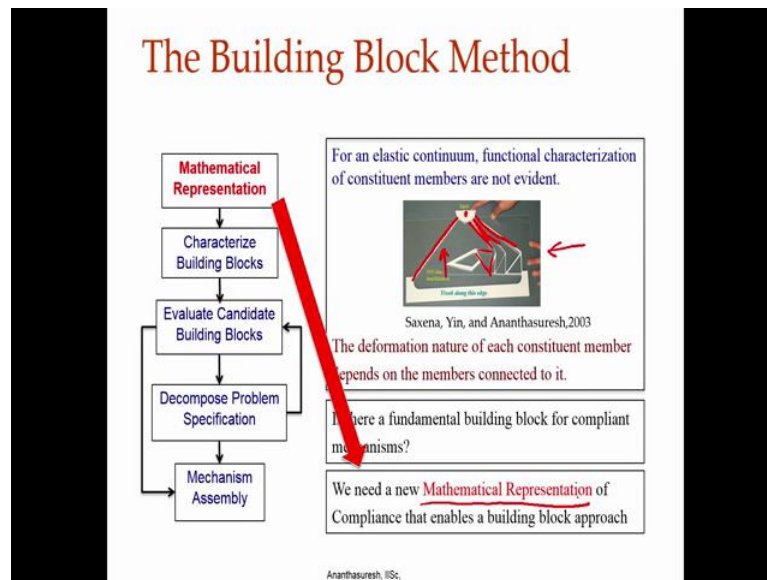
And it is best described using their own slides. So, here is what they consider they say that, if you want to look at this building block approach, they refer to a complicated system such as an automobile now we will have several sub systems there is engine, there is axle system, there is suspension system, air condition system, and so forth. So, when you look at a complicated system you decompose the problem into sub problems sometimes the sub problems are subdivided into sub, sub problems. So, smaller pieces where you look at a design, so, now, what they want to do is if you look at a compliant mechanism also as a system or elastic segments or what we call compliant segments how do you put them together.

So, that you can get this so, what they say is that you first characterize, first you need to have mathematical representation that is the first thing and then you characterize the building blocks, whatever building blocks you have you try to characterize them in some ways that you can assemble them together not physically, but of course, computationally and then eventually if becomes a physical one when you make it evaluate this candidate building blocks, whether they suit a particular function that you are looking at of course, function itself is decomposed into smaller functions and then you have this assembly and then go back and forth to get this that is what they are saying, and the question that is posed here would not it be insightful, the kind of decomposition that we do for a car can it will it not be nice if we do that for compliant mechanisms is what they ask.

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The Building Block Method

Mathematical Representation → Characterize Building Blocks → Evaluate Candidate Building Blocks → Decompose Problem Specification → Mechanism Assembly

For an elastic continuum, functional characterization of constituent members are not evident.

Saxena, Yin, and Ananthasuresh, 2003

The deformation nature of each constituent member depends on the members connected to it.

Is there a fundamental building block for compliant mechanisms?

We need a new Mathematical Representation of Compliance that enables a building block approach

Ananthasuresh, IISc,

And they take this example that gives a very large amplification from input that is here, to an output here, it gives almost 50 and 75, 150 can actually design it by taking a clue from this one. So, such a thing how do you do that, there are beam segment there is a beam segment here, there is a beam segment 1, 2, 3, 4 there and then the smaller ones, all the beam how are they assembled in order to do that what we need is this mathematical representation of each of these building blocks that are beam segments as they will as we will see, it is more than a beam segment what they call compliant dyad, which we had referred to in the last lecture as well. So, you need to have mathematical representation for these building blocks.

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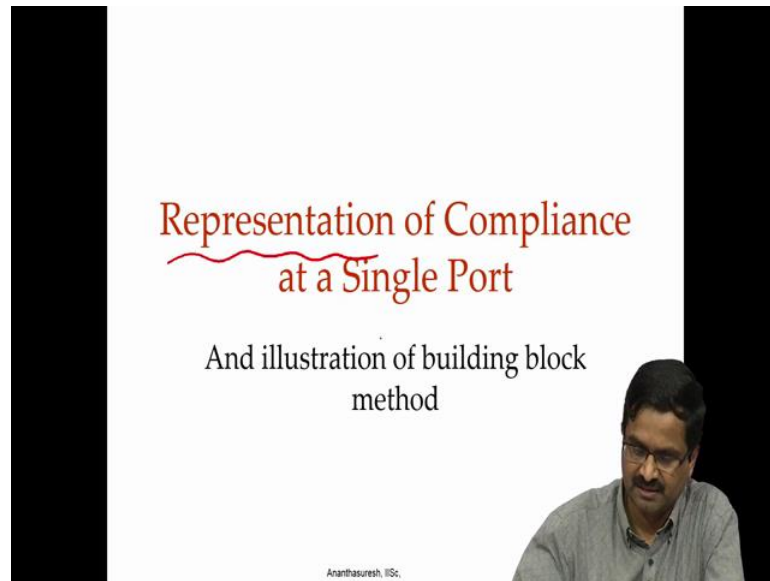
So, that we can put them together, so, the representation is the first thing that we want to do it is a single port it says that is if an elastic body at a particular point if you want to know what is the compliance you can actually get that

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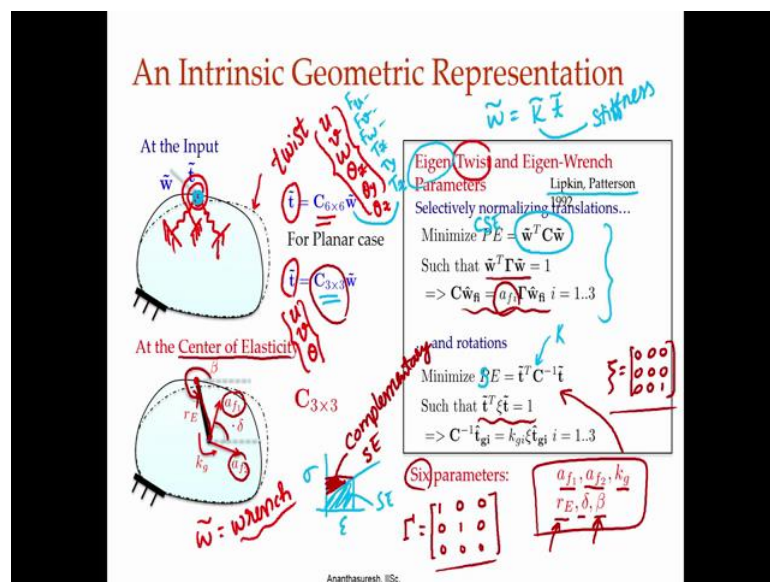
So, here is a representation. So, you have an elastic body which is shown here and it is deformed state is also shown in this dot dash line, dot dash curve in response to a force of a electric point it is a single port 1 point, what is it saying there are lot of symbols here let us go through it you have this t tilde, which is a vector which is called twist or eigen

twist, that comes later first this twist t is twist this is borrowed from the robotic literature, robot schematic literature where the twist vector is basically displacements.

So, when you take it as a 3 dimensional case a specially moving body, elastic body deforming elastic body at every point you will have a displacements, but here you are not looking at just the displacement at every point, you are looking at the elastic body as a whole more like what we say in Harvey Lipkins model, where there was an elastic there was a rigid body, which is suspended by a bunch of springs. Similarly, here if I had a elastic body I do not want to model this entire elastic continuum instead, I would like to represent that with some springs. So, if I take it as a 2 d problem that will be the twist vector will have displacement in the x direction we can call it u, and then v, and then in the rotation will be theta, rotation about the z axis that is what this will be that is called the twist.

In the case, of special case you will have u v w that is u is the displacement in the x direction v in the y direction w in the z direction and then there will be theta x theta y theta z this is for the entire rigid body that is you think is over here. Which is suspended by this spring that is how it is looked at so, otherwise if you take continuum analysis the displacement will be 3 in special case 2 in the planar case, but now we are using sort of a lump analysis where the continuum elastic body is represented by a bunch of springs that is what is lumped modeling.

So, that is what is done here twist is the displacements and rotations of the rigid body, which captures the elastic body by adding these springs where do the springs come that is where we come from robotics is the compliance matrix where w is called the wrench. So, just as we have called t the twist robotics literature the w is called the wrench because, wrench is the one that with which we apply force of course, wrench is a term that is used in the US, but here in India we call it a spanner. So, spanner with spanner you can apply torque. So, it is borrowed from that it is a wrench vector; the wrench vector corresponding to this will have the forces and the movements.

So, corresponding to u I will have a fu or fx or fv and fw corresponding forces and then there will be torque about x axis, torque about y axis, torque about z axis, that will that will be the wrench. So, what relates the twist and wrench or rather displacement and forces is the compliance matrix, if we take the reverse of it that becomes a stiffness

matrix that is we say f equal to k u or u is the vector, here also it will be w the w is the wrench vector if I do k times t that becomes, then the stiffness matrix for the rigid body suspended with springs. So, here we are not worried about this elastic body we are capturing the elasticity or elastic nature of the continuum with a bunch of springs as if there is a rigid body at the point where we are interested that is the theory behind this. So, in the case of 2 d it will become 3 by 3, 3 d 6 by 6 ok.

Now, when you have that you would realize that when I apply the force in one direction I will get something, if I apply force in other direction that is we have this fu fv and fw meaning f x f y f z depending on what I put or even maybe movements, if we add there how does this behave, how much displacement do I get, how much twist vectors do I get there will be one direction in which if you apply force that is some combination of fu fv f w you will get maximum displacement, and there will be another set where you get the minimum displacement in order to see that let us actually stake take an example and sketch something. So, let us get empty slide here to sketch something ok.

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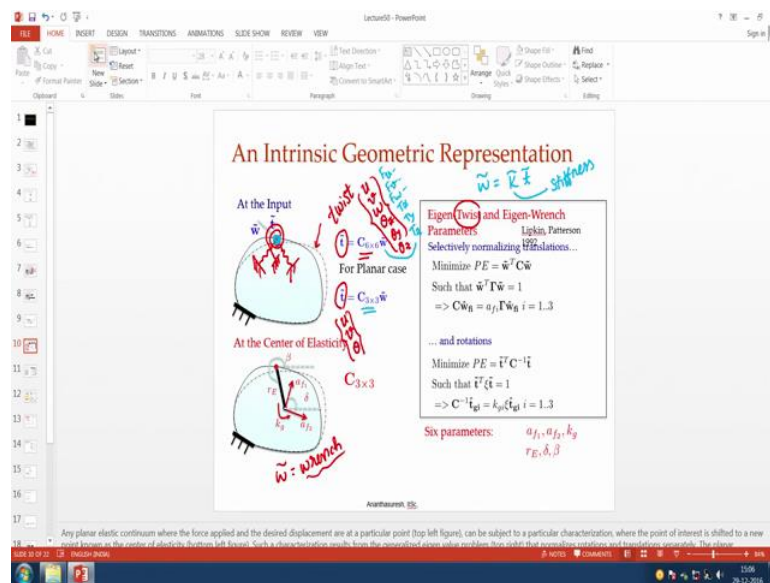
What we mean is that let us say, I have instead of continuum elastic body let us take say some beam like that, which is fixed over here our intention is taking arbitrary beam whatever it is.
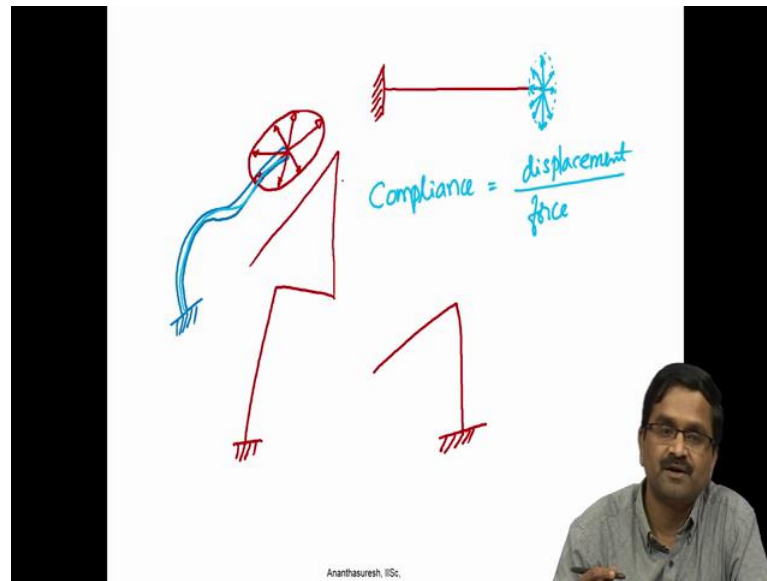
So, some beam like this, now on this beam if I apply force let us say in that direction let us say it moves so much, if I apply force in that direction it may move so much, to apply force in another direction it may move so much, so another direction it may move so much right. In all directions when you apply the same magnitude of the force and let us say that I am keeping the same magnitude, but change in the direction of the force in different things what I am showing here are the displacement maybe if I apply force here it will move more maybe not I am just saying that there will be a direction, which will have the maximum displacement there will be another one which has minimum displacement.

So, to make it simple, let us actually take a cantilever beam that is always good to start with a cantilever beam for a cantilever beam you can say that if I apply the same force in different directions let us say apply some force f in this direction it may move so much, it may this is the displacement moves so much, if I apply in this direction it will move very little because axle stiffness is very high if I apply at incline move it may move so much, apply this way it may move so much, apply this way it may move so much. So, what you get is actually an ellipse if I do all this in different directions there is symmetry so, it will be ellipse like this.

So, this cantilever beam for the same force rather stiffness also, because how do you measure stiffness we apply some known force measure displacement force divide by

displacement gives you stiffness or reverse of that gives you compliance. So, if our displacement is more displacement divided by force, we can call it compliance is equal to displacement divided by force, you can see that the compliance are inverse of this stiffness will be different in different directions for a given compliance segment, whether it is a complicated one such as this or a simple cantilever there will be a direction in which in which you apply the force, the same force will have maximum displacement will be another direction which is actually perpendicular to the first direction, which will have the minimum it is like what you get principal stresses.

So, if I say that maybe that will be clearer. So, you know that in a elastic body at a point there is a state of stress meaning that there will be sigma x, sigma y, sigma z, tau xy, tau yz, tau zx but then we will have a principal stress where there will not be shear stresses there will be only normal stresses there will be a particular direction the other direction will be perpendicular. In 3 d in the case, of 2 d you have sigma x, sigma y, sigma x xy or tau xy, then there will be principal stress direction where there will be maximum stress perpendicular to it will be minimum stress the same thing stiffness is also a tensor like stress and strain are stiffness and also compliance that is what it is about that is why here we are talking about eigen analysis right, it you do that eigen analysis that is posed like an optimization problem this what is Lipkin Patterson had done.

So, that post optimization problem where we are minimizing the compliance energy that is the wrench this is like f square by 2 k no that is complementary strain energy you can call it whereas, if you take c inverse is nothing, but k here stiffness these are like displacements. So, displacement transpose stiffness matrix and displacement gives you strain energy they are not potential energy this is strain energy, this is complementary strain energy, when you have stress strain if you have a curve area under the curve this way will be strain energy the other way that is you know should not show the thing, if I have sigma and epsilon like this, this area is what we call strain energy on the other hand I can also take the area towards this that will be complementary strain energy in the case of linear stress strain both will be the same, but if it is non liner they are different.

Anyway that is what is put here and you are normalizing this with a matrix gamma, which is not here gamma is simply it is all done for let us say 2 d case this is 1 0 0 0 1 0 0 0 0 meaning that we are basically solving eigen value problem only for translations here, and then we are doing it for rotations here this xi that you see this giggly thing that

is 0 0 0 0 0 0 0 0 1. So, basically you are doing it only for the rotation in which case you get the eigen analysis there for the rotation this is for the translation. So, what these two eigen value problem when you solve you get is what I just described that you would get a basically, what you call compliance ellipse in the case of 3 dimensions it becomes compliance ellipsoid because, just like your principal stresses if you do take in 3 d there will be one direction which is one extreme maximum value another one, which is intermediate the third one which is the least that is how ellipsoid we get when you have a tensor it is representation graphically will be ellipsoid like this.

So, that is what we are getting here when you do this you get this a f I, where I goes from one to three in the case of 3 d if it is 2 d that is what I have written here for 2 d you will get a f 1 and a f 2, these are the maximum compliance and minimum compliance because, we are dealing with compliance matrix we are talking about flexibility we can also take the reverse and we get the stiffness first we are talking about compliance ellipsoid where a f 1 a f 2, I mean there is k g which is the rotational stiffness that comes from the analysis over there that eigen analysis.

And this r e and delta and this beta or other parameter that come out of this we will explain what they mean. So, basically what you have at the 6 parameters, what you have in 3 by 3 symmetric matrix again 6, so, those c 1 1 c 1 2 c 1 3 c 2 2 c 2 3 c 3 3 those six parameters are transformed into something like invariant. So, this does not depend on the coordinate system that is Eigen analysis anyway we get these things. So, what you have here are these quantities that represent graphically the stiffness at a point.
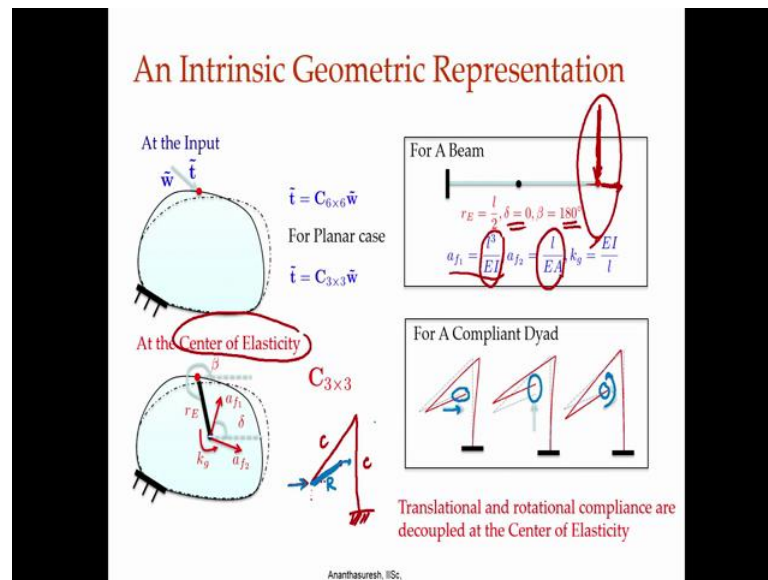
So, if you want to look at it the meaning of this is that if I take a cantilever beam, here then a f 1 a f 1 here turns out to be l cube by EI, it is a familiar thing that is if I apply a force in the vertical direction transverse direction you know that the stiffness is EI by l cube compliance is l cube by EI, that will be maximum compliance l cube by EI in that direction and the other hand if I apply the same force in this direction I get a different displacement which will be l by EI, a e by we say the deflection is pl by au we call it now we are talking about compliance.

So, we have to divide the deflection by force phi we get l by EI. So, this will be much, much smaller compared to this for a given beam because compliance in this direction in this direction is very low compliance compared to compliance in that direction, anywhere else it will be in between. So, here we will get the ellipse. So, it gives you that it also gives you that that beta is one eighty degrees meaning that our and delta is 0 delta basically says that the vector if I draw the ellipse here it will be something like this the direction of this is given by this one axis is along 0 minimum thing here and then beta is another angle, which is given in interpretation in the we will see that in the next class.

When you do that one other thing that we get is what is called the centre of elasticity. In fact, the meanings of those parameters are also here we can see that. So, this a f 1 or alpha f 1 is the basically stiffness extreme compliance, where not stiffness compliance values and r e here is an important thing what this r e that we get and this angle beta. So,

this angle beta what they represent is where you applying the force and then you are moving to something called centre of elasticity, the centre of elasticity is a point which decouples the rotation and translations or actually two translations also. So, and the delta is of course, this direction of that maximum compliance and minimum compliance. So, with those things let us actually look at this example which makes it clear.

So, let us say that we have this dashed one. So, we have what is called a compliant dyad which is fixed here. So, this is compliant, this is compliant now you can imagine that when you apply force let us say in this direction, here it would not move only in that direction it will have some displacement in the vertical direction also whereas, instead of applying there if you find the centre of elasticity, which for this problem lies over here if you were to add a rigid segment around here to here, just a rigid segment r you have compliant segment add a rigid segment if you apply a force in the x direction like it is shown here it will purely move in the x direction. If you apply force in the y direction at this point it will purely move in the y direction apply a rotation like it is shown here it will only rotate that is the decoupling nature that is basically this invariance also comes.

So, you not only get this extreme values are Eigen values and Eigen vectors, but you also get the centre of elasticity here. So, with which if you say basically what we are saying is if you apply a rigid one apply a force here at this point which is where it effectively comes you are not only applying the force, but we are also applying a moment.

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So, if you were to purely apply a force here this point could move, but by applying a moment you can compensate for the other motions that is the rotation and the translational other direction, if you actually see this here this is parallel moving the dash line and the line if you see here they are moving parallelly right; that means, that there is not even rotation there this is a rigid body that is attached, this is a rigid body that is moving like a rigid body the other motions completely decoupled.

So, this is the intrinsic geometric representation we are talking about where we can characterize the compliance ellipsoid with 6 parameters this a f 1 a f 2 k g, k g is the rotational stiffness as if there is a torsion spring here, that is what it is.

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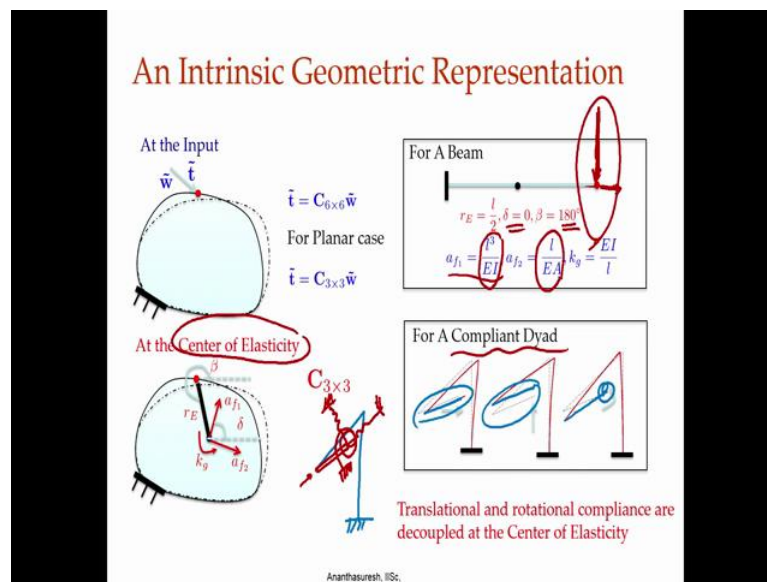
So, we go back to the analysis when we have compliance segment, I can simply see that this compliance segment that we have we can think of this instead of that we can say there is a rigid body let me show with a different color, I have a rigid body this rigid body is now suspended with some kind of springs here. So, there is a spring in the maximum compliance direction perpendicular minimum compliance direction and there is a torsion spring there. So, when apply force is in this direction or this direction it purely moves like that without having componently rather as you see here that is how the compliance dyad is characterized ok.
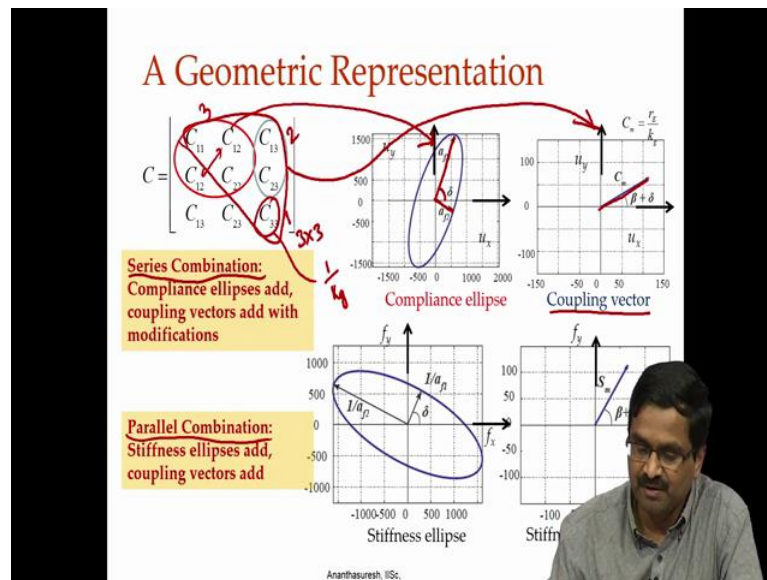
And that is what is shown here one more time, that in this compliance matrix which is the inverse stiffness matrix the 3 by 3 matrix that relates displacement on forces may say forces, forces and movement, which we call wrench and displacement as a twist displacement as well as rotation this 2 by 2 matrix gives you this that is there are three symmetric matrix c 1 2 c 1 2 the same thing here. So, these three parameters give you the maximum compliance minimum compliance as well as the orientation delta of that 3 parameters 3, here and then these 2 here, will give you what they call a coupling vector, coupling vector tells you how the movement and force are coupled rather this coupling vector is the one that denotes the centre of elasticity related to the point where you are applying the force.

So, this is the vector that actually tells you how to move in the point of application of force to a point centre of elasticity if you apply forces or movements there they give you decoupled motion x force will give you displacement purely in the x direction, and y force will give purely in the y direction rotation gives you only rotation node other 2 displacement. So, if I put x force I get only x displacement no y displacement no displacement and likewise right, but of course, these are all for instantaneous motion that is small displacement right then if you want to do this we have to do this analysis for all times steps in the non-linearity formation and you can characterize in this fashion. So, we get that and we also get this k g which is actually simply, the k g that we got the rotational stiffness is 1 over kg c 3 3 or rather 1 over c 3 3 k g so we have that. So, we

have three quantities that are coming out of these two it has the coupling vector and then one here total 6 that is what we have symmetric matrix where 6 quantities is there. So, we can represent any compliant system or elastic system using these 6 parameters in 2 d.

Now, when you have this you can do something more, which is combine them in series or parallel what do we mean by that, what we mean here is that let us say I have one compliant dyad like this, let us say that I have another one now if I combine these two what happens that is I take this one over there and add that. So, if I just try to sketch the same way here for this 4 segment one what will be the 6 parameters, how do you do that it turns out that we can simply do some simple algebra of adding things in the case of series combination this compliance ellipses add.

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And coupling vectors add with little modification whereas, opposite if I do for stiffness they simply add what they mean by that here is where we will see. So, if I have a segment one. So, let us we will say color so, I have this segment one and I have another one added serially to it series addition. So, this building block one will have 6 parameters of it is own building block two will have parameters of it is own, if I take the combined building block what do I get it turns out that we can do some simple algebra first of all the second building block may be two is rotated with respect to the frame that you would have chosen.

So, there is a transformation matrix you have to consider where this is the psi angle how it has rotated this is would have been you know fixed related to something there is a psi and there is also an l r, that is what was to be over here related to this it is displaced. So, we have the transformation matrix we have to take that and then do this series resultant of the 4 segmented one by taking the compliant ellipsoid of the b b 2 and then b b 1 transformed with related to this second one. So, when you do this you get the resultant one.

And likewise you will also get the rotational stiffness centre of elasticity for this segment was over here, for this one over here, the new one will be elsewhere and torsion stiffness there will be different it goes like a harmonic average and the way that new centre of elasticity lie is given by this how do we get this cf there is c 2 plus c 1 plus l r by kg 1, and then if you substitute a few things we will get c 1 and c 2, which is kind of shown graphically here how do you get this. So, the idea is that as you add more and more building blocks what happens to the characteristics of this new segment is easily obtained by just adding them like this if it is parallel we have a slightly different approach, but there is an algebra for this.

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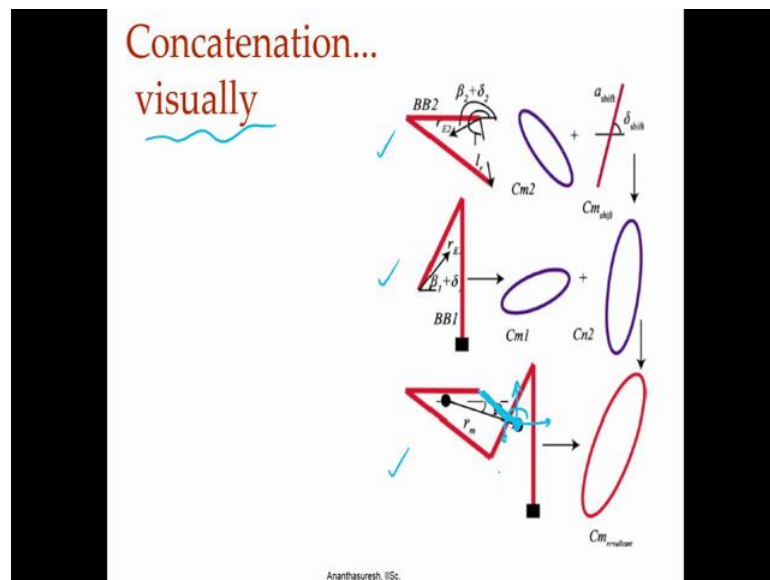
So, you can do that very easily. So concatenation visually if you look at it. So, you have this segment and this segment when you add them together you get this new one and also you will get a new centre of elasticity and if you attach that to the body and apply the

force with like a rigid one. If you apply force in the x directions move only in that direction only in that direction and purely rotate. That is we are talking about the rigid body suspended with the segments.

Now this theory we will stop here, and continue how to apply it in the next lecture.