Compliant Mechanisms: Principles and Design

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Lecture - 43
Non-Dimensional Analysis of Beams

Hello, we are in to the 8th week of this class and this week we are going to discuss a new design technique which is based on a concept that we introduced this week, which is non-dimensional analysis of compliant mechanisms comprising a slender beams. We have been talk about beams as being the crucial elements for plainer compliant mechanisms are for that matter even for special or 3D compliant mechanisms.

Now, we will use beam elements or beam segments that comprise the compliant mechanism to tell us something about non-dimensional that inherently exist in beams and how that extends compliant mechanisms and how we can use this non-dimensionality concept to design compliant mechanisms. Once you have a problem like we discuss last week we should be able to select compliant mechanism, redesign it or even redesign or just design annual that is design compliant mechanism newly for a given set of user specifications is what we look at and that comprise the 4th design technique for compliant mechanisms that we are discussing.

So, let us look at this non-dimensional a portrayal of non-linear elastic response of beams. So, the key words now are it is non-linear elastic response not just linear and we are going to non-dimensionalize the response. In this case elastic response, because compliant mechanisms we want to limit it elastic response because it has to be repetitive. So, if there is plasticity and some permanent deformation it will be 1 use 1, 1 use mechanism that is not the intention of any mechanism or most mechanisms. So, we want to retread our self to elastic regime and then talk about this non-dimensional analysis, but it is non-linear mainly because of large displacement analysis. So, let us look at the slides and then recall first how the deformation comp cantilever beam that is simple as been think of.

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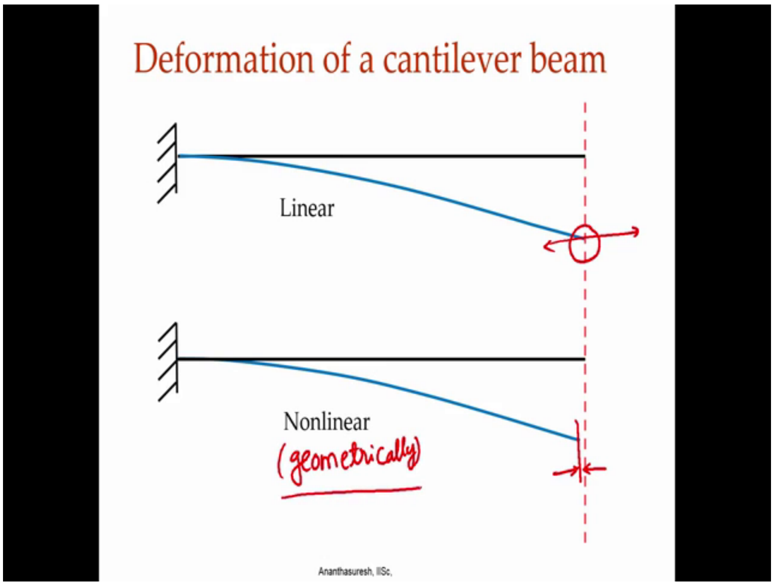

So, we have cantilever beam if it is linear analysis verses non-linear analysis, if it linear this point will not have any displacement in the axial direction, in this direction should not move whereas here you can see when you consider non-linear or what we call geometrically non-linear that is we try to write the equilibrium equations in the deformed configuration rather than the original configuration. We already discussed in lecture number 13, 14 where we considered the large displacement of x on compliant mechanisms, today we will revisit to find some non-dimensionality in the non-linear deformation of compliant mechanisms. So, if have linear there is no axial displacement if there is non-linear the point that is a free tip or loaded tip of the cantilever beam will move both in the axial direction as well as transverse direction.

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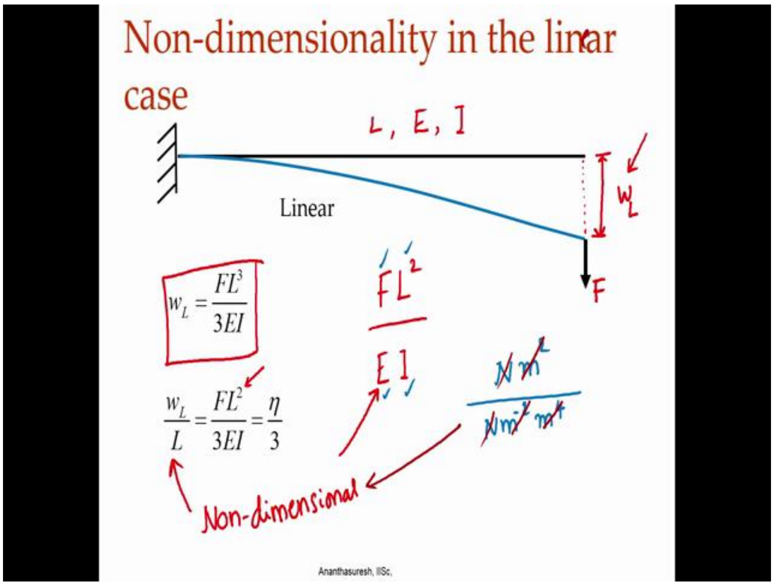

So, let us look at what happens if you just consider linear analysis first in which case we directly move only here no of movement in the axial direction, we all know or most of you may remember that is W L that is a transverse deformation at the loaded end that is given by f l q, if this is the force is F, length is L and young's modulus E and second moment of area I if we have W L is given by f l q by 3 E l now, it is easy to non dimentionalize this you just bring W L and below that you put this L.

So, that becomes L square now. So, we get F L square by E I that has to be non-dimensional because on the left hand side we have non-dimensional number, that is we are normalizing transverse displacement with the length of the beam W L that is this deformation are displacement we are non demensionalzing using the length of the beam itself, this is non-dimensional. If these non-dimensional this also has to be non-dimensional let us verify that. So, force is Newton L square that will be m square meter square and then E young's modulus then be Newton per meter square and then second moment of are that will be meter to the forth. So, over all what do we get this Newton newton canceled meter minus 2 M 4 that becomes M 2 M power 2 that becomes again non-dimensional.

So, f n square by E L is non-dimensional. So, this is no big deal that we can easily see that when you have elastic response in this case it is transverse displacement that can be easily non demensionalized sometime call it normalization also, so there is really nothing unusual about this. The fact that this non-dimensionality holds even in the case of non-linear deformation is interesting and that is what we will consider here, let us say large displacements analysis let us revisit it very quickly.

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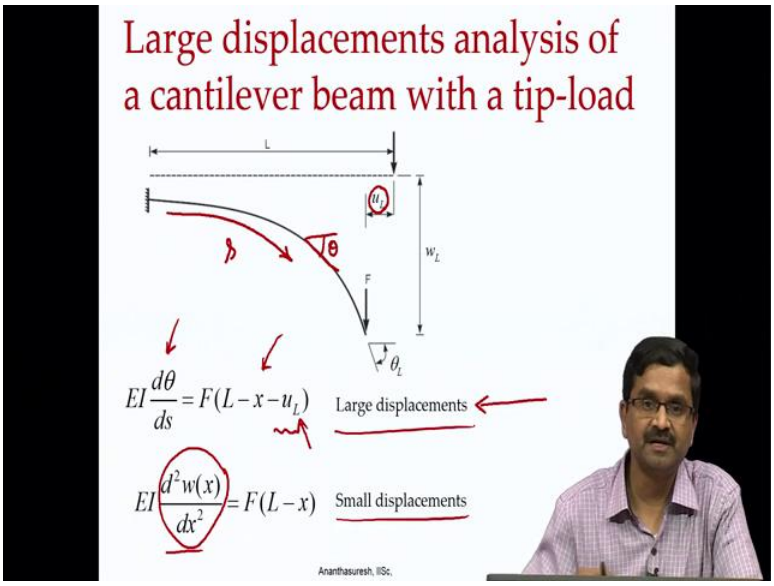

So, what will be the equation in the case of a cantilever beam which we discussed in lecture number 13 is that we take the curvature as it is greater change of slope along the arc length of the beam that is this is if we recall our arc length s. So, d theta by d s and s is slope any were. So, that is slope that is theta, so d theta by d s is there on the right hand side also we take in to the fact that if there is an axial displacement of the free tip is denoted by (Refer Time: 07:00) bell that has to be subtracted in writing moment a right hand sides.

So, we have when we take analysis we neglect in the denominator for curvature d theta by d s is this divided by 1 plus d w by d x square whole thing rise to 3 over 2. Now, we say the beam is not bending much meaning d w by d x is not a whole large. So, it is square will be negligible compare to 1 and that is what we get. So, whatever was in the denominator we I case it zoomed out show, we just say that what the denominator we neglected.

So, that becomes a linear and we also neglect this u L here that is a small displacement large displacement, it is non-linear equation were gone through earlier, so we quickly go to what we did.

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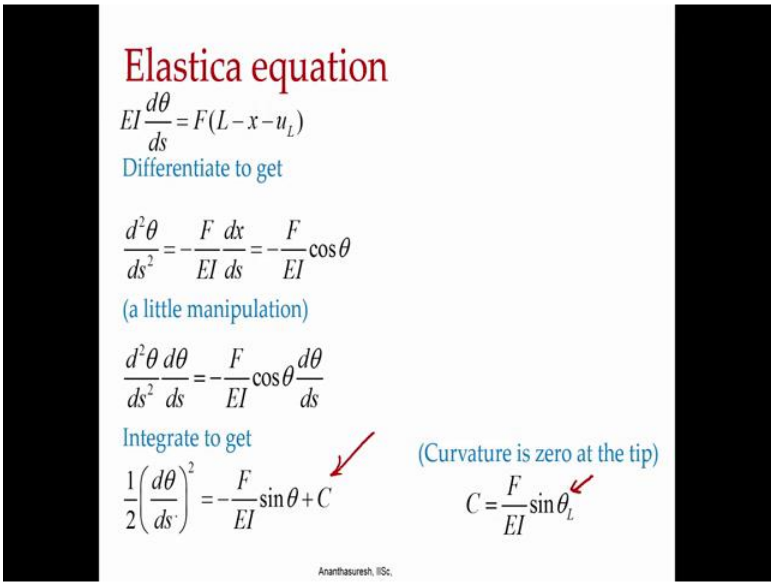

We start with this what is called in elastica equation and differentiated and say that this is similar to Kirchhoff pendulum with large oscillations whatever equation we get that is, what we have for the cantilever beam we had discussed all this at length that called undulating elastica. So, we differentiated, little manipulation and integrate get this constant we evaluate by putting this slope at the loaded end that is theta L here.

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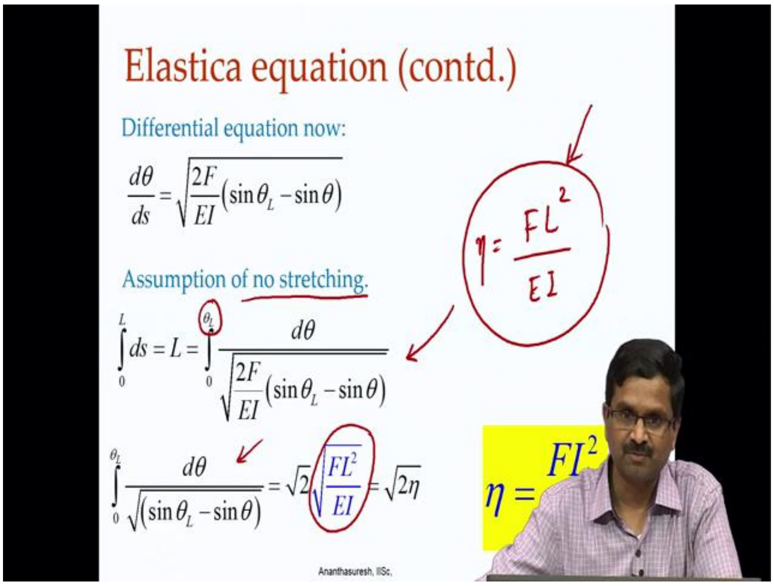

So, to evaluate this constant C we substitute this at s equal to L which case the curvature becomes 0 that is we can get the C theta L and we know the equation, we also assume that there is no stretching of the neutral claim and neutral axis of the beam and that gives us the equation, what was the differentiate equation? Now turned to be equation in just 1 variable which is theta L, may integrate theta L will you the 1 just scalar equation of length equation. Now we notice here what we had actually put at that time also in blue color which is what we called eta is eta that we have here is F L square by E I is exactly the 1 that we had when we did the small displacement analysis here, for determining theta L this is what again comes out because on the left hand side here is all non-dimensional because will angles are there right hand side should be non-dimensional which also already verified that this is non-dimensional. That is we put it in big bold letters at that time now we did change variables and talked about the elliptic integrals this equation.

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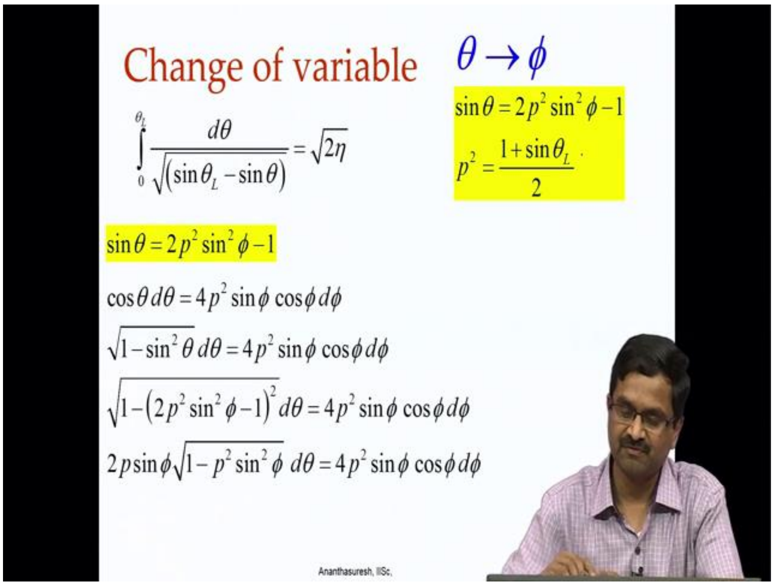

Now, with a change of variables that are given in these slides again I am going through quickly because we had done that in detail some time ago in this course and with all of these things we had finally; gotten these things to using these elliptic integrals of first kind and second kind complete and incomplete.

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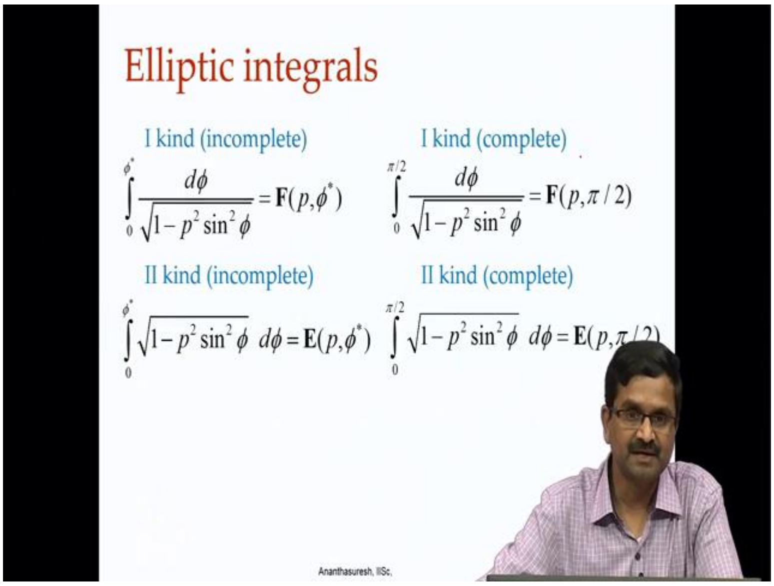

We had gotten the solution finally, that W L by L.

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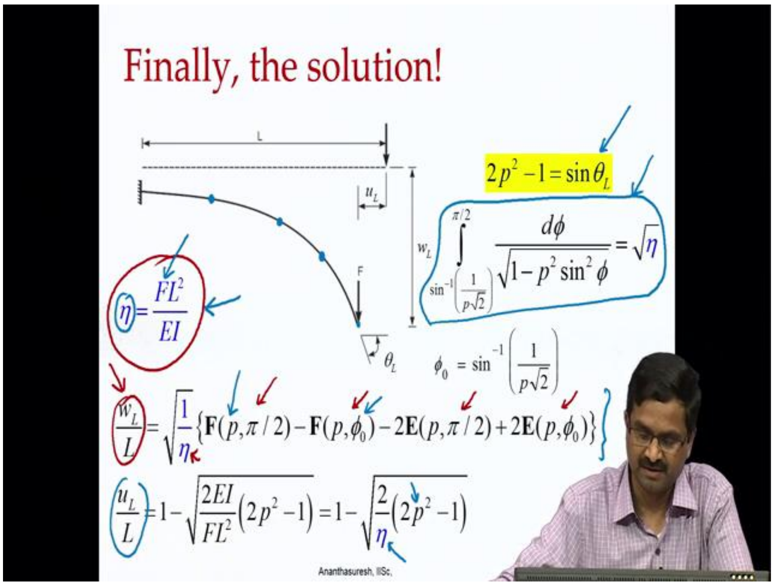

So, that is we are looking at this W L by L is given in terms of the elliptic integrals capital F first kind capital E is second kind, complete when the angle is pi by 2 like this and incomplete when it is some phi naught which is theta L equivalent with the change of variables and then second kind complete and incomplete. If we notice in this W L by L which is non-dimensional, so clearly what is on the right hand side is also non-dimensional for what does it depend on we see eta or familiar F L square by E I that is there and there elliptic integrals, but then we also see this p. So, we see this p and this phi naught we see this phi naught and again the same p and phi naught. So, what do they depend on?

So, phi naught at least in our rotation what was correspond in to this theta L with the change of variables that depends on theta L, what about p? The p again depends on theta L. So, it is all the same variable different forms and also we have this equation it that was our length equation let me put a box around it this equation is what we use to compute p. So, if we know value of eta we compute p. So, in other words what we find is that this entire non-linear equation for different values of p which again depends on eta it is basically function of eta itself.

So, this W L by L even in the case of non-linear large displacement situations also it is a function of eta it is a non-linear function, because as I change in eta there is F where I go from small value of F to large value of F I am going to get different displacement of this is non-linear, but even that non-linear function is a function of a single variable eta which contains the force, the size, that is length L middle property E and cross section property which is second movement of area, all the 4 things are in one quantity and that quantity is what determines a non-dimensional displacement and the same thing is true for the axial displacement also.

So, axial displacement if we see again it depends on eta and then p, p interned depends on eta. So, everything is a non function of eta. So, what we find is that non-dimensional transverse displacement or axial displacements are functions of eta itself that will be true for any point. Now we are looking at the loaded point, but instead if I take this point at this point at this point 1 can verify in this analytical solution that all of those are non-linear functions of eta itself, eta has force as I vary force that is going to change, but still it depends only and the combined value of this F L square and it by E I.

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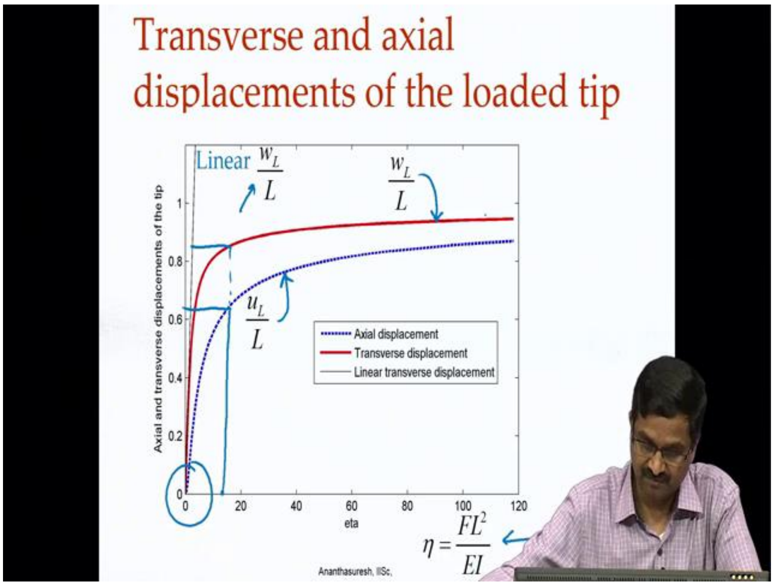

So, we can plot it now. So, we actually did that at the time itself we have eta shown here and we are showing this W by L this is the red curve, that is transverse displacement for the loaded tip and then axial displacement normalized or non-dimensional with the respect to L and then linear 1 is also shown here. As we can see linear 1 is tangential to non-linear as we expect we do not see that axial displacement for that we have to zoom in over here and then will see that that is actually because has to be tangential like this with x axis because small displacement case u L is 0.

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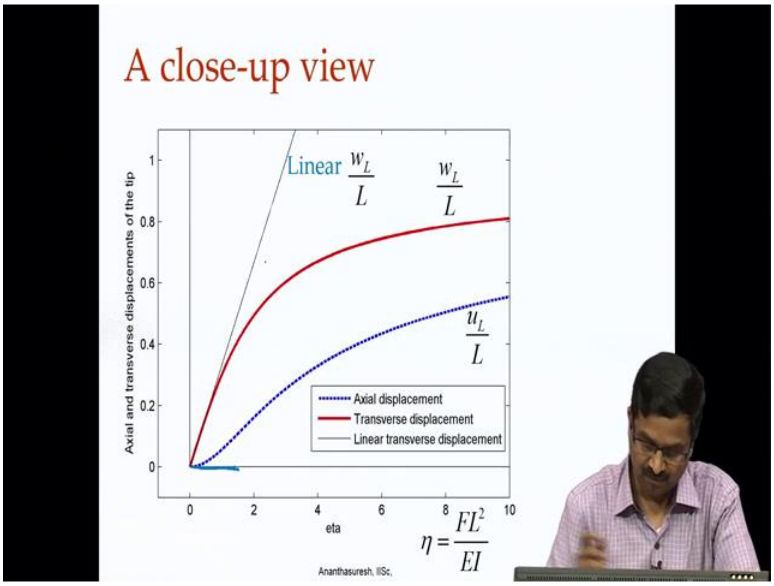

So, it should be a line like this and that is what is it is again tangential to the non-linear case just other transverse 1 is. So, we find that once we have a curve like this. So, whenever we have eta value for a beam no matter of what the force is, what the size is, what the length is, what young's modulus, what second moment of area cross section is, if I know the value of eta let us say I have a value of eta something like that I can immediately find out what this u L is and what this w L is non-dimensional form and get it.

So, once you have these curves no matter of what beam you take, what cantilever beam you take, transverse force applied at the at the free end and not the fixed end nor the loaded end but free end, whatever may be the case meaning that whatever is the length, whatever is the force, whatever is the material property E, whatever the cross section in fact, shape of cross section on to does not matter because I is what matters when you take that when you have this curves we will find that the displacement is readily available is large displacement, you can keep on increasing eta whatever we have this and we can find it there is no need to calculate again.

So, these things can now be called maps, in a way there Elastro static map or we can say Keneto elastic because there is a Keneto matrices involved here it on a displacement keneto elastic map of the deformation and with this non-dimensional factor.

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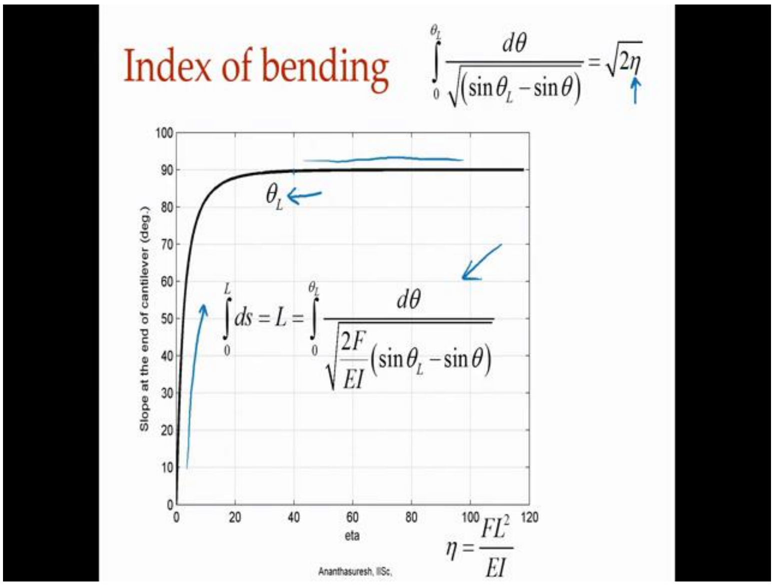

Now, what happens if you plot theta L that is the slope at the loaded end verses eta, we see that at eta increases the slope is also increasing it increases very rapidly in the beginning after about 40.

So, it is stabilizes to something is almost 90 degrees now very close approaches symbolically, but we can actually say if for a beam eta value is large then is bending more because is the cantilever beam here we talk we about. So, as eta increases we will have more and more bending unit. So, we can think of this theta L as well as eta is an index of bending eta smaller means that it is not bending much eta larger beam side it is bending a lot more, and you can see that eta can go from 0 to infinity of course, when you approach forty itself we are very closer to 90 degrees that is the beam has actually become almost vertical like that almost never be completely vertical will approach there.

So, we can see how a eta indicates the index of bending and again this is not whether the beam is short or longer made of flexible material verses stiff material, it is just that a combined effect of 4 things force length material property e and a cross section that is second moment of area all together put in. So, that all again goes back to this thing being dependent on just eta and it is the length not changing as the beam bends is assumption made in doing this.

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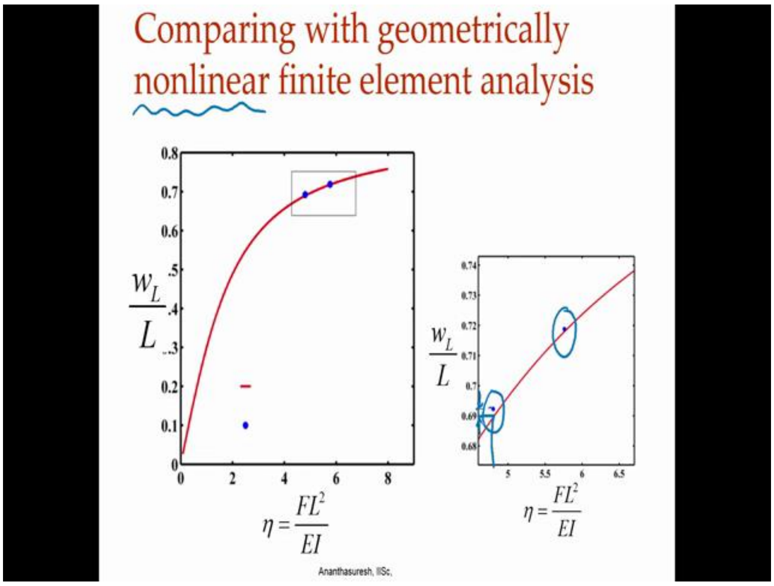

Now, if we were to do finite element analysis again non-linear finite element analysis we find that at least in this case again if I put W L by L verses eta these blue things are done with finite element analysis. We see that yes there lying on the curve, but if we zoom in a little bit we find that there is a discrepancy it is a small there not exact lying on it.

So, analytical solution is correct because you have done that if the assumption made that it does not made stretch, but here when that there it does happen there is a little bit stretching even for a cantilever which finite element analysis capture. So, we realize the limitation of the assumption, but to look at this is 0.69.7 if I were to take a eta value some were here instead of this being that will be that. So, there is an error who we know what this error is because the axial a stretching.

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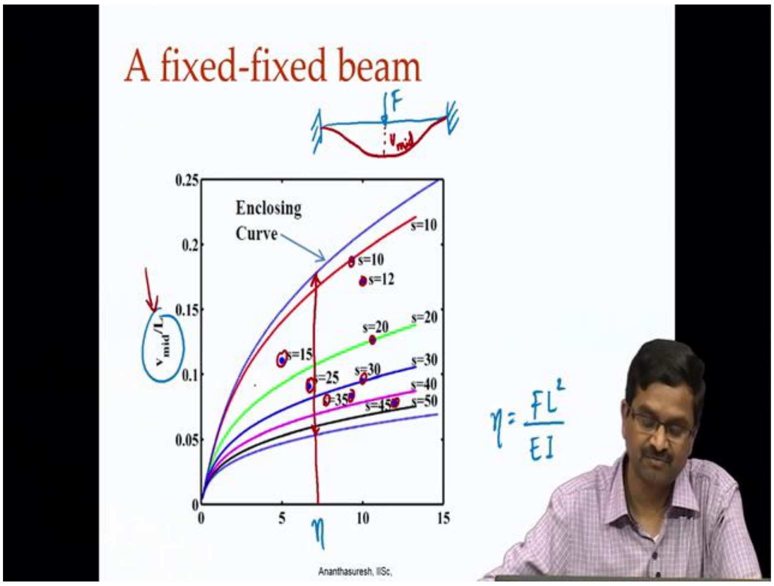

Now, instead of cantilever beam now let us move on to a fixed fixed beam when you take a fixed fixed beam we find that if there is this eta again theta is same F L square by E I, now if we what to do midpoint displacement.

So, have the fixed fixed beam fixed at both ends and the force is applied and the midpoint transverse force now, we have to see what happens to displacement when it deforms like this what is this? That is let us call it we made here midpoint as indicated again non demensionalized or normalized or non demensionalized beam it by L how does it go it looks like for a particular eta. So, we do not get the same value. In fact, there is a range. So, when you put all these dots here will come back to what this s is we put all this dots there not lying on a single curve like it was a apparently for on cantilever beam. In fact, cantilever there was a difference, but here we get a whole range of values.

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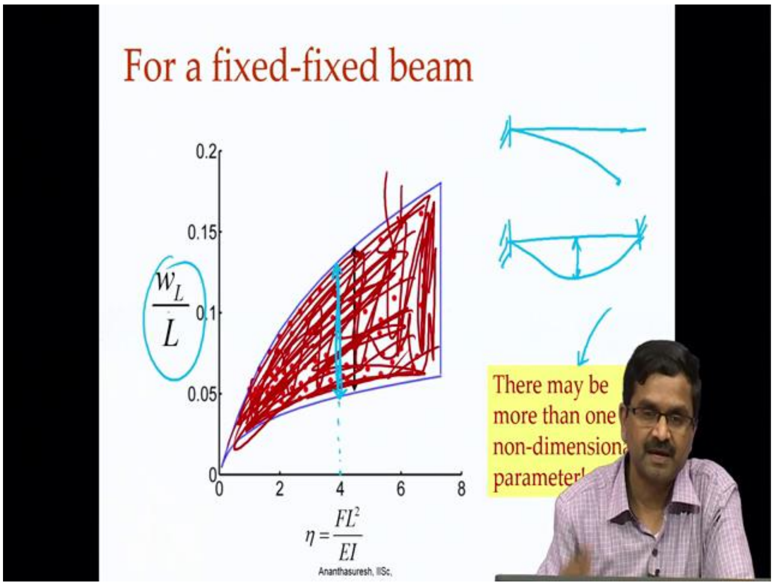

So, we get a number of points like this that is each point here refers to a finite element point. So, every 1 of them we have indicated now it looks like we get an entire space here. So, it is not a single curve, but entire space. Now, this can be really called a Keneto Elastro a static map. So, this whole thing it can be anything here. So, we can just really fill that an. Now, what happens is that if I take a particular value let us I take eta equal to 4 then this entire range of values is non-dimensional displacement. So, what does this mean? So, that there may be more than 1 non-dimensional parameter here for us to fix this problem, earlier we said that this non-dimensional display for a cantilever is simply a function of eta, but now giving eta is not enough because there is a range here.

So, is if see the difference for a cantilever beam we said that this displacement axial are transverse they are simply functions of eta, but here for a fixed fixed beam that is not true for a fixed fixed beam what is happening is if we deforms like this we are saying that, this displacement is not a function of eta alone because for the same value of eta a several values of non-dimensional cantilever beside little bit, but here it is more pronounced.

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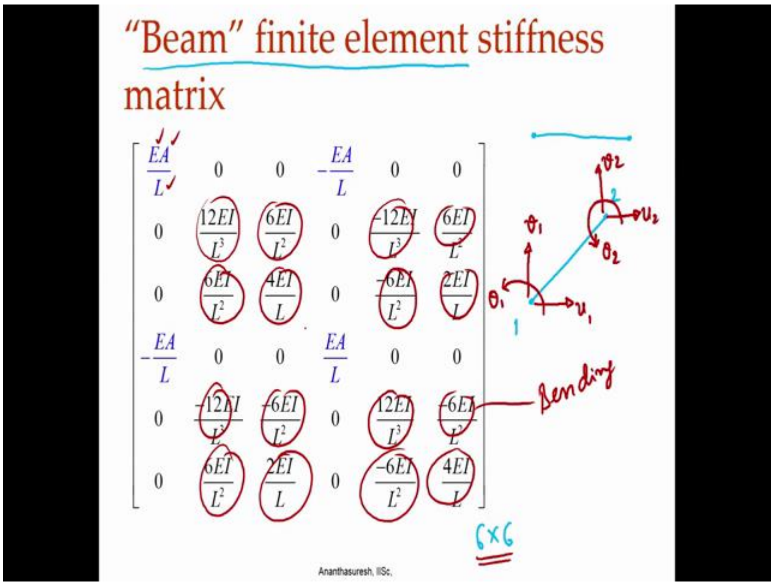

So, what is wrong here is the question that we ask and in order to see this we want to look at, we can do with a analytical 1, but that gets little involved and we have already done that cantilever beam to show that the function itself is dependent on eta as a whole were as for a fixed fixed beam that does not in to work is something wrong here. If you look at analytical solution fixed fixed beam we would find that it is still a function of eta alone, but we do not see it when we do finite element analysis over here and try to or even experiment whichever were we do we get n number of points and getting the entire a map here.

So, in order to resolve this instead of doping analytically let us do it numerically by using beam finite element. So, what we are showing here is the 6 by 6 stiffness matrixes beam finite elements. So, if I have a beam element in 2D, we have beam element it can be even in client. So, let us actually take it in client this is the node 1, this is node 2 we will have 6 degrees of freedom. So, there will be displacement we can call it u 1 and v 1 displacement x direction y direction and the rotation here that is call it is theta 1 and then this will be u 2 and v 2 of the second node we take it 2 node beam element and then here we have theta 2 rotation. So, we have 6 degrees of freedom that is so we have 6 by 6 matrix here.

The symbols here of familiar young's modulus area cross section this is young's modulus area cross section, length of the beam, I put this in blue to indicate that that accounts for the axial stiffness E A by L or A E by L and remaining things that is the things that I will circle this one, this one and all of the this, 4 by 4 that we have that corresponds to the bending here, this 4 by 4 and this 2 by 2 we have to put it blue this corresponds to bending. So, all of this is bending, since it is bending what we see is I second movement of area rather than area of cross section.

So, we have both axial and bending taken into account because we believe that it reflects reality better than the analytical solution because analytical solution for the E is of solving we assume that the length of the deformed beam does not change. Now, we do not make that assumption we take this.

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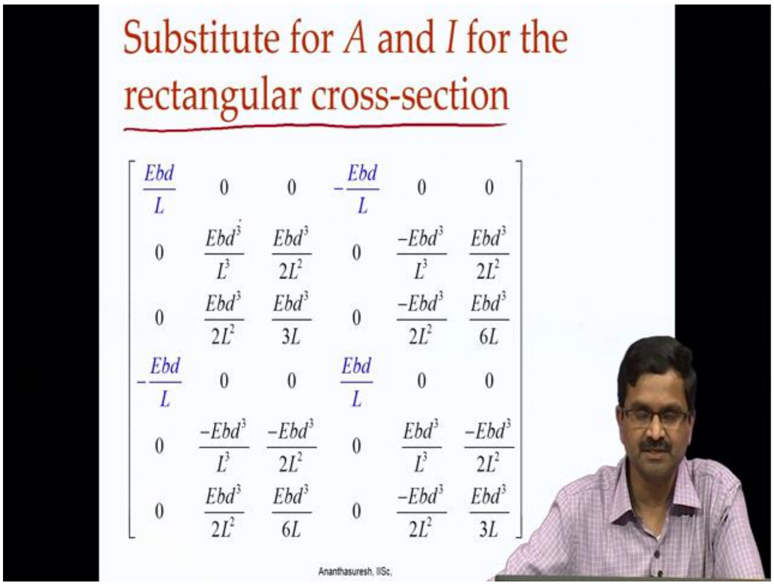

So, once we have a this stiffness matrix for a beam element or a segment, what we do is we substitute for A and I assuming rectangular cross section, where rectangular cross section we just substitute Ebd and I we have b d q by 12 and 12 was there got cancelled and we get this.

So, we taken this 6 by 6 matrix just substitute for area of cross section and second moment of area that is A and I, then we take something out we factor this out because we see something E b d q by l square w take it out leaving something in this matrix.

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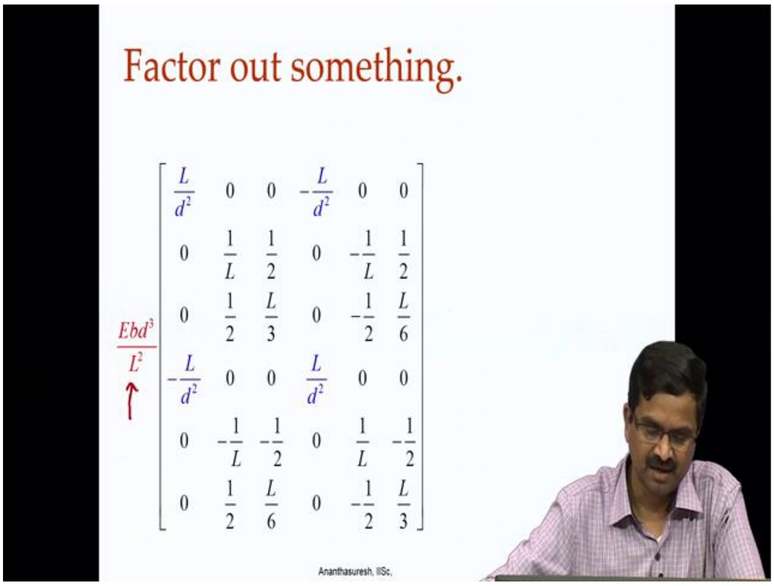

And then we also define something all slenderness ratios, slenders ratio is defined as L by d length by depth. So, d is the depth of the cross section which is in the plane of the deformation of the beam we defined that.

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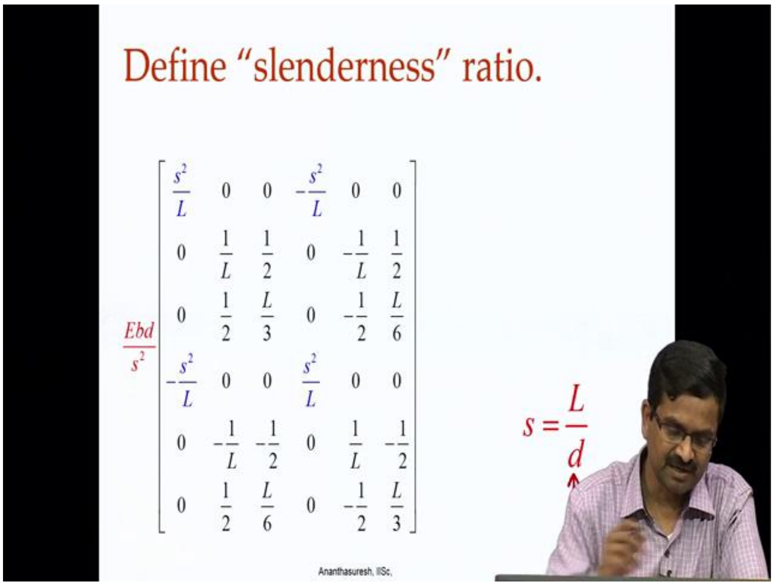

So, from previous slide to this all we did was we just L d square we multiplied by L that becomes s square divided by L, so we get s square by L here define this slenderness ratio.

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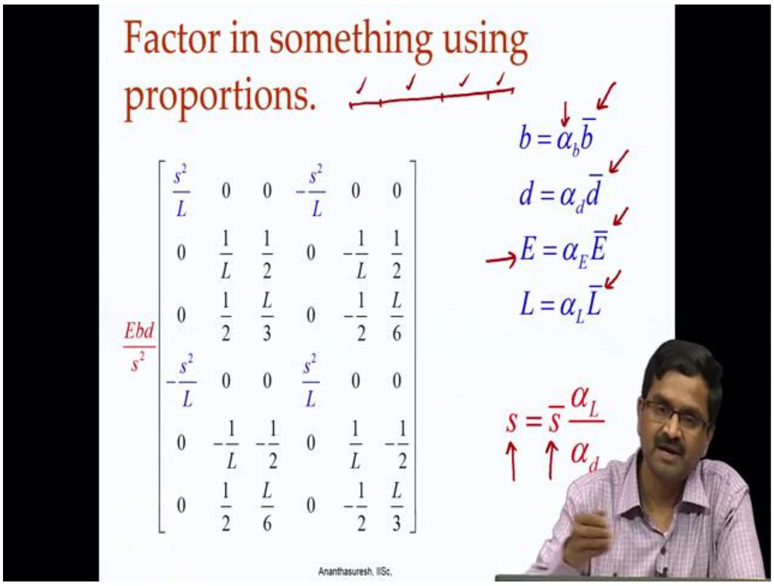

And then we factor this proportion so what we do is we have this now we just say the beam if there are let us say 1 long beam we break it up in to smaller beams. So, we say there is some average breadth, average depth, average young's modulus, average length different beam segments were long beam I can break it up in to smaller piece longer shorter whatever. I take average take proportion of each of these beam elements I have this 1 will have 6 by 6 were take this one, this one and this one, I just make this proportions b alpha b times b bar and same thing with d and E as if it is going to be made of different material properties, we do not do that but in general it could be made of different materials.

So, we have put that also in an average sense and the length itself, length can be different for different beam segments will we do this. First of all this slenderness ratio also will have this s bar and then this alpha L and alpha d or the proportionality things for different or multiple beam segments.

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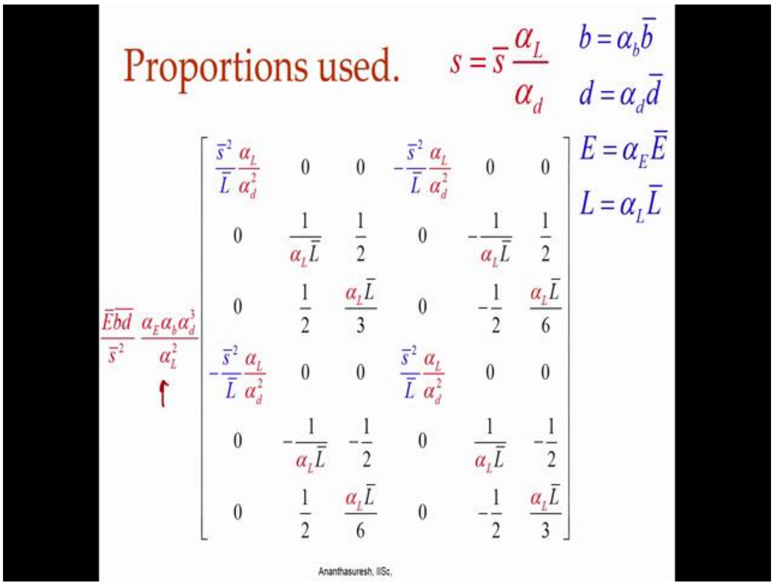

If we do all this what we get something like this. Now, we have all the proportionalities introduced here and inside as well and then we look at the force displacement relationship.

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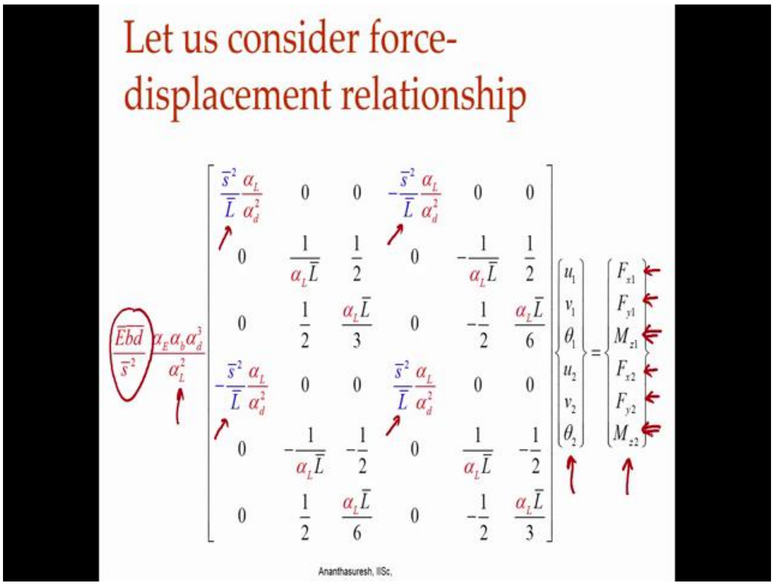

So, we have this stiffness matrix now which we have done all this things non dimensionalize, if we see inside except the average L that is there over here every were else it is all proportion which have just numbers and the here also we have this, and this inside all average values and these are the 6 degrees of freedom and corresponding forces or moments. Corresponding to translations we have forces that is this, this, this and this and corresponding to the rotations we have moments. We will see how this gives us a way of finding this non-dimensionality for general boundary equations of the beam and not just for cantilever we will pass here and continue in the next lecture.

Thank you.