Compliant Mechanisms: Principles and Design
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Lecture - 30
Continuum Element-Based Topology Optimization of Compliant Mechanisms

Hello, so far this week we have been discussing how to pose problem to solve compliant mechanism topologies for given specifications without assuming much about what is going to come. So, today we will continue that especially the previous lecture where we were discussing the elastic continuum topologies we had derivation that we thought that, if you had tried before this lecture that would be that would be nice, but let us see when if you did not we will just look at to complete that derivation and look at sensitivity analysis as well as optometric method to solve compliant mechanism topologies.

Looking at this today, we will discuss some implementation issues as well as the sensitivity analysis. So, we have the implementation issues we will look at that as well as sensitivity analysis that relates to this design of elastic continua are a compliant topologies.

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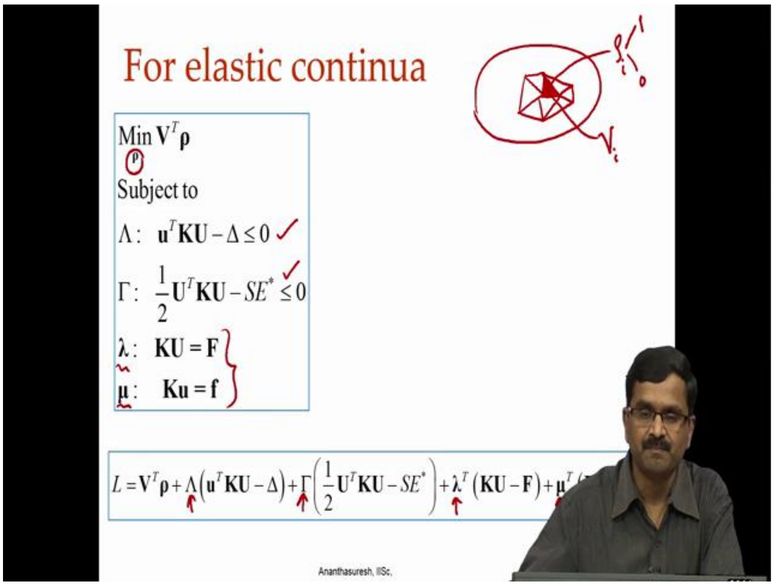

For elastic continua we pose the problem like this, in discretized form we could have done it in the variation calculus form that would require you to right all of that equations, whereas; now it is a just simple n variable optimization plot. So, we will look at this discrete problem and complete the derivation we are started in the last lecture. So, first thing we do is write the lagrangian where we V transpose rho, rho is the indicator functions sometimes gone a fictitious density it is better to call it indicator functions. So, when you have a domain, when we have a mesh in it. So, we can mesh like that, a lot elements like finite element mesh I am showing a rectangle, but in it have a element if I say rho i let us say this is the i-th element, that will have some volume three dimensional species, whether the material with there are not is indicated by rho i when it is 1 then there is material then it is 0 there is no material.

So, that that is how this algorithm works. So, it is going to decide rho variables to minimize the volume of material because, this v this will have a volume of V i, i-th element. So, that times indicate a function gives in a total volume what to minimize that volume subject to the deflection constraint and then a strain energy constraint and you have two equilibrium equations, one for the applied loads one for the unit virtual loads or what we called unit double load and corresponding lagrange multiplier vectors are also there. So, we write the lagrangian as shown in the bottom of this screened here. So, with in objective function there is a multiplier associated with the deflection constraint, multiplier associated with the strain energy constraint and then we also include this lambdas and mu vectors as lagrange multipliers corresponding into the equilibrium equations and to solve them we get a joint equations as we will see and as we had also see in the context of beam based or frame topology optimization.

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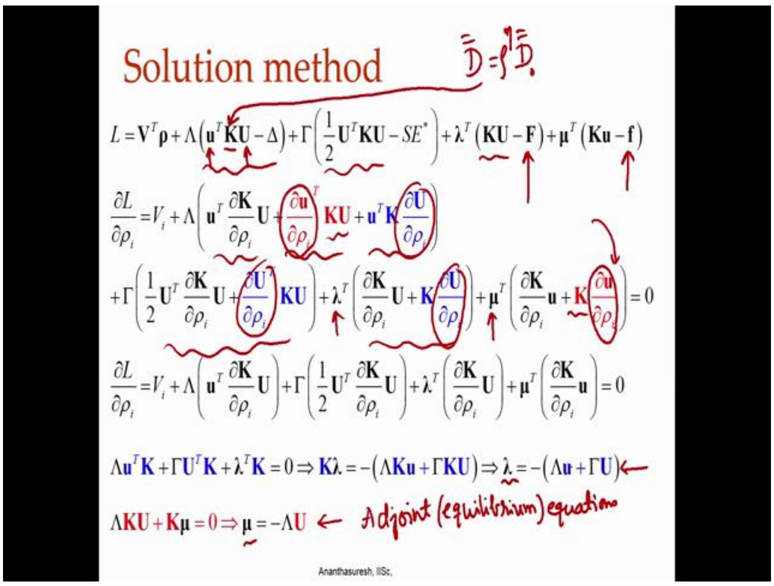

We take a lagrangian the solution method goes like this, we say that that lagrangians derivative with respect to each rho i, rho i where i goes from 1 2 3 up to n elements that you have in our domain. So, that should be equal to 0 this is standard are straight forward derivatives noting that this quantity K is the one that is going to have rho directly in it, because that is stiffness matrix that depends on the material property in this case (Refer Time: 04:29) ratio which we say is if you recall we had said that the D-matrix or every

element we have D 0 times rho.

If rho equals to 0 is as if it is made of a material that has very (Refer Time: 04:47) modules. So, that is not being there; and were rho equal to 1; that means, the materials properties that is D 0 directly to the stress strain matrix. So, we have this D equal to rho times D naught. And we also said that rho can have an exponent eta to push them to 0 or 1 it is small like a penalty parameter, because that is were rhos are and rho directly the D comes into stiffness matrix here, and indirectly it will be there in the applied load that causes the real displacement u capital U and then small u where the unit virtual load causes the displacements, they are all going to be dependent on rho directly or indirectly they stand that we write this equation. For example, this one term leads to three terms when you take the derivative, because u k small u capital K and big U upper case u all dependent on rho.

So, we have three things likewise, this will have also two terms because, u and k are there and this has two terms here assuming that this F and this f do not depend on rho. They can also depend on rho when the load is something we said body force meaning that, it will act and it will point that domain in which case this capital F and small f actually small f never will be a function of the design variable because, when you got unit virtual load that is we are applying unit virtual load at the point and the direction where we want the displacement. So, this will never be dependent on rho, but this applied load f can depend on load in which case there will be additional term over here, Ok.

So, these very straight forward, if we see we have put a few terms in red color and few in blue color two in red color three in blue, why? That reason is that we have this dou u by dou rho i and the dou u by dou rho i here, which we do not know when we solve the problem that is equilibrium equations, we would know capital U and small u for some given rho values the (Refer Time: 07:14) of optimization you start with the initial guess and then keep on refining them towards optimum, but we would no rhos in iterative process and at a given times. So, you can compute capital U and small u, but we do not know derivative with respective each design variable rho i where i goes from 1 to n.

We do not want to compute that. So, what we do is, what multiplies them including mu here and capital lambda there we make it 0 likewise we do not want to compute this also. We say that, we will find our this lambda and mu such that we do not have to compute this sensitivities have this state variables capital U and small u. So, when we remove them from here we have only this of course, we have lambda and mu which we need to find for that we get these adjoint equations. So, these two as we had discussed in the case of beams also they are called adjoint we can also add adjoint equilibrium equations.

So, which will helps us solve for lambda mu they happen to have similar structure to the equation equilibrium equation as we called them adjoint they adjoint them. So, lambda and mu we can get from these, we are simply avoiding finding the derivatives of state variables here that is adjoint method.

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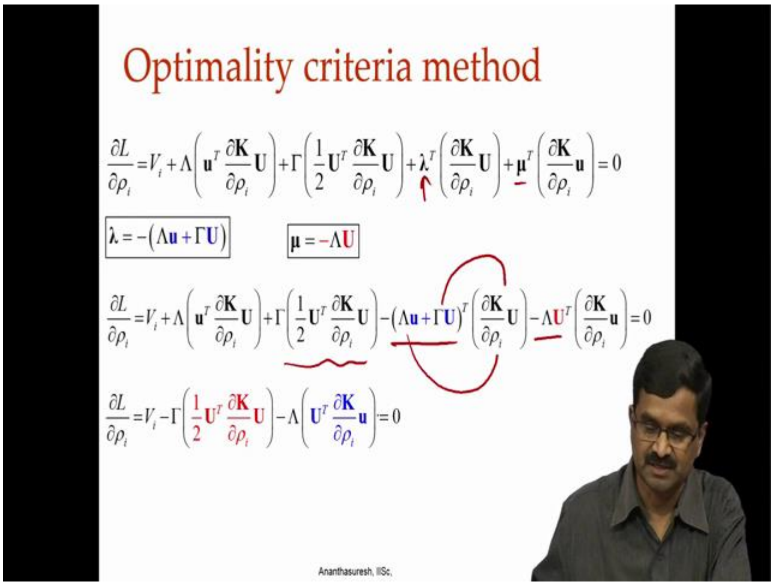

We have these, now we can substitute into this long expression we substitute for lambda and mu, once we do that we get this plainly substituting wherever lambda is I have this of course, the minus sign and wherever mu is lambda u transpose we have in this process we have also use the symmetry of stiffness matrix k, if you look at it closely you will that you would need to use that. So, what we get when we strike after (Refer Time: 09:39) because, lambda times U transpose dou K by dou rho i u is also here. So, if I multiply this and this, I get that full one this is there that gets canceled and then here also we have U transpose dou k by dou rho i u and that will be there when I multiply this with this, but this has a half that has minus overall become minus half and then this other one that we have lambda U transpose that also we can get. We get something that looks like this now, v i minus gamma times something minus lambda times something.

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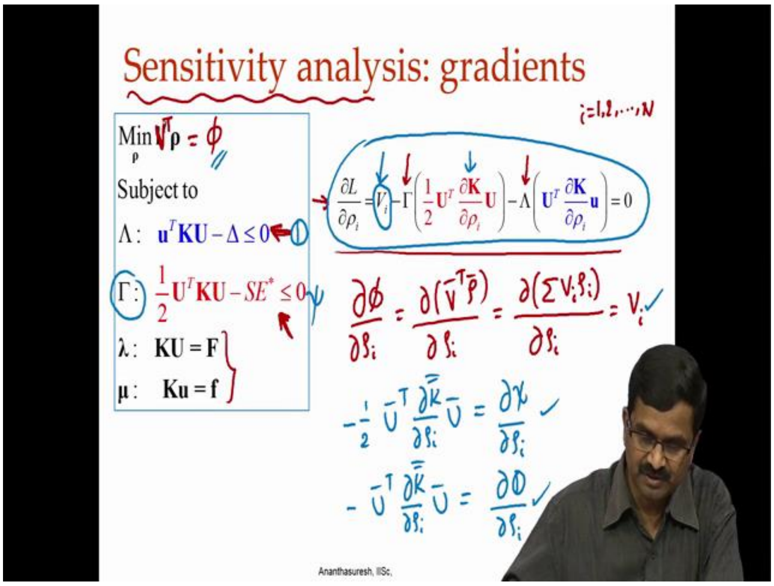

So, this particular one actually enables us to do sensitivity analyze at gradient by that what we mean is that when we have the problem and do the adjoint equations finally, this is what will be true for all i i equal to 1, 2 how many ever elements you have. So, you have to find rhos in such a way that this equations is satisfies for all i, i equal to 1, 2 N or unknowns are basically rhos, rho 1 rho 2 rho n and we have that many equations we can solve, but these are non-linear equations and we also have the unknowns that is this gamma and lambda, we cannot solve in one step we are do it iteratively we gets a non-linear programming problem.

So, usually you have this is a non-linear constraint of course, these are linear and these also linear in design variables, but because of non-linear constraint that we have here this and this a non-linear programming problem and you have do it iteratively we cannot do it in one step, even if everything is linear you cannot do it once that is why you have the simplest method, right for linear programming now it is a non-linear programming problem.

If you want to solve it now, we know the necessary conditions that is this equation we have two ways of doing it, one is called the optimality criteria method which we will discuss and it shows it is implementation today. The other is just use any non-linear programming algorithm that is called mathematical programming or non-linear programming the number of algorithms in the literature (Refer Time: 11:59) over a number of years and that implemented in several software programs and whatever is accessibility you can use, but when you do that you would need the sensitivities that is what we call sensitivity analysis which simply means gradients.

So, if have if I call this the objective function, let us say f you already have let we just use some phi or objective function then; when you use a mathematical programming algorithm then the algorithm would aspect you to give gradients if you do not give it is going to find it using finite difference method which will be computationally inefficient because, you have to port of each variable and find the derivative right instead of that. So, if I want to find dou phi by dou rho i in this case is linear it is actually quite easy because, it is actually not v here i here l here it should be v transpose that is a volume and if I want to do dou phi by dou rho i, because that is dou v vector transpose rho vector divided by dou rho i. So, in this case it is easy because, v i rho i is a summation. So, if I do this it is a volume of that i-th element this indicator functional fictitious density. So, it is respect to rho i what I get is v i, that is exactly what we have here.

So, we have that term that does not have lagrange multipliers in this equation, this is the derivative of objective function here it is easy for us to verify that is what we get, but this is true no matter what objective function you have here, it can be very complicated function it can be let us say integrated stress measure over the entire volume should be a scalar we want to have scalar optimization not a vector optimization scalar any scalar you have their when you do this procedure the way we have done the term that does not have lagrange multipliers will be the gradient of the objective function that is if I if you call it phi dou v by dou rho i.

Similarly, what multiplies the gamma here? So, gamma here is gamma multiplier there right. So, what is multiplying gamma minus we have to include the sign also minus half

U transpose dou K by dou rho i U this will be if I call this constraint, if I let us call this constraint psi then that is dou psi dou psi by dou rho i. So, the term that multiples gamma the multiplier corresponding constraint derivative is what we have, likewise; what we have here, minus U transpose dou K by dou rho i U is equal to if I call this constraint let us say we have use lot of them let us say I lambda mu psi let me call this you do not use theta for anything usually let us call it theta this is some scalar. So, this will be dou theta by dou rho i. So, our necessary condition here this one actually gives you the sensitivity or gradients of the objective function and all constraints.

So, this is quite easy because dou k by dou rho i is some the easily calculated, because you are simplistic stiffness quantum frame work we can easily take the derivative that and then after word system multiplication post-multiplication by u pre-multiplication by u transpose. So, that is the easy to do. So, it is easy to calculate these quantities now and you can calculate given to the mathematical algorithm and that will turn out the numbers give a solution that is one way, but most often is convent to use what we call optimality criteria method, there are 2 options here that is why the r is appearing.

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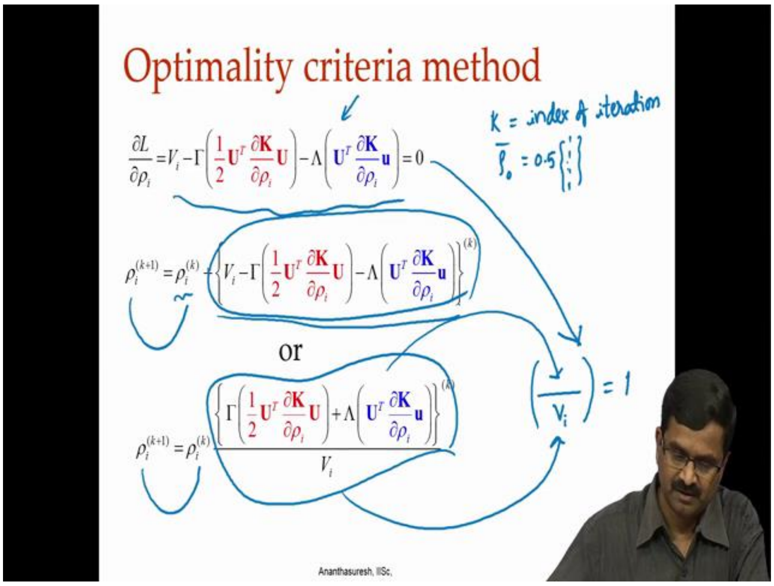

So, we start with that necessary condition that we have derived now and what we will do like this one is what should become 0 is iterative procedure we want this to be 0 and we want to find the values rho i meaning i equal to 1 n, rho 1 rho 2 up to rho n we want to find. So, what will do is we start with iteration. So, I would use k to the index of iteration.

So, this is index of iteration. So, you can start with the let us say k equal to 0 to be your starting point initially guess then you would have rho 0 as a vector most often structure optimization we start them to be uniform, uniform we do not make it one depend on how much volume we want to have we can give that let us say all of them will be 0.5 let us say 50 percent and you would make that, you will make 1 1 1 1 like that all there will ones n of them. You make the uniform next k plus first iteration what you do is you take rho i k and add then this, why do you add that like a fixed point method. So, this is the equation that we want to solve that is equation we want to make that 0.

So, what we do this we update the variables by adding to what is there k iteration when it algorithm converges that is left to the fix point method how it works let us assumed converges then where it converges this whole quantity would have been driven to zero by converges what we mean is that rho i k plus 1 and rho i k should be close to each other when will be close when what is being added is 0, we added becomes what is added become 0 that is actually our equation solving the problem. We will be we (Refer Time: 18:33) updating rho 1 rho 2 rho 2 up to rho n. So, all of that we do. This is one way and this could be slow of a adding we could also take ratio multiply.

So, when you have this equation from there we can get this in a form that something divided by something is equal to 1, that what is something, is what in numerator what is here we denominator which is simply V i, something vi equal to this whole quantity this gamma times half u transpose look at the dou rho i u minus that is actually plus where the other side gamma times transpose dou k dou rho i u right. So, now, if you take V i down stairs then that will be equal to 1. So, we can say rho i k plus 1 equal to rho i k times what is supposes to become 1 when the algorithm converges meaning again rho i k plus 1 and rho i k should become close to each other toward satisfaction that is what stoner criterion. Then this quantity that we have would have reached 1. So, we can do either addition way or multiplication way of course, there are some small adjustments we have to do if it is changed the design radically mean that you all start uniform all rhos are equals.

Now, we suddenly when you update it like this will get completely different image some elements would have disappeared meaning, they are gone to 0, some other gone to 1, some other gone between the step is drastic algorithm may have troubles converges it oscillator right, we want to do it slowly that is put a parameter that advocate I want to whatever quantum adding I will not add as much I have only 10 percent of it or I can raise it to a power and accelerate it also, I can do either way.

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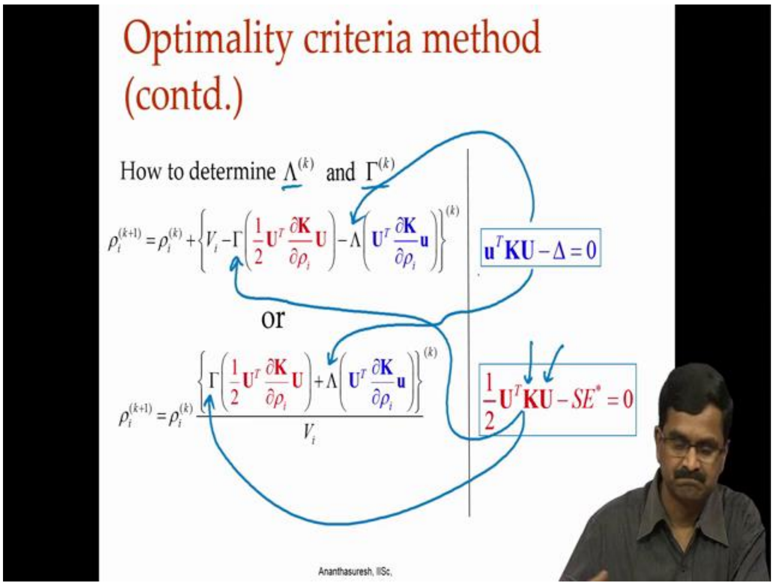

If we do all that we have these tool now, the how we determined these two lambdas and gamma, that two scalars we do not know what those are at the k-th iteration. So, that requires for update. So, further we have the equations we have the deflection equation, we have strain energy equation. We need to use this equation to find let us say this is to find lambda and this is to find gamma I am not saying multiplication addition method either way. This is to find this gamma and this is to find this lambda in either way. Whether, you additional algorithm or multiply algorithm we can use is this equation and do easier set and done because, these are non-linear equations. We already said that deflection constraint in terms of design variables as energy constraint is in terms design variables are not linear equation and non-linear equations it cannot become apparent here, but there are non-linear equations. Because U depends on rhos directly K depends u depends here not only k directly small u and big U lowercase u and uppercase u depends on rho. So, that non-linear equation, but you can do that using whatever numerical methods that you would want to use.

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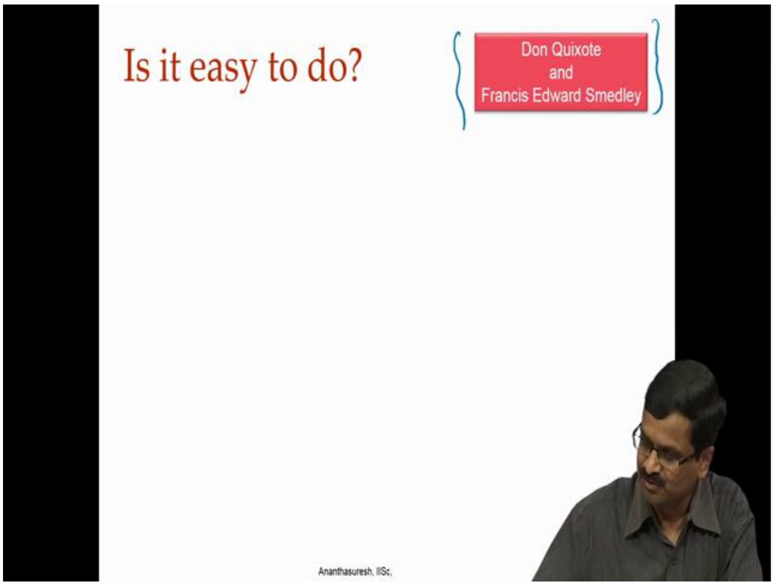

So, is it easy to do? It is actually not, but at same time the theory simple whatever we have discussed we have understood that actually implement, but whenever implementing numerical algorithm even a simple Newton Gauss algorithm if you do not understand you may not be make it converge right, but the theory is exits for Newton Gauss algorithm are algorithm like that here also we have enough theory to back is up we should confident that you can implement I understand you can do and I have put this in two names Don Quixote and Francis Edward Smedley you know that there is the reason for it because whenever we go to numerical analysis in synthesize design methods like that I would like to code that these people Don Quixote and Francis Edward Smedley had said this all is fair in love on war not making fun of that, but I am saying that I want to add design of also there, all is fair and love war design when you design when you design something whatever works we can do it.

So, we have the theory now we have this how determine this lambda and gamma in the iterative process we have to understand what we are doing and then we can accelerate it by raising to power and multiplicative factor here adds something do whatever you want and get that I am not saying whatever you want that is legal whatever is legal mathematically you can do and get the solution to prove to you in the next lecture we will actually taking implementation and show how this algorithm was this are exactly what is implemented the algorithm you will have the files yourself to play in with math lab.

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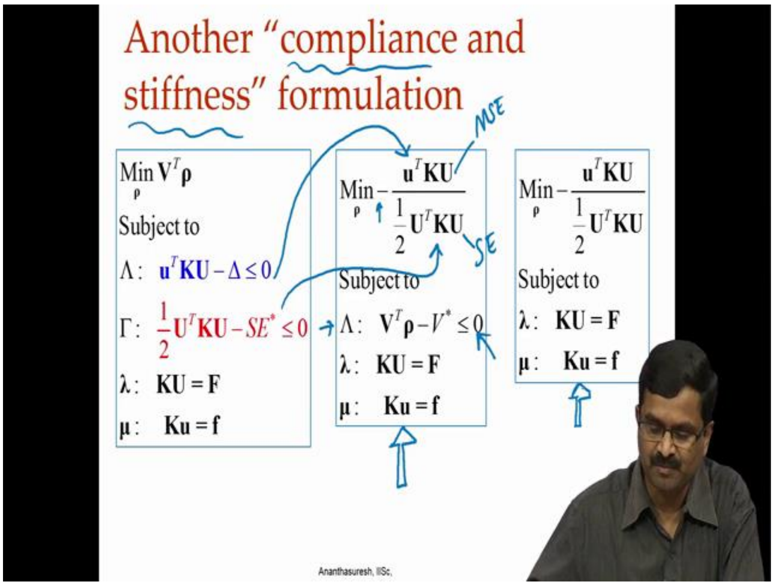

So, always remember on Don Quixote and Smedley. Before we end this let us actually look at a small change in what we are doing basically, we are looking at compliance and stiffness formulation. So, we had them as to constraints while we are minimizing volume of the material, instead these are problem that we have discuss so far, what we could do this we can instead of having two constraints which are both non-linear and the linear constrained to the objective function linear expression of objective function let us push the linear expression that is the volume to be constraint that is what we are done here, that becomes linear V transpose rho that is V i rho i sum that should be V star then upper gone the volume and then two of these that is this deflection and this strain energy there captured into one expression. This the formulation sense we will minimize negative notice that negative there negative of mutual strain energy that is what we had called mutual strain energy and then this is strain energy we had minimizing the ratio negative because we want maximize deflection. So, minus MSE we have put SE in the denominator that we minimize it you again maximizing.

So, this formulation captures making the structure or compliant mechanism to have large displacement to the point where you wanted and have strain energy as low as possible, that is how formulation. And further we can also see what happens if we remove the volume constant all together that also we tried, in implementation that I am going to show in the next lecture we are going to solve this problems. We have volume constraint and then this ratio is what will be minimizing with negative sign that mean that ratio is maximizing. So, instead what I want to show quickly here what happens if I remove volume constraint then we get what we called optimality property in the last two lectures I had mentioned now will see.

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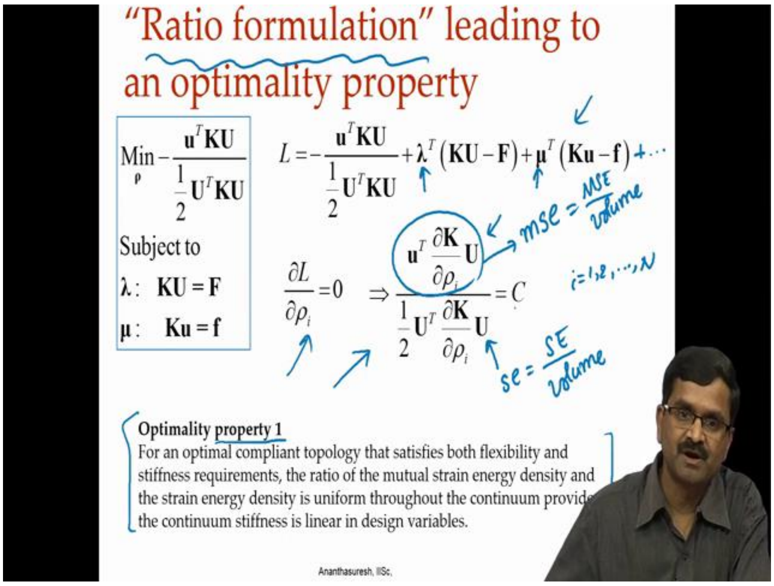

So, this property we can actually see we take up the problem there is no volume constrain now we are simply maximize in the ratio or minimize the negative of that ratio MSE by Se and write lagrangian, now this lagrangian multiplier other than what we have in the equilibrium equations they go way because you get adjoint equations and if you do this we get something like this that is the ratio of something and something is equal to constant for all i.

So, these equal to i equal to 1, 2 N if you do this constant that is what you will get you can try it you can write out the whole thing and see now, what is this quantity here it turns out to be mutual strain energy density that is MSE by volume of that element each element has a different volume right. And similarly this is strain energy density that is strain energy divided by volume of that element, what says is that ratio of these two constant throughout the optimal design that is what we can all like a property, because we have to know what it is that an optimal compliant mechanism will satisfy this ratio formulation as we can call it that gives this property we can pass it and read it understand what it means what it says it is the result what we have here.

That ratio of mutual strategy density that is a contribution of an element towards mutual strategy mutual strategy is a scalar every element contribute a little bit similar strain energy scalar every element contributes elastic structure when you applying notes there is going to be strain energy stored different points are going to have different strain energy and we are saying is that different point will also contributes towards mutual strain energy, if we take that density ratios that is going to be constant we get is similar results when you do stiff structure optimization which says that strain energy density should be uniformed through the structure and what we call fully stressed design, here also we have analog. So, we are taking stiff structure problem and adopted into compliant mechanism we says similarity here automatic property.

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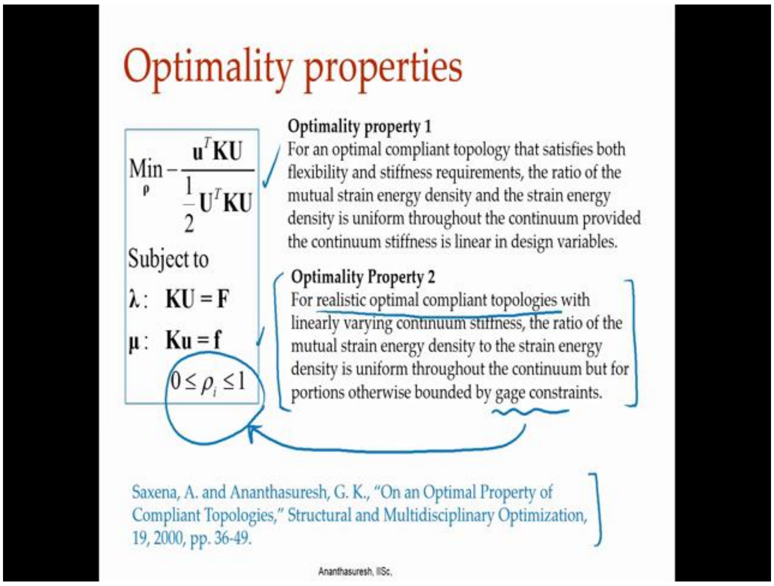

And then most often we will have this upper and lower bounds in fact here we defiantly have them rho i cannot be any number you want it has between 0 and 1. So, that constraint will changed property a little bit that is put as optimal optimality property 2 so these are called gage constraints. Gage constraints just size are gage that is design variable cannot we have lower bound upper bound here it is 0 and 1 there indicator functions are fittest densities will have this bounds. So, where the bounds are our necessary condition will not have just what we have shown here will have also plus the multiplier times those upper and lower bounds, rho as lower bound that is the constraint then multiplies associated with it and then upper bound then multi associated with it will also get added here.

So, they will again come back here in which case you becomes constant it will be straightly different that is what optimal property when you get realistic compliant topologies where some elements are disappearing meaning rho equal to 0 some of them are reaching the full value of 1 there actually present in between thinks do not have a problem, because there rho is neither 0 not 1 they will satisfy property 1 whereas; this will satisfy property 2 that is what it says, this can be verified in fact this what are the criteria method or (Refer Time: 30:07) programming methods try to achieve the details of that are in this paper you can look at that paper.

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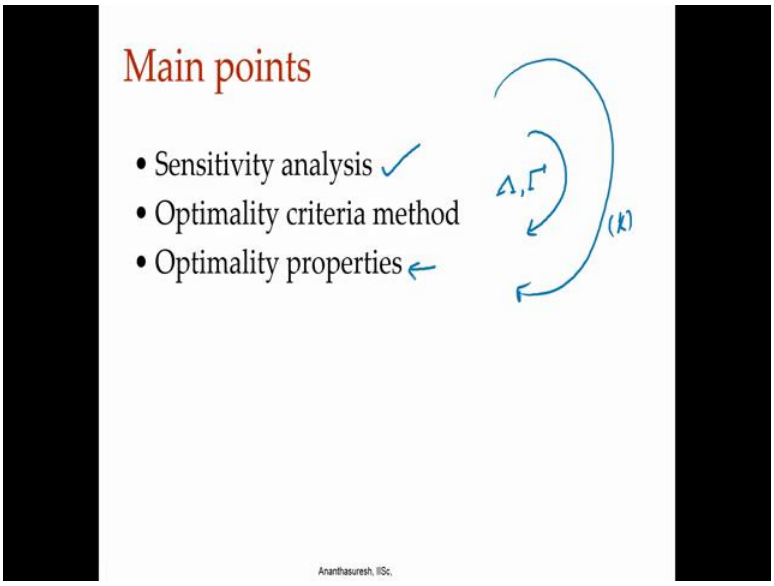

So, now let us end this lecture by capitulating the main points. So, one is that we can do sensitivity analysis very easily we are the necessary condition we showed that we get necessary condition whatever does not have the lagrange multipliers will be the gradient on objective function and now if you take multiplier like lambda here whatever multiplier lambda is a gradient of the corresponding constraint. So, we get all the analytical expression for gradients. You can use this analytical expression and go to a mathematical programming software and do it, it will be very efficient in algorithm is good most of them over a period of time would have matured themselves.

So, you can actually use it and if you want to do it your own then automatic criteria is the best way. So, we have the rho k plus 1 rho k we have update formula what we need to compute there other lambda gamma which are evaluated using the corresponding equality constant there is inequalities they become active and become equalities they scalar equations you will solve them. So, there will be outer loop and then will be inner loop in that, there will be outer loop where you are updating this k iteration number and there will be inner loop to find this lambda and gamma in our case. The inner loop we can do and you get the answer and we also discussed this optimality property very quickly and that is very important when you see a design you should this optimal or not we can do that stiff structures we can also do the compliant mechanisms.

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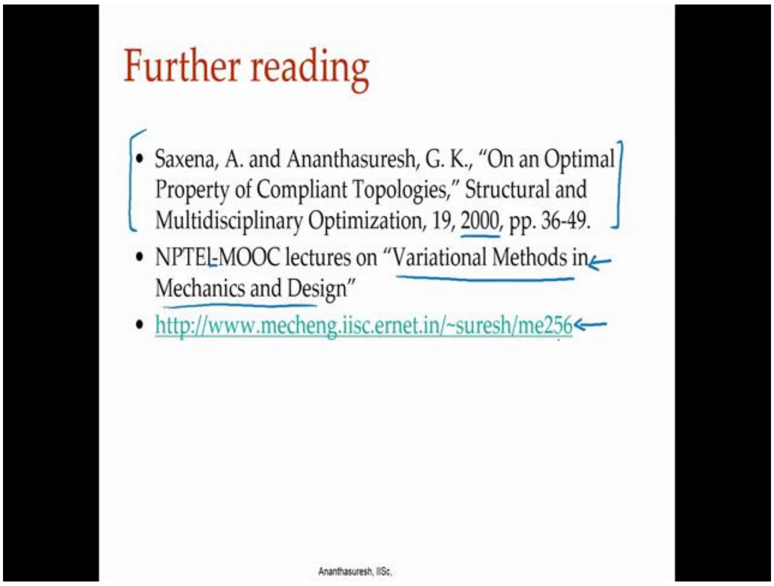

So, that is how this algorithm works now will see the implementation of this algorithm in mathlab and the further reading for this you can look at this paper, but usually research paper will not have a lot of detail although there are simple examples and then some results of the continuum things are there this paper is very old, but you can also look at some of the lectures where you can understand the optimization. So, that this property is derived yourself there is a NPTEL lecture variation methods in mechanics and design and also course that teach there are lot of material there you can understand all that then you understand what is there in the lectures of this week. Now we will look at the in the next lecture we look at the implementation of this solve some problems.

Thank you.