

Compliant Mechanisms: Principles and Design
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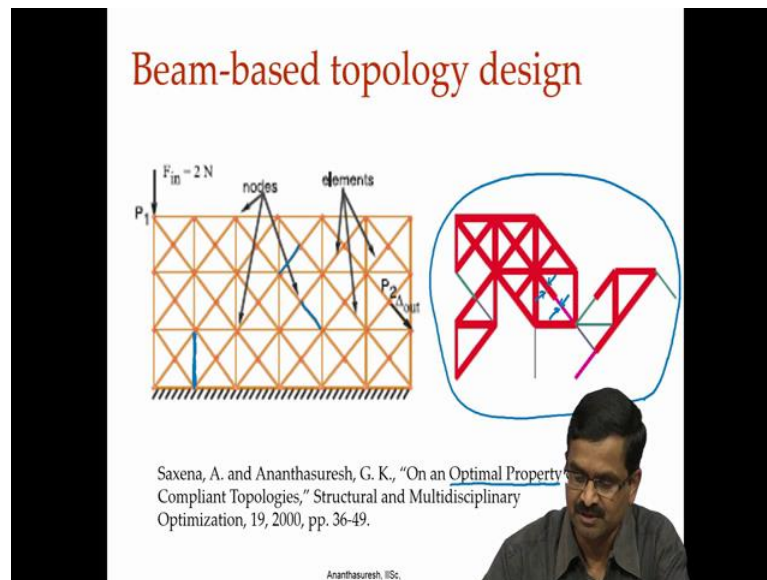
Lecture - 29
Design of elastic continua for desired deflection

Hello, in the last few lectures we have been discussing compliant mechanism synthesis using optimisation, which instantly leads to what we call topology optimisation. So, far we have discussed that in the context of truss element based ground structure as well as beam element based ground structure where we can get topologies with which we can make working designs of compliant mechanisms.

Today we will extend the same concepts based on mutual strain energy which is a consequence of principles of virtual work, we will do that this will be the third time where we visit this concept after examining it on the view point of trusses and then beams now will do with continuum elements meaning that we can basically generalise the concept where our design equations will be differential equations will go into little bit detail about sensitivity analysis as well where if you do not want use what we called optimality criteria method instead want to use mathematical programming methods you would need gradients or what we call sensitivities.

So, will discuss that also today little bit more theory and the next class we will look at some implementations and really understand how to apply what we have learnt. So, let us look at this design of elastic continua for desired deflections, so that has been our focus when we talk about compliant mechanisms we always say that we want to design a compliant mechanism for desired deflection because without deflection compliant mechanism does not make sense, it has to move to act like a mechanism.

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Just to recall last lectures content where we used a ground structure which comprises beam elements and similar thing we had taken for truss elements except that in truss elements we would have joined you know this would have joined, that would have joined that and so forth. So, allowing overlap which we said for beam elements does not make sense in practise, so we take a different ground structure where we do not join every pair of nodes in the ground structure, we only join a few points to avoid overlap.

So, if you ask the difference what is that makes beam elements, let us say more appropriate than truss elements, we can see that whenever we have an element like this where it is fixed here that is different from a truss element which would have had a pin joint there right that would have had a pin joint here and here, that would lead to basically rigid body linkages and whereas, if you take this now as a beam element we have instead of this pin joints, we have fixed connections. So, we have fixed connection in this case over here and over here and over here. It effectively means that if I want to replace this with a rigid body and a joint like we do with so rigid body model, there will be a torsional spring also there, torsional spring over there.

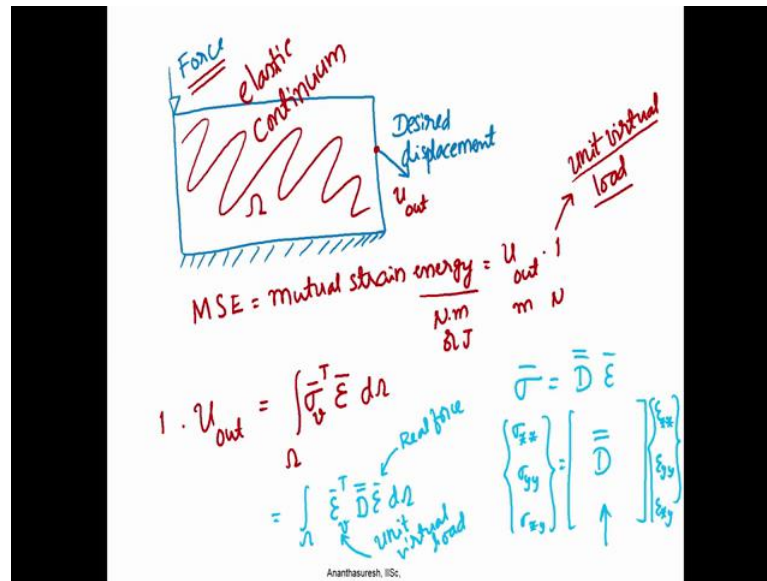
So, it gives the ability to bend let us say this element bends, it has to maintain that slope and that slope over that it has to bend in some fashion right, so every element here will be bending. So, you capture the behaviour that you would get in a continuum, let us say we take this in the last lecture we saw the practical realisation of this design, where we

were able to make this out of one piece by smoothening that you do c and c machining all the sharp edges will become filet. So, we get one big plate that we take; right one plate and then we cut out all this holes to make this mechanism. When we do that, what we are doing is a continuum; now that continuum is being represented with the beam elements as shown here so; obviously, continuum model if you do that is going to be closer to the reality than beam element or truss elements.

So, what we want to discuss today is; how to go from here which is working with discrete elements truss element and beam elements. Now let us go to continue elements directly and one more thing before we leave this is to note that in a ground structure such as this one in a particular element for example, we do not have this element that is missing here right, why is it missing because its area cross section has gone to 0. We ensure when we specify the strain energy upper bound a c star and the deflection that we want and the force that we apply and the material property that we give, we have to ensure that we get a solution where areas have cross section will not be imaginary.

In other words square root sign had a quantity we wanted to be not negative and we also ensure that it is not 0 also, but it can get to 0, it should ever be negative it can become 0 in which case the corresponding area cross section of an element will be close to 0 in which case we say that the element is missing in the ground structure, that is how which element should be there, which element should not be there all that is decided by the algorithm. Now we have discrete elements, we are taking out some the rather algorithm takes out some of these and puts a remaining ones. If we ask what is that the element that are remaining some are them are wider say for example, this is wider than this one which is narrower. So, what is that; that is common to all of those we would find that there is something that we can call an optimal property of this which will come about when will do little bit more analysis that also we will try to do.

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So, let us move on to something closer to reality which is to use continuum elements, by that what do I mean in this case now I would just take the domain, it can be any arbitrary shape domain. And then we say that just like in the previous problem, we say that bottom edge is where we can fix and we are applying a force here; this is our applied force. Now we say that a point somewhere here has to move in that direction, this is the desired displacement; we are hoping on this desired displacement because this exactly what differentiate a stiff structure designed problem from a compliant mechanism designed problem, previous in the truss and beams we had a set of r ground structure of elements.

Now will not say that may not be the case no discrete element, we just say that this whole thing is a continuum so, this whole thing is one continuum then we have to see where we would like to put material, where we want to put a whole right.

So, this is the continuum or what we can call an elastic continuum because in elastic body we have interested only elastic response of this compliant mechanism. How do we do this problem, now first thing we have to do is; we have to write the mutual strain energy because that gives the deflection the point. So, I would like to know what this deflection is, so if I call this it is by some symbol; let us, I just call this u_{out} in the other point where we want the deflection and in the direction in which we want a deflection. If I say u_{out} we know that this mutual strain energy is numerically equal to u_{out} . Why do say u_{out} , numerically equal to because these energy this is displacement even it is

different, but we do note that this is to be multiplied by 1, which is the unit virtual load. We are reinforcing the concept once again here; this is a unit virtual load, so this is meters, this is Newton that gives a Newton meter or joules energy.

So, how do we write this you have for this problem, any general elastic body two dimensional, one dimensional, three dimensional does not matter, this u out at a point is once again given by this principle of virtual work because if I imagine a virtual load, unit virtual load in that direction in that will be the external virtual work because this into virtual load is external that should be equal to internal virtual work, which we have to integrate over the entire domain. When I put this ω like that I mean a domain, it can be one dimensional domain, two dimensions or three dimensions just domain, but integrate.

What do we integrate, the work done by the internal forces when we apply this unit virtual load or rather the unit virtual load will cause internal forces everywhere in the domain and those forces doing virtual work over the displacements caused by the given load. So, when you put a force there; it is a real load that is going to cause displacements everywhere, you take those displacements and conjugate them with the internal forces arising due to this unit virtual load, we get the internal virtual work.

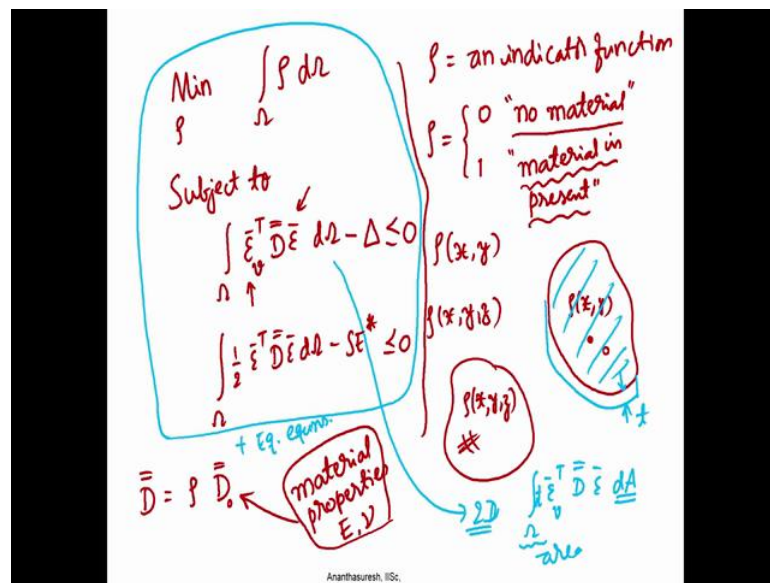
So, we have the strains due to the virtual load let me call that as ϵ_v , so since we are talking about 2D, 3D in general, so I will put strain as a vector that it will have the normal strains and shear strains. If it is 2 d problem we will have ϵ_{xx} , ϵ_{yy} , ϵ_{xy} with the 3D problem will also have two more normal stresses ϵ_{yy} ϵ_{zz} and two more shear stresses ϵ_{zx} ϵ_{yz} . We will have all of those three normal stresses, three shear stress in 3D, two normal stress and shear stress in 2D.

So, will have this we have to multiply this with the, so let us actually change this will say ϵ is due to the real one whatever force is applied and will make stresses to be due to the unit virtual load. So, because we imagine that displacements are real due to the real force and supply whereas, stresses are internal forces are due to the unit virtual load whichever way you do it does not really matter because it is just that this also can be expressed in terms of ϵ , $\epsilon^T d \epsilon$ where d is the stress strain relationship. So, this one is a scalar because both ϵ and σ will have the same dimension when you take in a product will be just $\sigma^T \epsilon$, this is the

mutual strain energy. If we also say that sigma is equal to d times epsilon then this would become symmetric in the sense that we will have this sigma transpose in that case it will become epsilon v transpose d this is the matrix epsilon this. So, this is strange to the real load, this is real force and this one is due to unit virtual load or force.

So, let us say in two dimension to be clear, let us write it as sigma x x, sigma y y, sigma x y we have this is d d by three matrix times will have epsilon x x, epsilon y y, epsilon x y these are our material property matrix, that is what we have.

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Then we can write this mutual strain energy in this manner all, so once we have this we can go head and close our problem our problem is to have as we have said earlier we want to minimise volume. So, let us say we denote variable rho to indicate the presence or absence of material; that is the key to topology optimisation where we convert a topology problem to material distribution problem. So, we say that the volume that we want we simply say this is the differential volume d omega; we have to integrate over thing we will get a volume, but now we are putting rho that becomes our variable, this is if it is truly to be a function of x and y 3D; x y z then let us call this rho an indicator function which indicates when rho is equal to 0 or 1, if it is equal to 0; it says no material there that is material is absent; no material. So, here there is material present material is present if you look at that now when rho that you take in the case of 2D, it will be a function of x y.

So, when you have a domain arbitrary domain, ρ can be defined everywhere inside if it is 3D, I will have this as the function of x, y, z . So, now have three dimensional object as the star that indicate the it is like a rock the some bulky thing, so there will have ρ which is the function of x, y and z . Then at every point whether there will be a material there or a whole, it will decided by the value of ρ , that becomes our unknown, that is the minimising volume and then we have subject to our displacement constraint to that degree of freedom where we desire that, so there we already did this mutual strain energy. So, I can write there functional or integral here which is, what we just did $\epsilon^T v$, transpose d , ϵ ; this is due to real load, this one real and this is the virtual we need virtual.

So, have that and then these we just with internal 3D; if it is 2D we just add thickness and then take care of this one, this I have written for 3D. If it is 2D, the difference will be this; let us see this were that if it were for 2D what will do is write it over this domain becomes now two dimensional domain, so this strain instead of having six component it will be only three components then we will have the same thing ϵ . So, here I would write d a at times thickness, so now it will become it is a 2D problem, so it will have some thickness everywhere.

So that is thickness t , we change that $d v$ will be $d a$ and this will be now integration over the area that is this two dimensional area that is only difference. So, we do not need to worry about anything else as to its 2D or 3D other than thickness coming there and instead of having six strain components in 2D there will be only three strain components and same thing is stress, so will have this. Now we say this particular thing minus δ that we specify should be less than equal to 0, you want it to be δ are small; smaller than that and then we also have strain energy which is easy to write. So, this is half $\epsilon^T d \epsilon$ over this minus $s e$ star this specify less than equal to 0 that becomes our problem.

Now if you see the design variable is ρ , but where is it; it is actually d , so what we do is we write this d to be ρ into what we can call d knot by d knot what I mean is the actual material properties. The elastic case is material properties is going to be Young's modulus e and presence ratio they will be there in some manner depending on this 2D or 3D within 2D, whether it is a play in stress element then strain element all of those are the axial material we have different combinations of e and ν as this three by three

matrix in 2D, in 3D there will be 6 by 6 matrix relating six strains and six stresses that is basically the material model you assume and they will be there. Now we take that multiply by rho, when you multiply by rho we can see that wherever rho is equal to 0, then the d varies 0 meaning that there is basically vacuum, there is no material that is exactly what we want; no material when it is one, it becomes d nought.

So, the material is present that is how we bring in the design variable to this problem. Now this problem we can solve just like we did the problems of tresses and frames that again goes back to our calculus of variations, but a small difference when we talked about tresses and frames, we wanted to know whether the static determinate or indeterminate whether indeterminate, we also had to include the corresponding equilibrium equations. Whether determinate we were able to get this mutual strain energy as well as strain energy directly in terms of the loads that are given where we could compute the internal forces right whereas, when I start a little indeterminate you cannot do the continuum structures are invariably indeterminate, so we need to add also to this equilibrium equations.

So, we had equilibrium equations for two cases for the real load as well as for the unit virtual load, if you do that let us rewrite this problem.

(Refer Slide Time: 21:26)

$$\text{Min } \int_{\Omega} \rho \, d\Omega$$

$$\text{Subject to}$$

$$\int_{\Omega} \bar{\epsilon}^T \bar{D} \bar{\epsilon} \, d\Omega - \Delta \leq 0$$

$$\int_{\Omega} \frac{1}{2} \bar{\epsilon}^T \bar{D} \bar{\epsilon} \, d\Omega - S \bar{e}^* = 0$$

$$\nabla \cdot (\bar{D} \bar{\epsilon}) + \bar{b} = \bar{0}$$

$$\nabla \cdot (\bar{D} \bar{\epsilon}_v) = \bar{0}$$

$$+ \text{Boundary conditions}$$

$$\bar{\epsilon} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \end{Bmatrix}$$

$$\begin{matrix} \bar{u} \\ \bar{v} \end{matrix}$$

$$\begin{matrix} \bar{u}(x, y) \leftarrow \\ \bar{v}(x, y) \leftarrow \\ \bar{u}_v \\ \bar{v}_v \end{matrix}$$

virtual

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So, now minimise we have over the domain this and rho is a function is our variable and subject to our deflection constraint which is epsilon v transpose, d epsilon v over this;

minus delta less than equal to 0 and then half epsilon; no v there just epsilon; transpose d epsilon, so this is not v here right this will be $\frac{1}{2} \epsilon$ this is real $\frac{1}{2}$, this minus $\frac{1}{2} \epsilon$ less than equal to 0 and epsilon that we have here that is strain which will depend on the displacement in the case of 2D, 3D we can write let us if I take in the case of 2D will be three strains; one will be $\frac{du}{dx}$ that is a normal strain.

Then we will have $\frac{du}{dy}$ that is normal strain in the other direction then will have $\frac{du}{dy} + \frac{dv}{dx}$ with a half, it depends on how write d matrix, but let us just write half for; if we do the same thing symmetrically, it will become half will disappear where you come $\frac{du}{dx} + \frac{du}{dx}$ half of it which is $\frac{du}{dx}$ and same thing here is the shear strain and normal strain. So, we now have the function u as the function of x and y also v as the function of x and y . This is the displacement in the x direction, if I take this way x and y and v the displacement of y direction. If it is three dimensional case; we also have w which is a displacement of of the z direction, then we will have a strains expressions also change, so now we need to have governing equations for u, v, w ; let us together, let us call that basically \bar{u} that captures u and v displacements that is due to the real load and then will also have u^* let us not get confused with this v is just to indicate that it is virtual.

So, we need to have governing equations for that, which we can write which we can write governing equations that that you will have to do if you recall that a way. So, we say divergence of this d matrix that we have with epsilon that will be stress and if you have any body forces you put that like inertia forces initial forces like gravity or centrifugal force you would do that equal to 0, that mean governing equation likewise will have governing equation for; d unit virtual load epsilon v and there will not be any body force there, equal to 0 and then you have boundary conditions also plus both of them will have boundary conditions which can be forces acting on the surface or some displacements being specified like fixed points will have all that is the continuum way of writing the problem.

For ease of understanding, let us convert this to discretize problem, it is not discrete problem like truss and beam elements will be discretize problem.

(Refer Slide Time: 25:16)

$$\text{Min } \bar{P} = \sum_{i=1}^N \rho_i V_i = \bar{\rho}^T \bar{V}$$

Subject to

$$\Delta: \bar{u}^T \bar{K} \bar{u} - \Delta \leq 0$$

$$\Gamma: \frac{1}{2} \bar{u}^T \bar{K} \bar{u} - S E^* \leq 0$$

$$\bar{\lambda}: \bar{K} \bar{u} - \bar{f} = \bar{0}$$

$$\bar{\lambda}_v: \bar{K} \bar{u}_v - \bar{f}_v = \bar{0}$$

$$\bar{K} = \bar{B}^T \bar{D} \bar{B}$$

$$\bar{D} = \sum \rho_i \bar{D}_i$$

$0 \leq \rho_i \leq 1$ (desired)

\bar{K} = Stiffness matrix
 η = penalty parameter

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In the sense that whatever we have written in the previous slide will now write; now the rho will become rho bar; that means that if I have some continuum, I would divide that into finite elements everywhere let us I take triangle elements, I will have you know something like this triangles, so arbitrary whichever way you mesh x will mesh and each element will have a rho; rho i if there are n elements in the discretize domain will have rho 1, rho 2 up to rho n. Now when this rho I, e is equal to 0 i-th element will not be will not be there, there will be a whole will be equal to 1, there will be a material right when you have that then we say summation of all this rho i that will be the volume.

There are number of elements is n that will the volume and then we say subject that true our mutual strain energy which is u v transpose where you do unit virtual load and get displacement and stiffness matrix k whatever we had earlier epsilon transpose g e epsilon, now it will be u v transpose k u due to the real load that minus delta is less than equal to 0 and then will have half u, transpose k this a stiffness matrix u minus s e star is less than equal to 0 and then will also have the equilibrium equations which were in continuum form, now it solve discretize form k u minus the applied force f vector equal to 0 and then we will have the k u v that is the displace due to virtual you need to virtual load then I can put that as f v equal to 0.

Basically f v will be, but in this case if i say this is the force applied and I want this point to move like that, this is the displacement desired right there will be 1 delta or less and it

will be active it will always be delta then only there this f_v will be equal to 1 everywhere it is 0, these the problem we need to solve.

Once again where are the design variable ρ it will be in k because this k is this stiffness matrix that is the one, this is the stiffness matrix and that is the one that will have the design variable because if for those of you recall finite element analysis. This k is assembled form of local element stiffness matrix is where that k will be what will have b transpose d b and what is b ; b will be strain displacement matrix. So, we have strains coming from displacements they b matrix relates them and d again is our things, that is related to ρ times that d 0 that we wrote in the case of continuum elements. So now, we can solve this problem where I can (Refer Time: 28:47) as per necessary condition and then solving it. And we have to write the Lagrange multipliers like before for trusses and beams, so it may look intimidating, but if you understand trusses and frames; it will be quite straight forward because the concepts are readily transferable.

So, we have two Lagrange multipliers and then we have λ which is now a vector for this vector equation then will also have another λ v shall we go back and look at trusses and beam problems, we have essentially the same frame work except that now it is the continuum element elements will be there or not there depending on the corresponding value of this ρ_i for each element and the ρ_i 's actually go here, but now other trick that people use is to put a exponent there which is called the penalty parameter what it does is, if the value of ρ that is close to 1, it will be quite large or when you write it raise it to a exponent say 0.9 will be considerable value even if you let say η equal to 2, it will become 0.81, but other hand ρ is close to 0 is 0.1 when you raise it to a power it will become 0.01 it becomes much smaller if it is larger, it will still stay close to 1 and if it is close to let say 0.99, if you multiply by 0.99 again when you square it η equal to 2 r actually qubit will be even better.

So, what this penalty thing would do is to push things that are close to 0 to become closer to 0, which are close to 1 become closer to 1 eventually and it will stay close to 1 right that way we ensure that this ρ which is to vary between 0 and 1 will be pushed to 0 and 1 so that material will be either there or not there are not been intermediate state that is what we want to avoid, that is what this η which is big penalty parameter does this is the penalty parameter. It penalises material if it is close to 0 and if it is close to 1, it actually encourages it and become part of the final solutions. Finally, we will get some

whole in this topology forms these optimisation problem statement, then I have to solve this we have to write the Lagrangian.

So, we write Lagrangian we have this sigma rho i were i equal to 1 to n because at these the volume that actually I should actually put this v i. Let us go back, it is not just y, we have to multiply the volume of that element. So, we have too many vs here let me use capital v i that is the; this particular thing has a volume; volume of that let us indicate that with v i why v i are (Refer Time: 32:18) rho transpose v.

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$$L = \sum \rho^T v + \Delta (\sum \bar{u}^T \bar{k} \bar{u} - \Delta) + \Gamma (\sum \frac{1}{2} \bar{u}^T \bar{k} \bar{u} - s e^*) + \sum \lambda^T (\bar{k} \bar{u} - f) + \sum \lambda^T (\bar{k} \bar{u} - f)$$

$$\frac{\partial L}{\partial \rho_i} = 0 \Rightarrow v_i + \Delta (\dots) = 0$$

$$\left(\frac{\partial \bar{u}}{\partial \rho_i} \right) = 0 \quad \left(\frac{\partial \bar{u}}{\partial \rho_i} \right) = 0$$

So, now I can just write it everything in matrix form, so I will write rho transpose v and then I have this lambda times u v transpose, stiffness matrix k; u minus delta plus that plus this gamma half u transpose k u minus s e star and then we will have this lambda transpose k u minus f plus lambda v transpose k u v minus f v you need to virtual load that is our Lagrangian, so we have the Lagrangian.

Now, we need to take derivative of this with respect to our rho i each element we have to get in equation for it, if I do dou l by dou rho i that we get to 0 then you have to see where all will have rho, the first term we definitely have rho that will simply become v i because that summation of rho I, v i if derivatives equal to rho i; I will get v i and second one lambda which we had done for truss and beams we have to recall what we did.

So, now where will ρ be; ρ be actually in here, but indirectly it will be in here as well as here. So, we have to take the chain rule and keep applying, we will get several terms similarly over here and over here and over here we have and this one and this one. In all of those will depend on ρ_i we have to write that we will get a lengthy expression equal to 0, but you do not have to worry about because when we solve the problem, we will be able to solve for u and u_v for assumed ρ distribution and then what we do not know would be $\frac{d u}{d \rho_i}$ and then $\frac{d u_v}{d \rho_i}$ by v by $\frac{d \rho_i}{d I}$, here are the centre of the displacements due to real load, as well as unit virtual load this one you actually do not compute.

Where when you get this long expression here which will do in the next lecture continuing this is equal to 0, everything know put in here and will have some term that contain these whatever multiplies these two. We say that should be equal to 0 and when we do that what we get will be equations with which we can solve for this λ transpose and λ and λ_v , we can solve that we call the adjoined equations again what we had done for the beams case will also do that.

So, what will be the; before you come to the next lecture is to write it all out, so all the derivatives you write and then separate out a terms that contain $\frac{d u}{d \rho_i}$, $\frac{d u_v}{d \rho_i}$ equate them to 0 then you will have equations to solve for this λ then rest of them what you get, you will get something interesting when you eliminate those terms by choosing λ transpose; λ and λ_v what you get will be quite interesting, we will get something which will be useful to interpret for sensitivities also sensitivity meaning gradients or objective function, the constraints we have objective function over here, we have the objective function it is objective function and then we have a constraints.

So, we have the one constraint which is the mutual strain energy, another constrain strain energy how do they change when I quote up one particular ρ , that is whether it is increased or decreased slightly how do the objective function constraint in that the gradients we can get them all from this analysis which we will discuss in the next lecture and then also look at the implementation where you can where we implement this optimality criteria method and then solve the problem.

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Main points

- Mutual strain energy ←
- “Compliance and Stiffness” formulation
- Optimality criteria method ←
- Sensitivity analysis ←

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So today to just summarise, we have understood the mutual strain energy in the context of continuum interpretation. In fact, we wrote the continuum formulation variation framework have been switched over discrete one, but those of you who are familiar with variation calculus you can actually write differential equations rather than the discrete equations they have written, but what is important also notice that we are still sticking to our compliance stiffness formulation there is constraint on the displacement, there is constraint on the strain energy

So, this is to avoid those unphysical situations where areas of (Refer Slide Time: 37:42) become imaginary. Similar thing would also happen in the continuum thing, but you can see it as clearly as you see in the case of trusses and beams and whatever we are to start doing this lecture leads to the optimality criteria method, and then it will also tell us how to do this sensitivity analysis. So we will continue this in the next lecture.

Thank you.