Compliant Mechanisms: Principles and Design
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Lecture - 28
Design for deflection of beams and frames

Hello, continuing with the previous lecture on classes now we move on to beams. Beam as a single beam where we want to design it for given deflection and then we also look at what we can call a frame ground structure. We are talking about ground structure in the last lecture will continue. And also look at couple of examples to see how this method will be useful to designs and compliant mechanisms comprising a few beam segments. So, today the topic is design of beams and frames for desired deflection.

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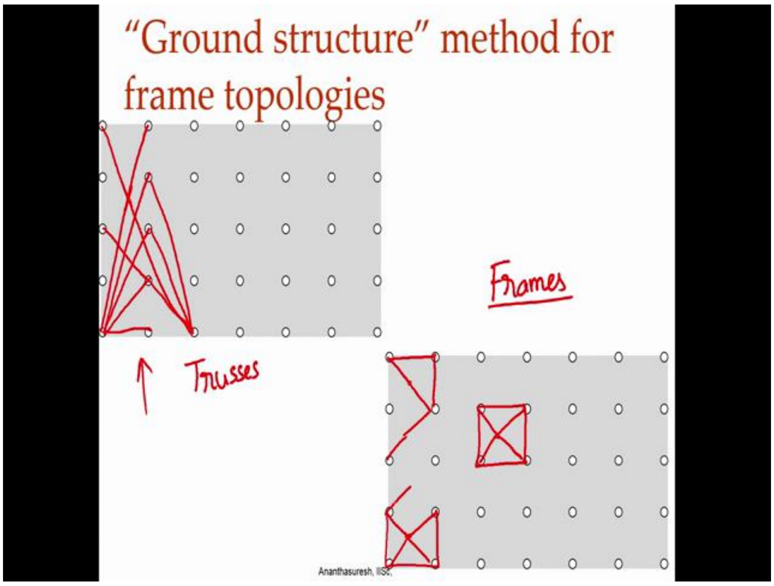

We are talking about this ground structure for truss topologies at the end of last lecture. So, where we said that we can join every pair of points in the cases of trusses, means I can have a these things and these and these and all possible ways every paid of them we can connect in the case of trusses. So, if I take two elements here there over lapping that is perfectly all right for trusses, because you can place all these elements in parallel planes and things will be just find because truss elements they only contract or stress, of course, other joints rotation is freely allowed.

When you come to frames there I cannot let things over lap. So, like we have shown in this thing for trusses which we are discussing last time we can let elements over lap there is there one over the other when you project the motivate two d plane because, we can place them in parallel vertical planes. In the case of frames we cannot do that, we can still do it, but the problem would be their they tend to twist as well, because there will be bending out of plane when you put them parallel planes and the behavior may not be same as what it will be they are all to be in the same plane.

So, in the case of frames where will now have not hinged elements like trusses, but actually frame may be welded to one in other. So, it will like this. So, you can also introduce a mix side load and put a few more elements like this so that we create more possibilities for resulting topology. Every one of them we can to the midpoint and do this, that will be the frame ground structure. And then algorithm may to choose to remove that, this, this, this and to make a mechanism. So, that is possible. So, it can do by removal one at a time or few reach at a ration it can give a compliant topology for a frame or which we can call a compliant mechanism with slender beam elements.

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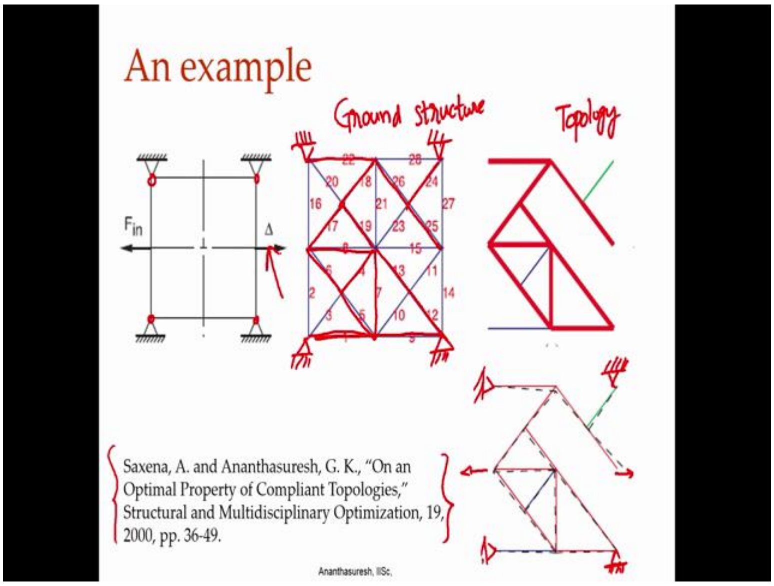

So, let us look at this example. So, here we have a specification it is taken from this particular paper, we will be discussing more about what was in that paper. Let us say we attaching over here they can be ping joints because, they are fixed at that point not really rotation is not prevented it is just a hinge joint, but rest of them we can define a frame ground structure like this. This is called ground structure because, there is like super structure in which all possible elements are indicated and algorithm would remove like if you see here this particular let us draw what is actually there in this is the topology that came up.

So, this element is there is in a blue because that is a little thin and then this is there, this is there and this is there, this is there, this is there, this half and half those are there, this is here, this is there and this one and this one. So, these pretty much what are there. So, here it is fixed there, fixed here and there is fixed here and fixed here. So now, we can see how the optimal topology would come about will be solve this as a frame problem will discuss a details, but we are showing the examples first that it will give something like this which as you can see when we apply force in that direction the output is moving in this direction, here the dash line in this figure is a deformed profile will solid one is what algorithm gave.

These in blue and there is this little thing is there right. So, this one; these are in blue comparative other once because they are little thinner there and the others that is how it was drawn. So now, we can see that you can get an optimal topology that if you fix you know these points as it was assumed in this problem now this particular one when you apply force here it will be move in the other direction as desired here. So, that is the design for desired deflection.

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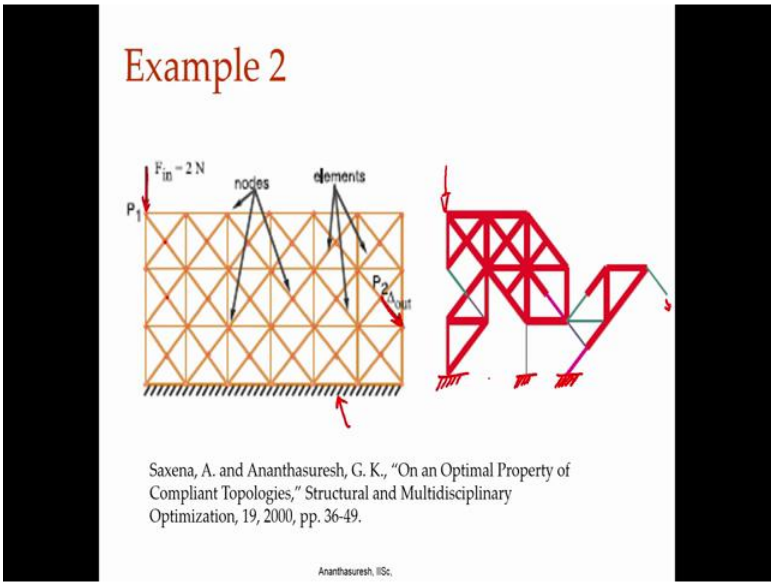

And let us look at another example where you have ground structure which is as it is shown here there are nodes at all these points now, here input is this way and we want output in a direction like that. So, this point should be move like that. How do we give that the unit load that we specified it should be at that point in that direction that is where we desire the deflection some delta? So, then the algorithm gave a result like this.

So, here it is actually fixed not like a ping joint it is actually fixed like a cantilever because, that is what is given in this specs here now this particular one if I apply a force there this point will move in that direction the point is over here that will move in the direction, and whether that does it or not we can actually take the solution we can do a simulation if we want the black one here is deformed one these dotted red once are the un-deformed once. So, if I apply a force here this actually would be moving in that direction.

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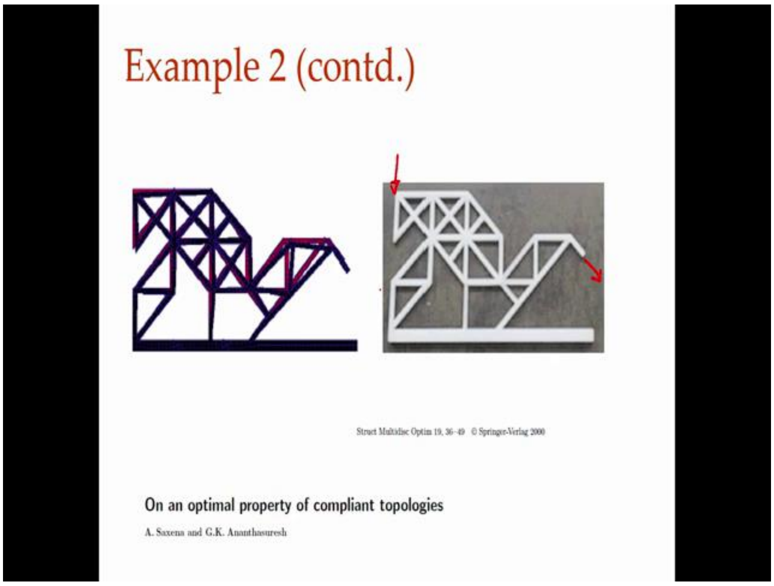

So, I have a mechanism for you to see this, this prototype was built exactly the way optimization came up here we can actually see if I apply force at this point in the downward direction is should move in that direction. you can see the output this used to call it a peak up mechanism it looks some out like a bird. So, it will go like that just like what optimization gave. So, we can synthesis topologies without assume anything if you go back and look we start over here, where we assume nothing other than the design domain where should be fixed where apply force and then we ask or we desire that output in the sense certain fashion we get the solution.

This parallels the truss design that we have already discussed now will just repeat that for beams. So, that the ideas that we have discussed will get rain force, because they also become relevant we considered continuum topologies where we have more continuous structures and not truss or beam elements like it is here.

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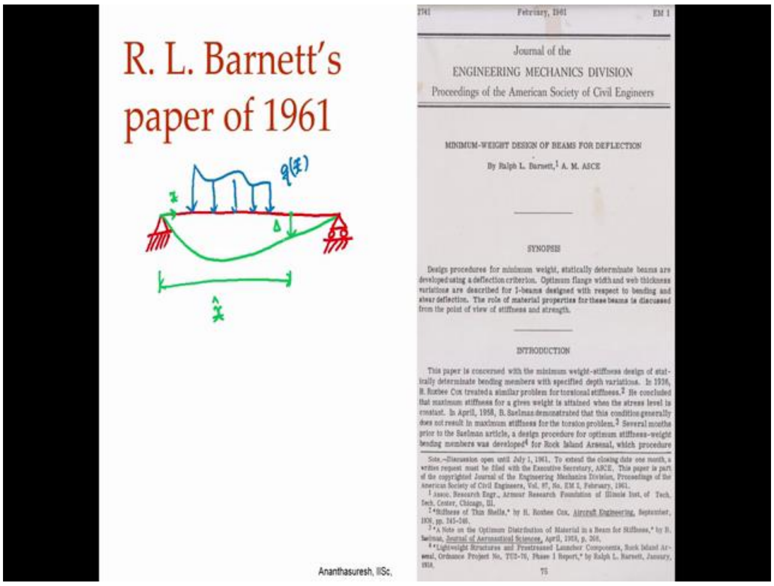

You go back to Barnett's paper gain because; it all began with his paper his work. So, what here considered was if I take a straight beam one beam first and if it is statically determinate let us say on that we considered some loading. So, from variable load general load so that we can write in general expression q of x under this load if the beam, let us say deforms in some way at a particular point we want this deflection to be delta that is what we want at some point that also has to be given some x hat because, x we can assume was like this.

So, we want to design this beams. So, that at some x hat the deflection is delta and we want to have minimum value of the beam that is a problem that he had looked at Barnett had considered.

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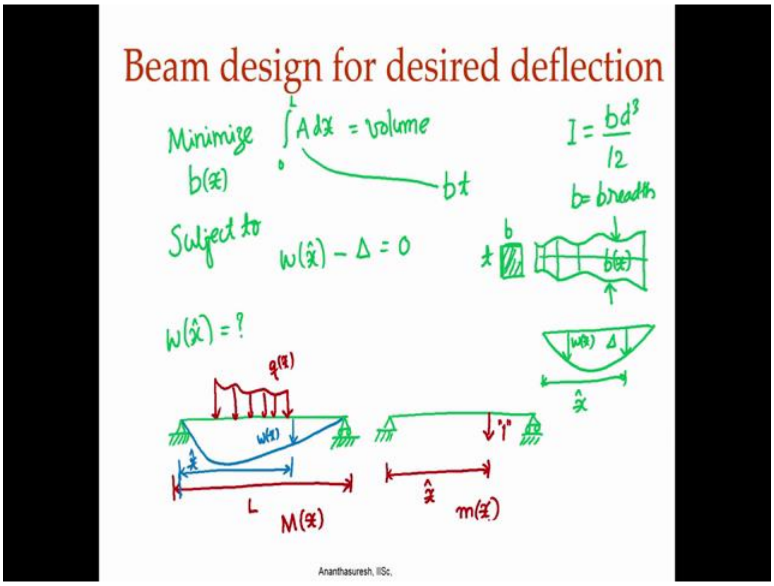

So, if you write that problem. So, we say we want to minimize the volume here you would try to pose it using a continuum beam rather than discrete one will come to discrete one later continuum we need do it our unknown function would be let us say width profile of the beam the beam is discrete beam it has a width that is when we take second movement of area we say b d cube by 12 where b we can call it width are is better to call it in breadth is a dimension which is perpendicular to the plane or deformation with the beam.

If that is a one that is if you have a beam we look from the top the breadth can vary like this, this is b of x at every place it will have variable value here is something here is more less and so forth that is a b of x that is a unknown function there becomes a calculus of variations problem. So, here so the functional as we call it the integral here 0 to l a d x that is the volume of material that is what you want to minimize volume of the beam and subject to or deflection constrained, we want to say the deformation the beam is like this.

Let us say if it is simply supported where it goes like this at a particular value they should be delta and that particular value is x hat if you say we say w. If this deformation in general is w of x transfers displacement we say w at h hat minus delta equal to 0 less then equal to 0 whichever where you want to say you want to put that constraint. So, we want to solve this problem we want to find b of x area here will be this area will be b times a t thickness of the cross section.

We assume a, let us say rectangular across section here rectangular across section. So, this is t and that is p this is incline and this is the thickness or d which are way since the take in derivative. So, is not, but use d let using t here. So, when you have this problem in a before we solved when need to see how you can get this w x hat how do we get that that is where we use the principle of virtual work are our method of unit virtual loan that is if I take the beam there will be where taking a simply supported beam, because a determinate to one we can solve for forces without having to compute the deflection. Let us say there is some loading on it there will be some loading on it which you can say q of x in the length. Of course, this pan of the beam and let us say this one deforms and this particular manner let us call that deformation w of x at a particular x hat that is form here to here this is x hat now these are all the same.

So, x hat, how do we get that expression. So, what we do in this case is take the real load and we also take the virtual load we take this same beam simply supported now we apply to virtual load at the point this will be unit load and show one at the point where we are desire in a deflection which is h hat, if we do that we will get the bending moment in this case I can call it capital M x and in get bending moment.

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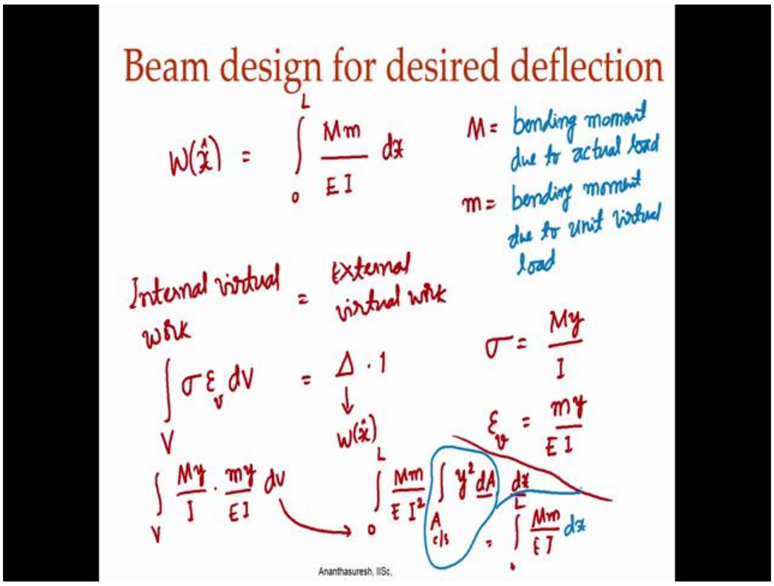

Here I can call it small x small m of x we know how due to the bending moment it is starting determinate to one not knowing the area of cross section is just find, because it is independent of that we get m and n if you have this, this w at x hat can be written as an integral as a functional 0 to l capital m small m by e i d x.

So, once again this m and small m the capital M denotes bending moment in the beam due to actual load bending moment due to actual load and this is bending moment you need to virtual load bending moment due to unit virtual load two different times you have to do and that gives this w x hat how does it come about. So, for that again we look at principle of virtual work which says internal virtual work I just repeating what you did for trusses. So, that we remember internal virtual work is equal to external virtual work; external virtual work what you have here we have a unit load in the a direction and the point where we desired the deflection delta there unit virtual load at that point whatever displacement that it moves due to the actual applied load that will be the external virtual of delta times one delta are what we can call this w x hat we want it w x hat to be delta.

So, w x hat dot one and what is the in as a external virtual work internal virtual work internal virtual work in general we had said the actual displacement times the internal forces that arrives little unit virtual load, that in the case of continuous systems we can write it has sigma epsilon v d v the entire volume we have to do sigma epsilon v d v that is what you have. So, what is sigma that is due to the actual load that we can write sigma is m y by i, where i is a second moment of area y is the point where you are measuring stress from the neutral plane in the y direction that is sigma, what is epsilon v it will be bending moment m and to y again to i. Since we want strain now due to unit virtual load the e also well come in the denominator that is what we have.

Now, we substitute that here we want to entire volume I write sigma which is capital M bending moment to the actual load and then y by i and then x l 1 we will be small m y e i to d v right. So, now, this one we can see that in the cross section you have certain things m m and i e do not vary because, that the cross section you have bending moments and you have cross section second moment of area. So, I can take them out I can split in two parts I will do from 0 to l length.

And then I write m m by e and i square there is this part which is the area of cross section because we need to do I will have y square d a. And then I will write d x d v is written as d a times d x when I do that what I get is 0 to l m n by e i and then this quantity here this quantity the definition of second moment of area that is i that i and i get canceled leaving only this much right.

So, that is basically is out of d x hat. So, whatever be needed we got it as m m by e i d x 0 to l. So, let me 0 to l.

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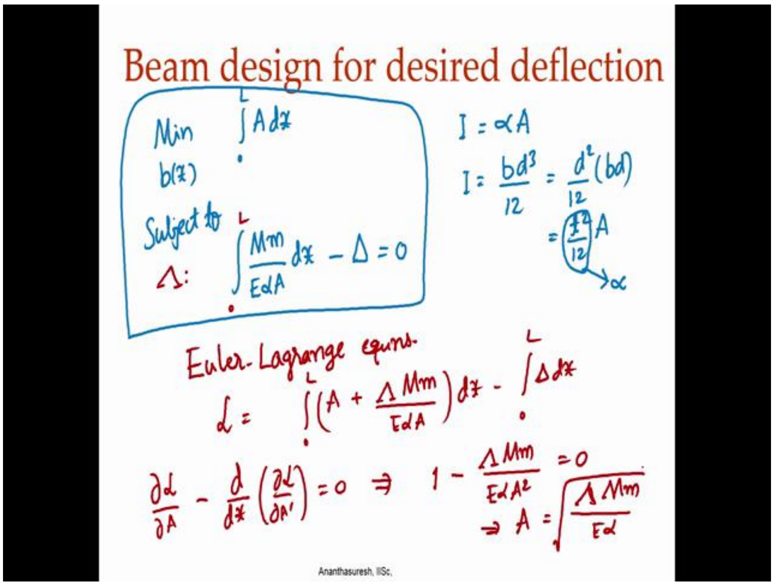

So, I can now write my problem minimize with respect to d of x integral 0 to l a d x that is volume subject to our deflection constraint where I have m m by e i. So, what we assume is that I will say is proportional to area of cross section that is true when you take b the breath or bit of the beam has the variable. So, I can write i alpha a alpha is something in the case of rectangular cross section we know i is b d cube by 12, but I can write as d square by 12 where d is our thickness d square by 12 into b d that is nothing, but area.

So, in our case is going to be come t square by 12 into a right. So, this t square by 12 let us call some alpha some no anywhere in thickness we have that. So, I can write it has e alpha a d x minus delta equal to 0 this is our problem statement this calculus of variations for which we write Euler Lagrange if I do that I will get the solution this is do that you are not familiar with Euler Lagrange equation you have to look up for that.

So, Euler Lagrange equations for this one we write the lagrangian and then write Euler Lagrange equations. So, when we do that we get this equation. So, first we write lagrangian which will be 0 to l a and there is a lambda as we had earlier a plus lambda times m m by e alpha a d x minus of course, there is 0 to l. So, this is also limits of 0 to l and then delta t x that it has the area which is like a constant. So, that we contract worry. So, now, Euler Lagrange equations are dou l by dou a minus 12 if we actually no need d by d x of dou l by dou a prime and all that, but here a there is only a. So, this equation is simply dou l by dou a if you do what you get is the first one let us to 1 because area is there plus actually becomes minus because a is in the denominator.

So, you get lambda m m by e alpha a square equal to 0 and that gives us area cross section area cross section is going to be this lambda m m by e alpha square route. So, you get area cross section like we did for trusses.

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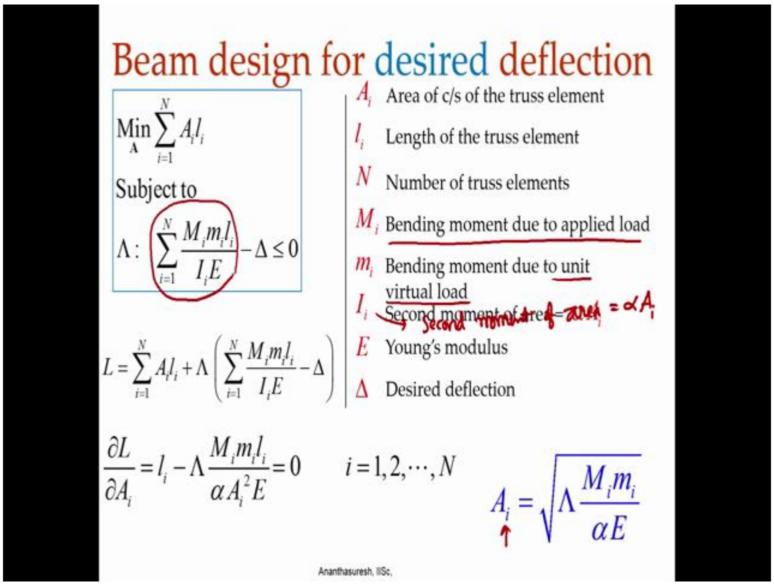

So, let us look at that not for one beam one beam is there and then how do you get lambda lambda we can get in the previous one by going back to this constrained we know area now for that contains lambda we can single equation lambda we can solved for that we can solve for lambda here now let us move from one beam to several that be there in a ground structure minimize a i l i. Again volume that we are doing respective a i now not one this a now will have several of them there will be a 1 a 2 the cross sections up to a n.

So, in a ground structure and your taking the sum of all of those things for or inter up integration now we have l i, because the assume uniform cross section m m by i i e 0 to l we do just l i comes here the length of that beam element in a ground structure each beam element has uniform cross section you take this, then all the things where defined as before for i-th element you have bending moment due to the applied load bending moment due to the unit virtual load. And this is second moment of area that is not written here second moment of area which we also assumed to be equal to alpha times area cross section as we jested is in delta is desired deflection. You can write the lagrangian it is now find where it optimization and then we do this dou i by dou i equal to 0 that gives us the area of cross section.

Just like what we had a continue system for it is for each bema member how do you find the lambda.

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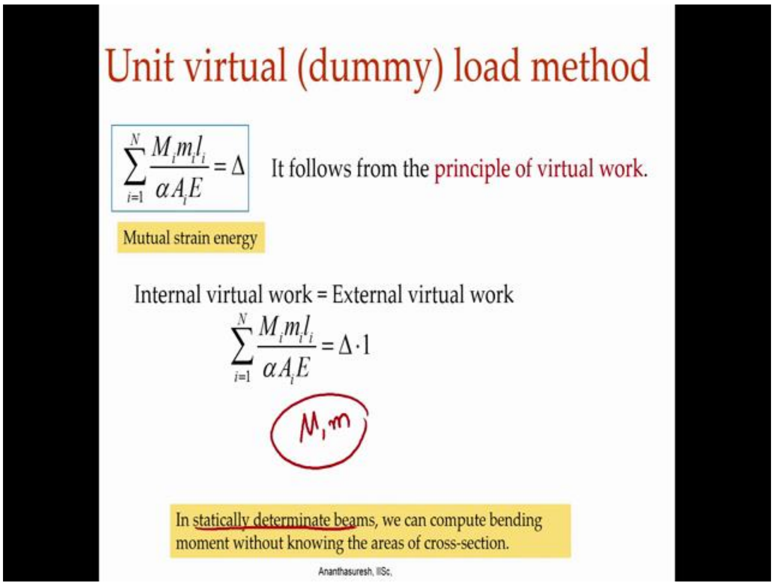

So, we go to the deflection expression there is mutual strain energy as we had defined we can use this that just repeating what we have is statically determinate beam we can get m and m easily if it is statically determinate in determinate. We have to make the modification that we cannot pose the scroll in terms of bending moments, but pose in to transmits displacements will do that.

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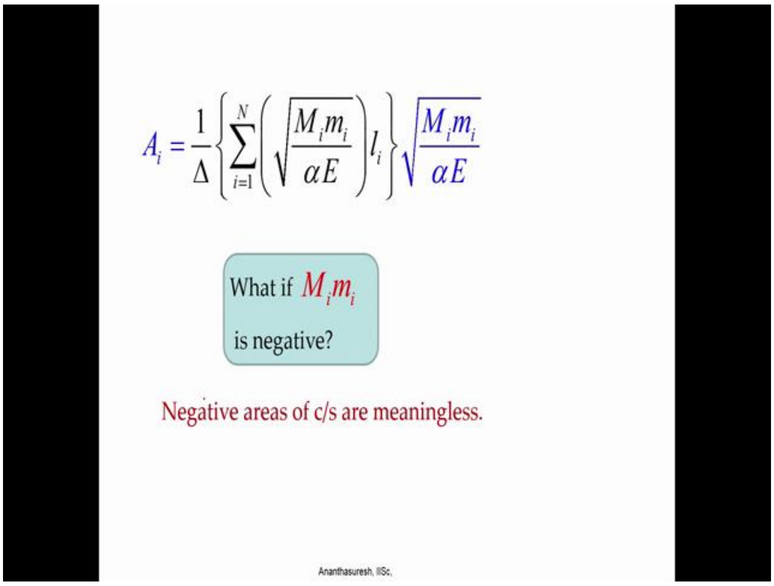

And then we also has this question what if the product capital M small m is negative that can happen when that happens at some point area of cross section will not be real because, it is send is square route sin and this is after solving for lambda this is actually square route of lambda.

So, we can solve it provided this product is positive always, but it need to not be for whatever applied load that you have whatever desired deflection direction you want. So, because negative areas are imaginary not negative areas really it is this quantity becoming negative then becomes areas are imaginary. So, they are not a load.

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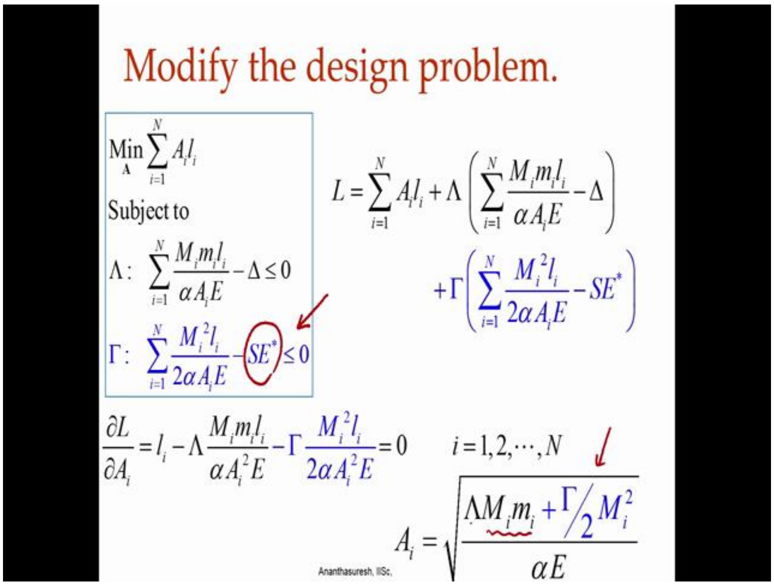

So, just like for the trusses we modify the problem by putting strain energy constrained strain energy is going by m square l by twelve for a e for a beam element. So, we put that constraint and we solve the problem then we find like we did for the trusses we get this extra term which is m i square that is bending moment to the applied load with applied load square. So, if you choose this s e star properly we can make gamma large enough. So, that whatever negative thing we will be there multiplied by is lambda over all be communicate faster.

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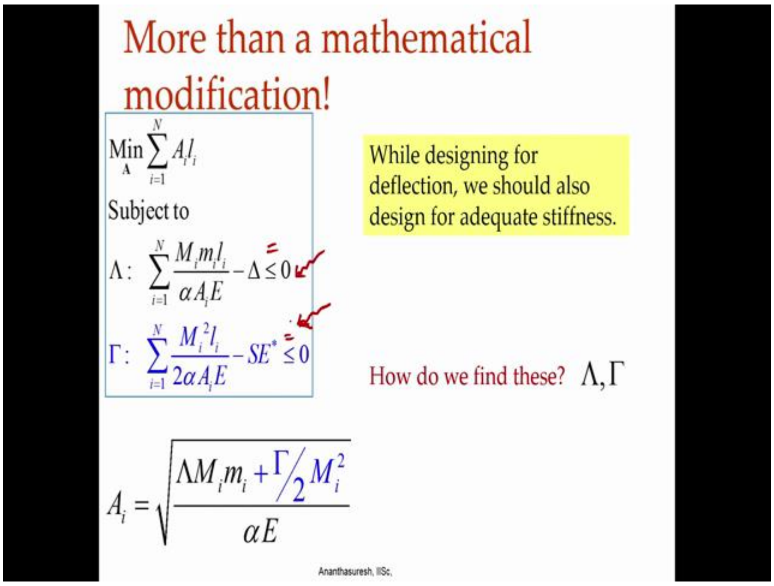

So, we tried to modify the problem. So, that in all situations we get the solution how do you find these lambda and gamma sin has before we have to solve this equations when they become active this become equal to this becomes equal to you would have area of cross section in terms of this lambda and gamma and known quantities m i and capital M i small m i you can saw. So, this is not just a mathematical modifications is actually a physically meaningful, one will be want certain compliance here we also we took would certain stiffness requirement that is why a mechanism to make sense.

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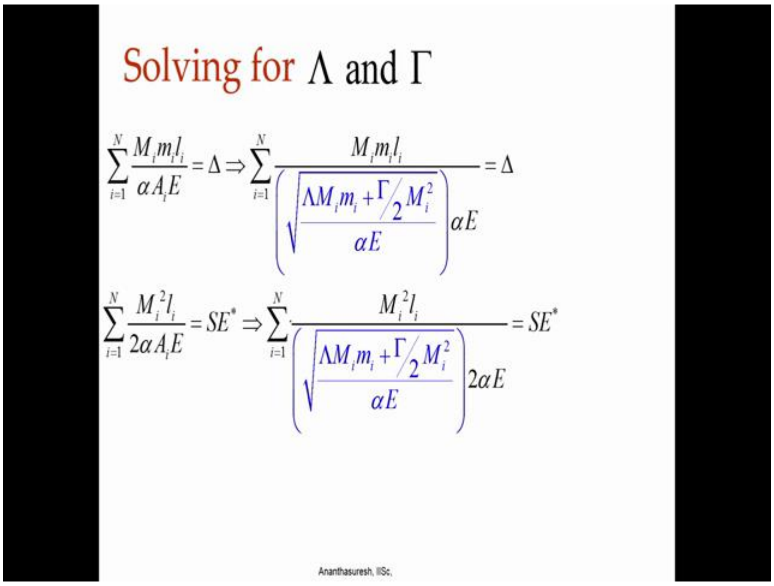

So, with these two equations if we solve for lambda and gamma we are done.

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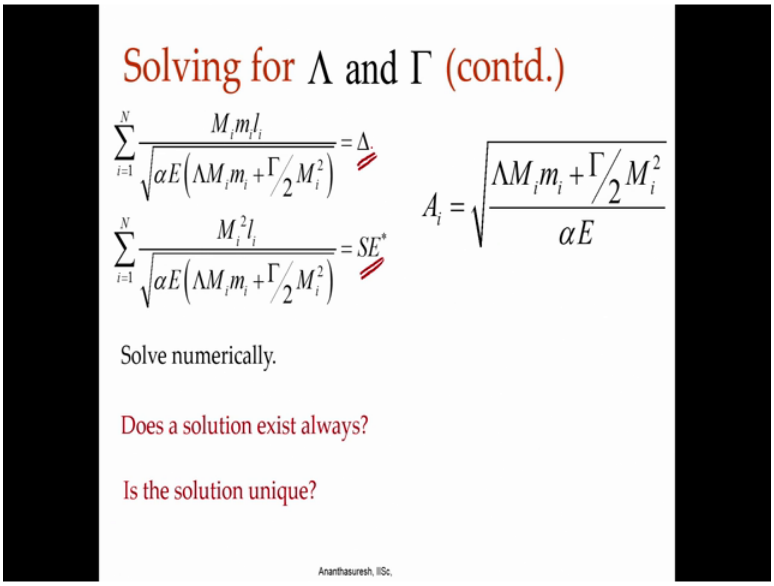

This is when we have statically determinate things where if you solve numerically and solution exist always need not it depends on the values of SE star and delta and which points your applying force in what direction you will ask for the deflection in all that. But in a solution exist it do not be any unique either, because then only in equations we are taking about the same argument has we had for the trusses.

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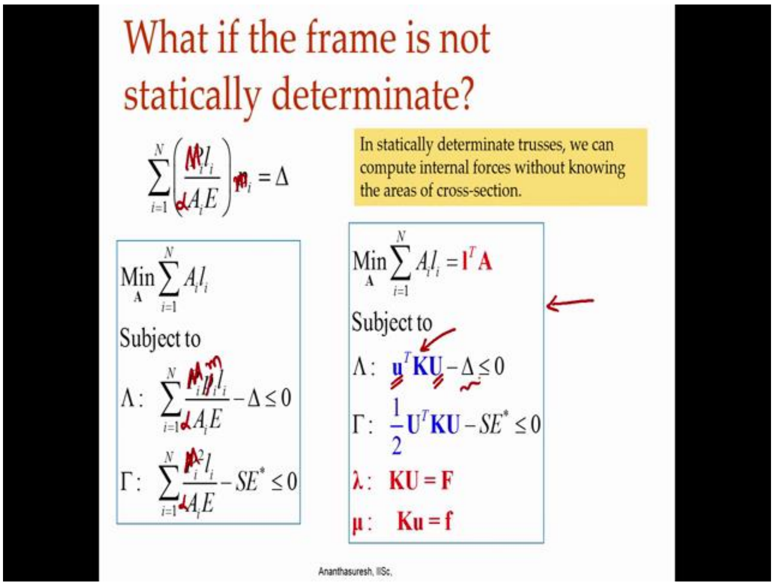

Now, we can also go to statically de indeterminate case where this type of doing where we had p's for the trusses now they will be replace with m, m, m and this will be the alpha will be coming because, area moment of inertia second moment of area we are calling it proportional to areas. So, alpha A i that is what we have instead now we can write in terms of these also becomes capital M and small m and this alpha will be there is a trusses we copied.

Now, this one of course, it does not change it just the stiffness matrix k now will be for the beam elements of frame elements truss of it will be exactly the same. In fact, will be the same for continue structures also where only our interpretation of capital U which is the displacement vector under we applied load and then small u which displacement work can be a unit virtual load afraid at a point in the direction in which we want a deflection delta you can pose a problem this way and saw not for now trusses this is for frames.

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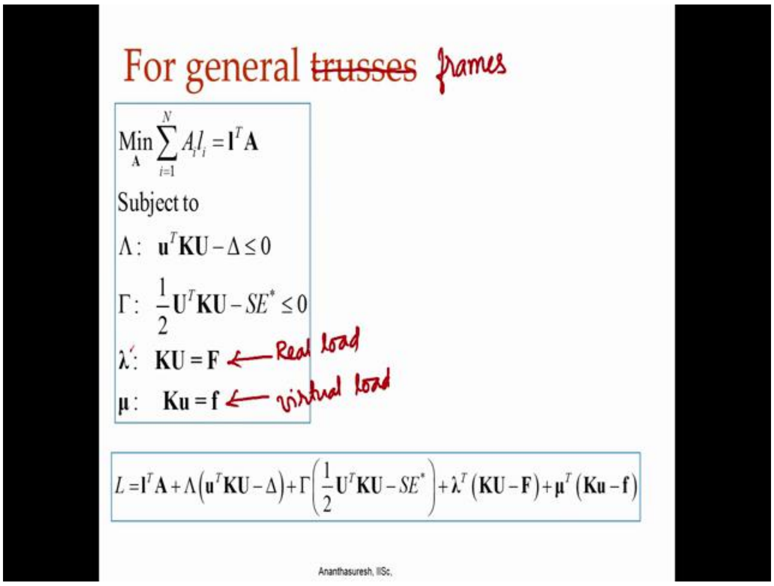

So, we have this we can do just as we did for the trusses and we can get this solution like we did earlier we get what we call a joint equations which give you this multiplied corresponding to be state equations that we have for the real load.

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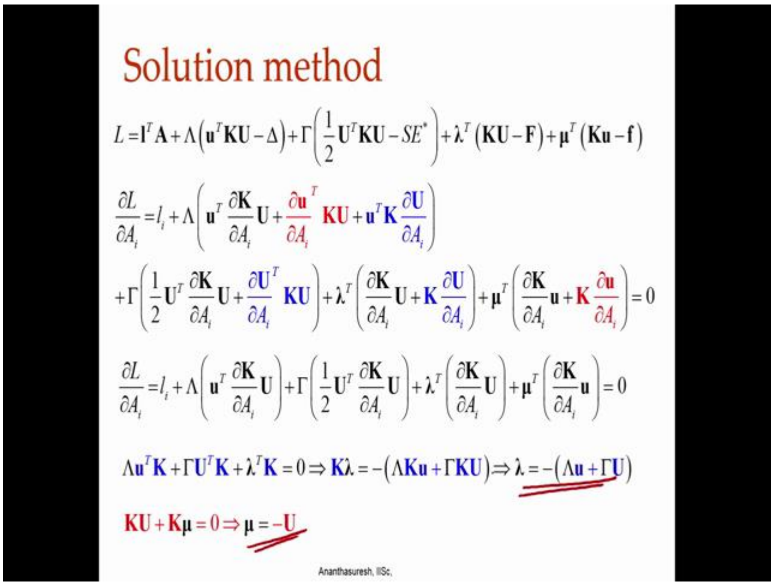

And let us go back in look at the, this one this is the equal equation for the real load and this is for unit virtual load.

There are ways to evaluate these as we done for the trusses the same thing course if we slowly look at all these expression that we have we understand what we mean.

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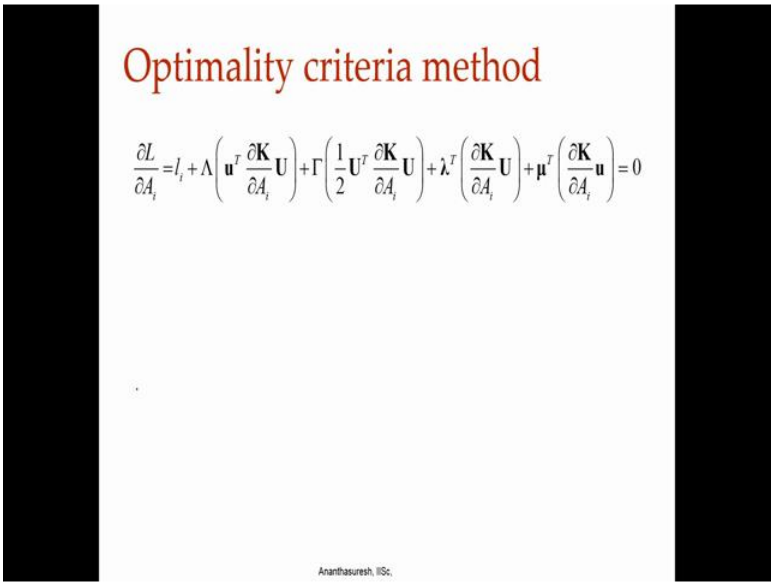

And then we put it all together, because the optimality criterion which can be used to fund area cross sectional of each other thing in the ground structures.

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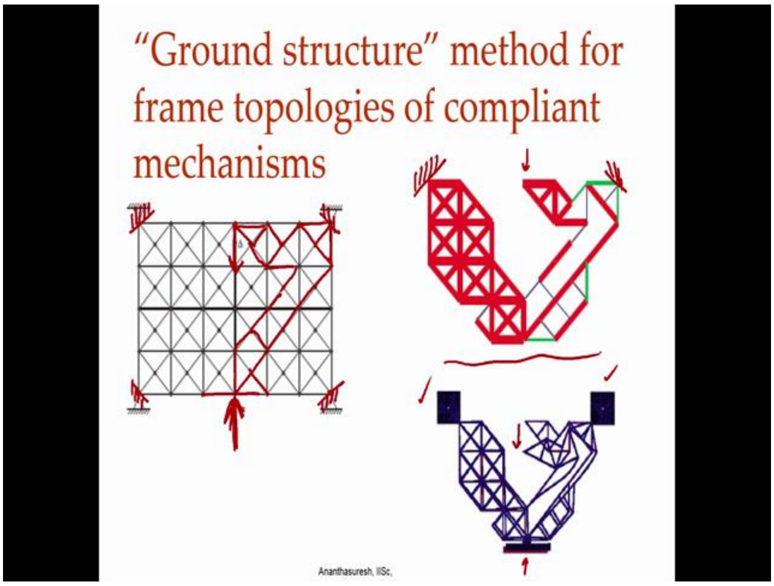

So, let us look at how to understanding the method let us take this ground structure for something like these where we applying the force this way we want this point to moved on it is non-intuitive when it is pinned at these points, actually it is fixed not actually pinned it is not like this these actually fixed like that look at that is how the structure that we are going to have it is actually fixed.

Apply the force it should come towards use is non-intuitive applying a force like this and your asking for 0.2 moved on like this that is exactly what we get. So, this is fixed here he did not use a support here it is using only supports at this point and this point you can actually see when your pushing up is actually coming down. So, you can see that mini elements are removed right from here we have this element, this element we can trace from based on this and this course all the way and this is there other colors other than red shows that they are thinner narrower than the other once that we have here relative wet is also shown in this diagram. So, it is somewhat like this I am just trying the element that is there other things are actually removed.

So, this is there, this is there, this is there and one of this is there and a few of these are there you we can we can see what are removed what are remaining right, and this one I have the mechanism for you to see of this words. So, here the exactly what we have, in fact, we compared with the figure and what is shown here the elements are exactly the way they are when I apply the force that and pushing it up here like that and you can see that this point is actually coming down just I see want I am pushing here that point is coming down it has a amplification factor of both seven that came from this solution notice that it is actually a symmetric. So, asymmetry was introduced to in this thing and it gave the solution like this.

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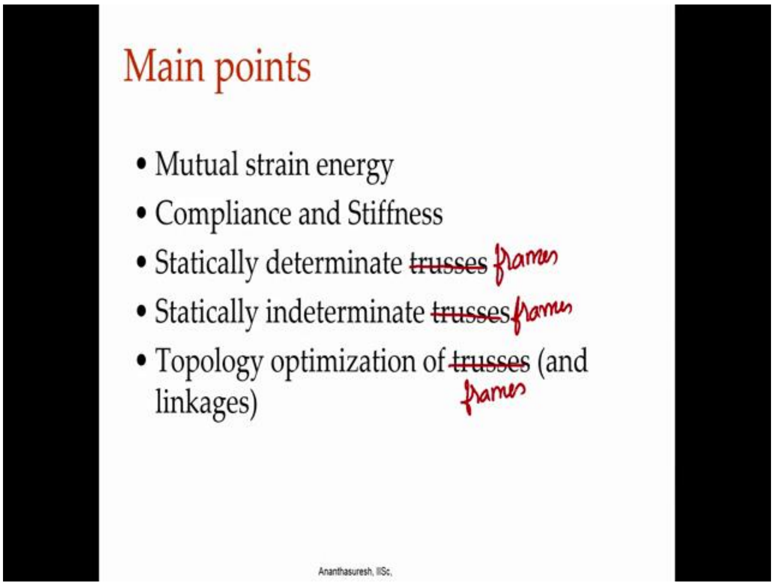

So, we can remove the things here and get the optimal topologies right. So, to conclude again we have use this mutual strain energy compliance and stiffness formulation, both statically determinate frames now not trusses and indeterminate also we can handle we can do topology optimization of frames which will also little linkages as we will see in a future lecture.

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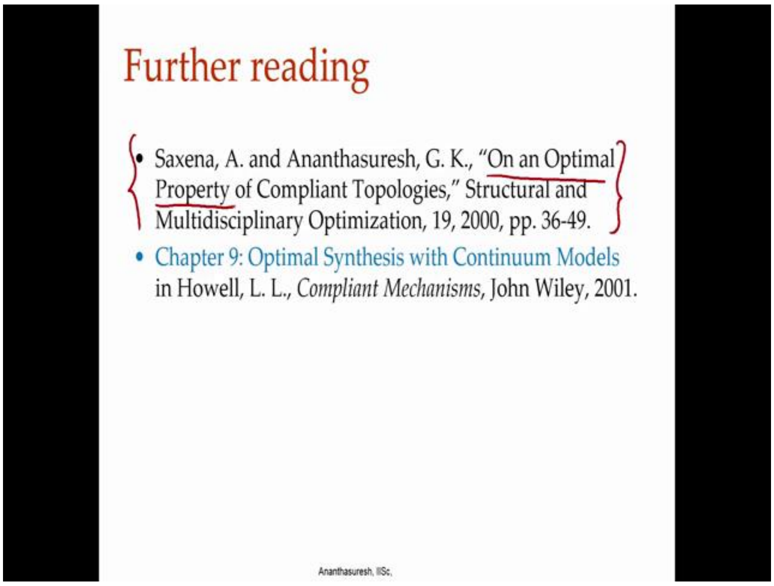

And the reading will be from this paper that had considered way something and optimal property which also we will discuss later to see, how we can generate these examples which are taken from this paper, where we can say what is the property of such optimal topologies it will have a ratio of two things constant throughout the structure; as we need discuss in the future lectures.

Thank you.