Compliant Mechanisms: Principles and Design
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Lecture – 27
Design for deflection of trusses

Hello, continuing with what we discussed in the last lecture, which was about how to pose the optimization problem for Designing Compliant Mechanisms. And 1 concept that we looked at was this mutual strain energy put forth by Sheldon Prager, which started in a paper written by R.L. Barnett, as early as 1961. So, let us look at an expansion of those ideas in the case of truss designed first later we look at beams and then the continuum the next lecture today we look at truss design for compliant mechanisms.

So, again the key word here is desired deflection, that is what differentiates it from stiffs structure design we are designing trusses not to make them stiff, but we want to have some deflection at particular points to be prescribed by the designer and the algorithm or up mission method should obey that constraint and give a solution.

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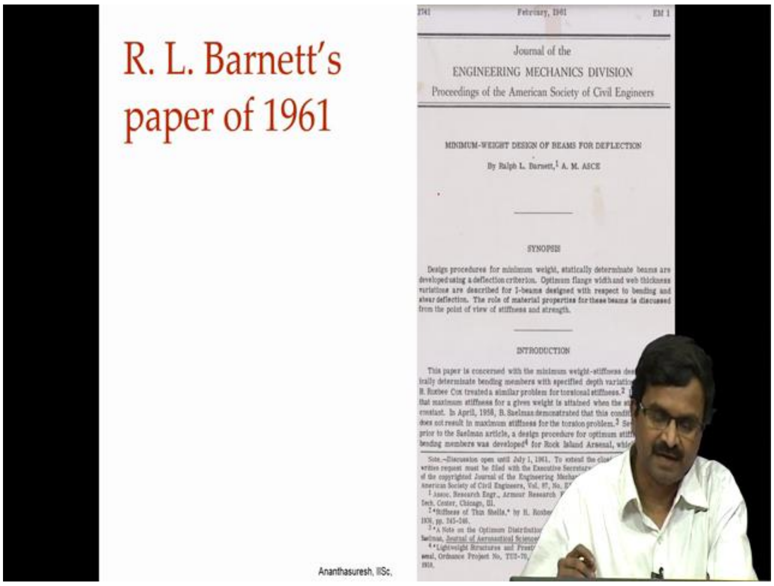

So, ones again lets recall that this 1961, paper by R. L. Barnett has pose this design for deflection this earlier paper.

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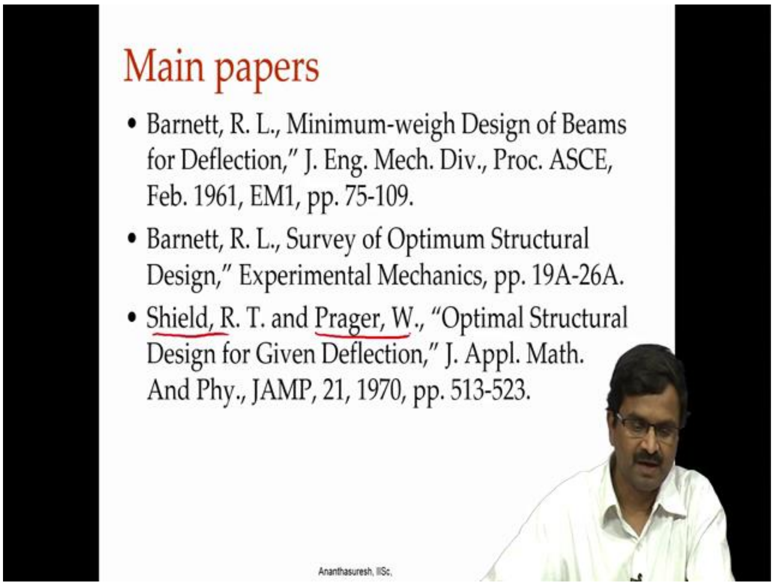

This was followed by A 1 of his own papers little later, and Richard Shield and William Prager, basically made it more formal by defining the concept of mutual strain energy which will discuss in little bit more detail today as compare to the last lecture.

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And there also A number of papers that came in 1970, basically if you look at the years of these everything 1970, 71, 75 and there where many papers which were mostly related to the group of William Prager.

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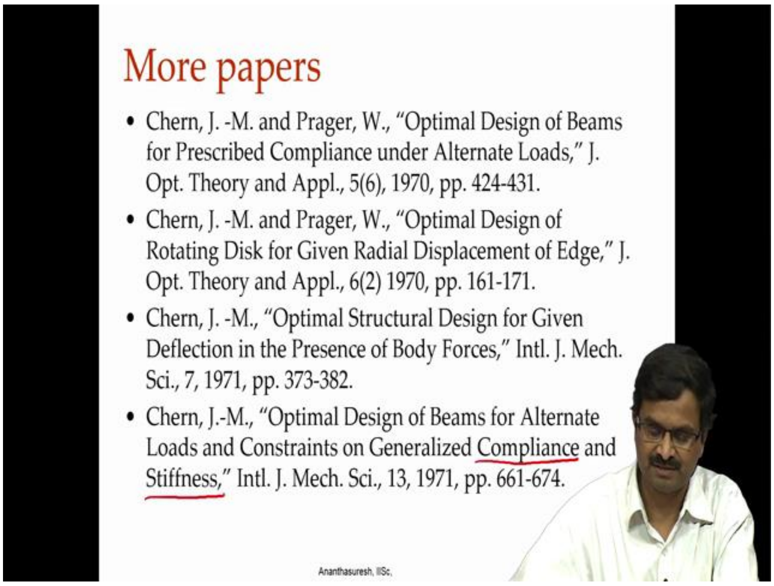

And also Chern again coauthored with Prager we have that looked at this design of mostly beams and disks regular structures for desire deflection prescribed deflection including body forces as you see in these papers, that forms the basis where you would like to design for both compliance as well as stiffness.

This is an idea that we will discuss in detail, why do you want to worry about stiffness when you are designing for compliance. By compliance we understand it as flexibility or deformation when you are doing that why should be worry about stiffness also this was something that we had left out at the end of the last lecture, today will discuss at an little bit more detail.

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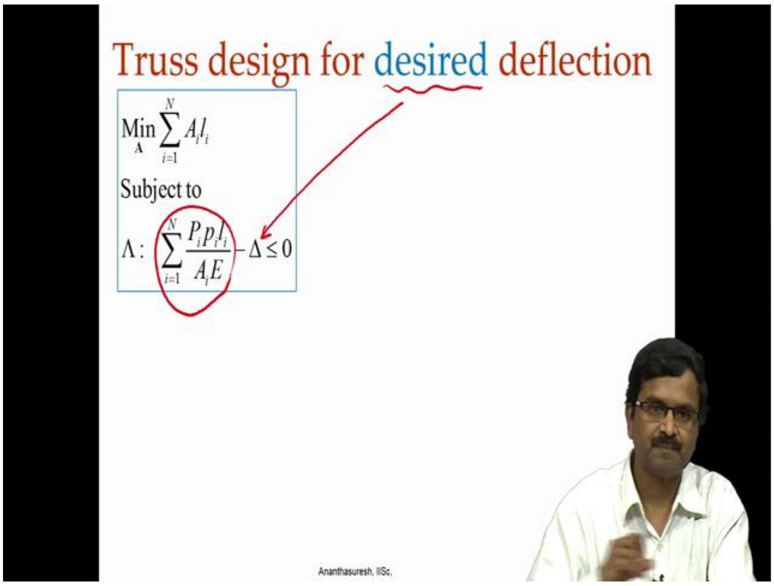

So, we considered this problem in the last lecture where again the key word is desired deflection, that is your delta at some point in the truss we have big truss we have there are loads acting they can be whatever, kinds of load they can be body forces they can be just loads that are point forces a different points in the way decease or nodes of the truss 1 degree of freedom we choose and say that at that point we want certain deflection. Here it is posed as an inequality constraint with delta be in the upper bound of the deflection ok so, over here if you see this is some set we should take note.

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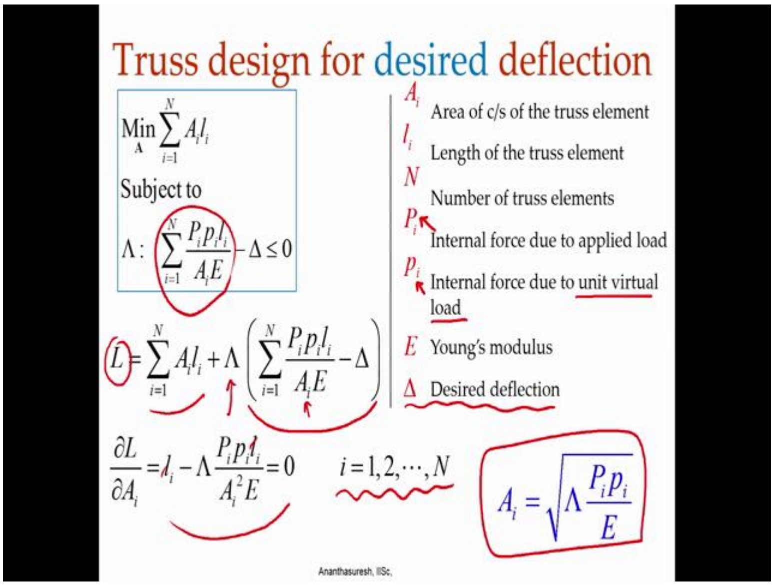

And this is what we call mutual strain energy, but before that lets look at various variables that are used here, A i stands for area of cross section of the i-th element, and L i is A length of the truss element, n is the number of truss elements that we have and P i, capital p i is the internal force due to the applied load, and small p i is the internal force in the i-th member, due to unit virtual load and E is youngs modulus of material that the truss elements are made of, and delta is the desired deflection.

These are the variable that are in the problem and we also solve this problem and we call this mutual strain energy, will get to that and given optimization problem we write the lagrangian by adding the objective function and lagrang multiplier times the constraint itself and then we write the necessary condition, which is this lagrangian gradient with respect to the area of vector this capital A is A 1 A 2 A n. If you take any one A i then dou L by dou A i will give in expression that looks like this we are taking simply derivative of the L, that is given over here and when you do that the first step we only give L i, because it has A 1 A 2 all thus A 1 times L 1 A 2 times L 2 all that sum we take there with respect to A i we will be left with L i the other one since, A i is the denominator that is that we get lambda capital p i small p i L i divide by A i square E that is equal to 0 this is true for all is i equal to 1, 2, n from here.

We can actually get a close form expression for area of cross section. So, A i is square root of this lambda times p i p i by E i. So, that is what we get and L i of course, gets canceled in this equation if you see this L i is here or here, this expression we raised concern last time that what happens to this p i p i where it is negative.

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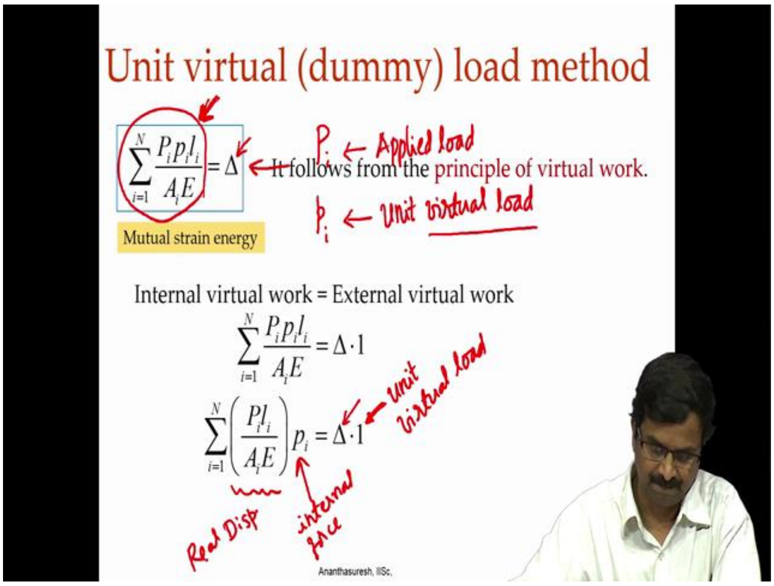

So, before we discuss that lets look at what does this mean right. So, we are saying deflection at a particular point or a degree of freedom is given by this summation, that summation the quantity that is there is called mutual strain energy. This is something that the paper that I showed put forward, put for, and put forth in 1970 time, why it is called mutual? Because there is capital p i and there is small p i as we said this is due to the actual applied load this is the internal force in the truss members and other is due to unit virtual loads, sometimes it is also called unit dummy load method that, dummy meaning it is not there, but you just imagining.

So, whenever you see the word virtual immediately you connect saying that it has under within principle of virtual work. In fact, this mutual stain energy stems from the principle of virtual work. So, what does principle of virtual work say, it is says internal virtual work is equal to external virtual work. So, what is internal virtual work here, we look at the internal forces and multiplied by the displacement. So, what is the internal force here in a way we can look at this as let me actually see this. So, this one is the displacement of each truss member, because when you have a truss member we know that the deflection of that is given by P L by A E right.

So, that is the displacement or deformation of for the truss member i-th member, because you are summing from I equal 1 2 n that is the displacement or deflection it will be extension or contraction of the truss element multiplied by this small p i, which we can call the internal force, internal force due to the virtual load that we are applying unit virtual load the unit because we are putting that 1 here.

So, internal work, virtual work is internal forces doing work over the actual displacement real displacement; this is the real displacement due to the applied forces. So, we take internal forces multiplied by the corresponding real displacement that gives you the internal virtual work and that should be is equal to external virtual work and delta is at that point, where we are interested in finding the displacement that is displacement there over there we apply this unit virtual load right. When you apply the unit virtual load we get the external of a these external system now this internal that is what it is.

So, if we see if you write delta here and then put this is displacement, this is energy because it is like p square by A e we have units of energy here you have 2 forces P i, capital P i small p i how can energy equated to displacement the reason is that there is actually this force here, unit force delta times that force will again have units of joules that is work on energy left hand side is also energy. So, this displacement being shown as a summation such as this comes from principle of virtual work as simple as that.

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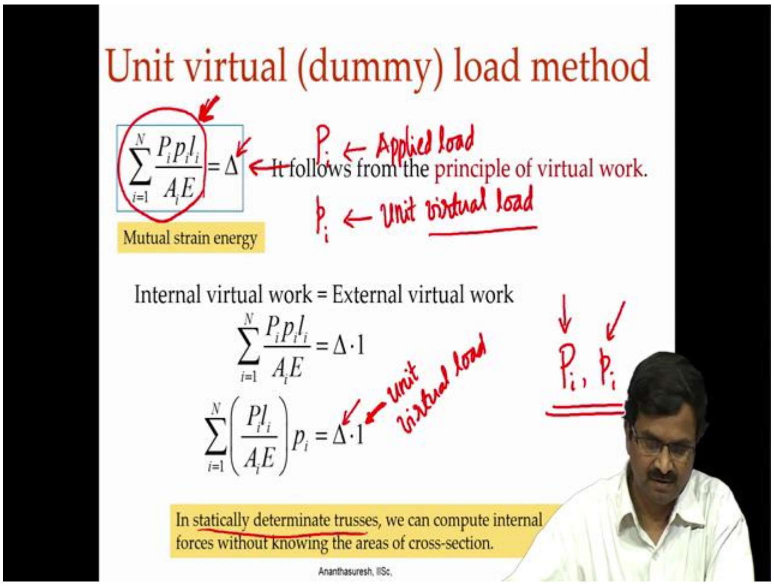

Now, this is true for both statically indeterminate things as well as determinate trusses; however, computing this capital p i or small p i for given loads this is for actual load and this is for the unit virtual load, when you apply it on the truss right you need to compute this member forces internal member forces that is possible to do without knowing the areas of cross section.

If the trusses statically determinate if it indeterminate you cannot do that here, when we pose the problem we assume that the trusses statically determinate where we knew that we can get this p i's and small p i's in capital P i's without knowing areas of cross section, because we are synthesizing the truss using cross sections as variables you do not know them. So, you should able to calculate them that are possible first statically determinate trusses.

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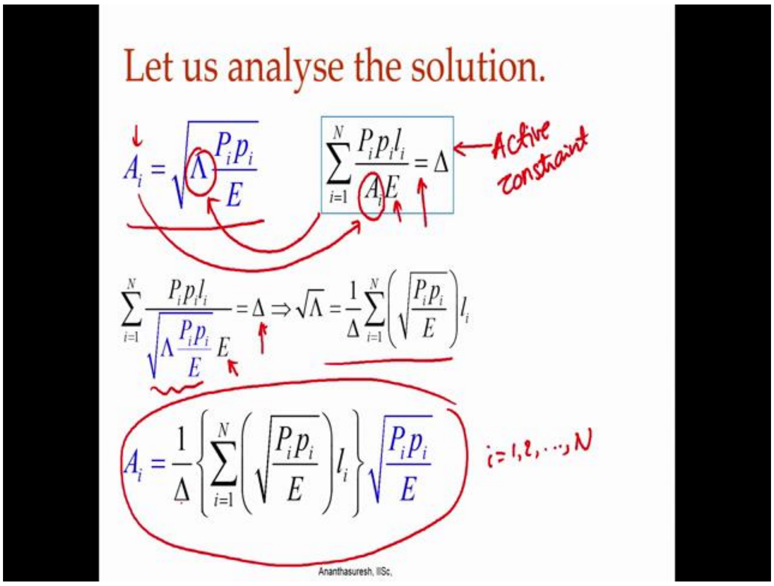

Now, let us analyze a solution, we have area of cross section given like this and we still have the unknown lambda and that can be solved using the constraint itself, the constraint is active, if it is not active meaning that if you use the complementarities condition here. So, will say this lambda times the constraint is equal to 0 either the constraint is 0 or lambda is 0 that is the complementarity, if let say constraint is not active which is what we have shown active meaning there is equality sign here the constraint is active rather than being less than or equal to now we say it is strictly equal to if it is not then lambda had to be 0 in which case area will be 0.

So, that wants bake sense, so we say constraint is active. So, if want to find this lambda we have to use this active constraint meaning this expression that we have is equal to delta that is precisely the degree of freedom will be equal to whatever delta that is specified or prescribed. So, we put that into our equation so we have this constraint here we have, A i this A i expression that we have whatever A i that we have will going there which is exactly what is shown in blue color over there.

So, sigma p i capital P i small p i and L i divided by A i, which is square root of lambda capital P i small p i by E, and then E is there over here, that is as it is that is equal to delta. Now if you work it out square root of lambda will be given by this expression which is what we need to come put back into A i area of cross section when you do that finally you get a solution that looks like this.

So, this is now everything we know proscribe delta, because we starting determinate truss you know capital P is and small p is you already know the lengths of that truss elements and we know the youngs modulus and we know the number of truss we can get area of cross section for any members. So, this goes from i equal to 1, 2, n all of them close from solution right, we get the solution without in iteration or optimization search it need it here.

So, it is just nothing is needed we have a close form solution for this you can also see that as delta is more area of cross section b less and when E is more again you have cross section is less and so forth. We can see what happen if it is a stiff material process flexible material you want large deflection was a small deflection we can see how area of cross section changes.

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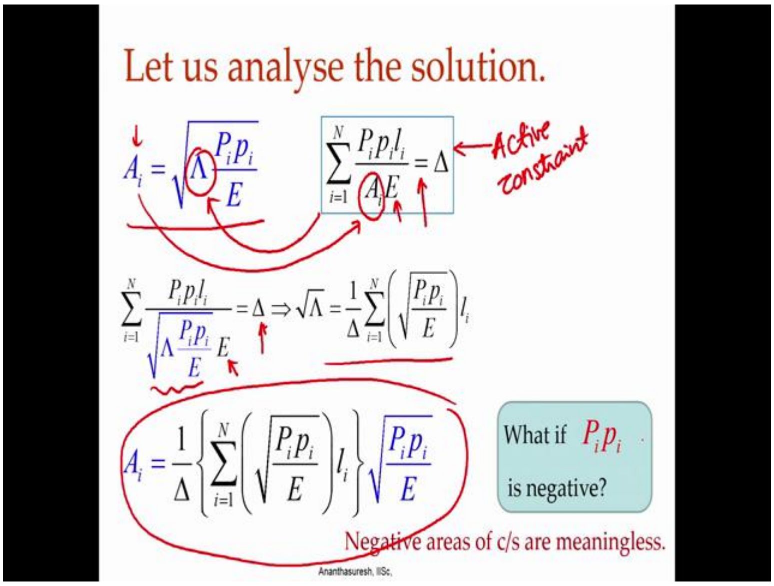

Now, what if this product P i p i is negative because it happens to be in the square root sign and that can very well be negative because capital P i small p i are due to the real load and unit virtual load depending on where you want the displacement prescribed that can lead to at some point capital P i small p i having opposite signs because, in a truss each member can have either compressive truss or console truss that is internal force is negative or positive.

So, you can have any sign so what if this P i p i product is negative. So, what do you do in that case, does it mean that we do not have a solution for the compliant mechanisms that we are trying to solve or the truss if you think? So, that is the question that we need to have because negative areas of cross section are meaningless. So, that is the problem so, when you takes stiff structures there is unique solution when you try to minimize the strain energy, but when it comes to compliant mechanisms we have this problem, which was noticed by Barnett when you are trying to do this design for deflection for some values of deflection you have a problem that you do not have a solution.

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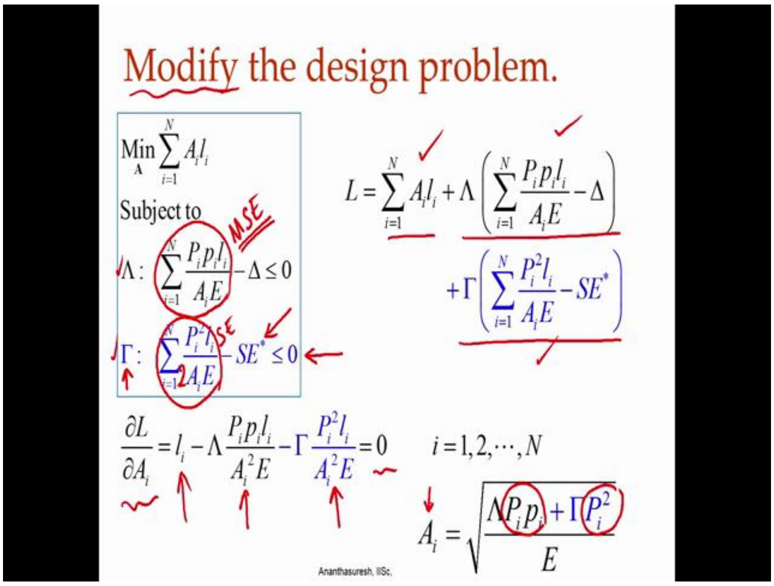

So, in those cases the affix as to be suggested that this came much later where we include in additional constraint now we modify the design problem. So, that original problem which is in black here, we put another constraint saying that this strain energy now you see why we call this mutual strain energy MSE for short whereas, this one is strain energy because its p square L by A e and since, strain energy there should be a half also that 2 will be there this is strain energy, where the mutual strain energy does not have 2, it comes from two different load cases actual load and then unit virtual load whereas, this comes from the main applied load that is strain energy.

So, when we put this we see how the problem that we raise that is the product of capital P i small p i being negative goes away, because whenever we pose an optimization problem which ensure that. So, solutions exist and we can find it. So, previously we could not get a real solution because amplitude becomes imaginary. So, that problem had to be modified and that is what this does.

So again for this problem if you are at the lagrangian, objective function the first constraint second constraint which is in blue corresponding multiplier, which is this gamma that we have put and you take dou L by dou A dou A i and equated to 0, because that is necessary condition. So, again you take the derivative with respect to A i all this expression first one, second one, and third one we get three terms as shown here earlier the first and second we already had last time itself. And now we can the new constraint you get an extra term it is similar its P i p i it is p i square. When you do that you get an area of cross section as it is shown at the bottom of the screen, where A i is square root of lambda times p i capital P i small p i plus gamma times capital P i square by E. Now notice that this is square; that means, this always positive an according to optimization theory the Lagrange multipliers have to be non negative. So, there not going to be negative why the.

So, if you choose gamma, you do not choose gamma rather we choose the strain energy upper limit if we choose at appropriately the gamma can be made large enough that even if this capital P i small p i product is negative you can make it overall positive. So, area of cross section remains meaningful meaning that we get a positive area of cross section.

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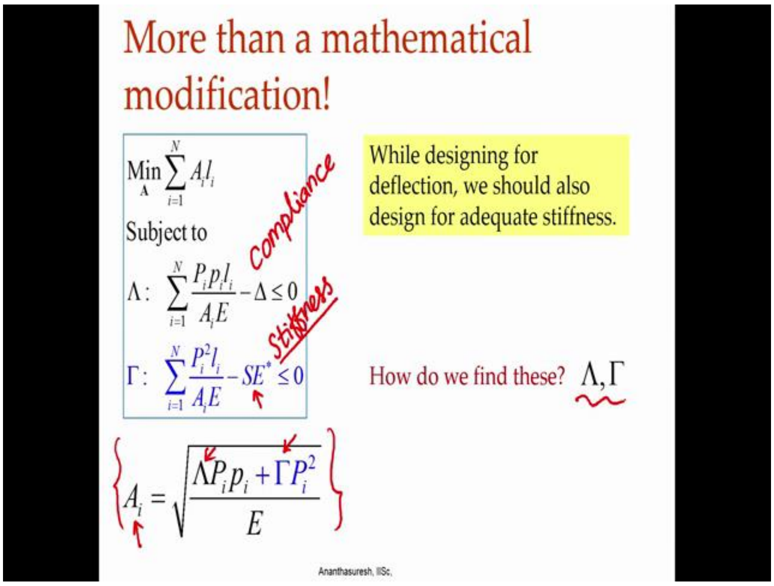

There is a trick it may look like it is only a mathematical theme, but actually it is more than that it is not just mathematical this modification, it is to do with accounting for compliance which is our flexibility requirement in terms of deflection at a point at the same time we have stiffness. So, when you design a compliant mechanism it does not make sense to have a lot of deformation, which you could if you want you can get infinite displacement by making some crucial part of the structure very flexible or almost nonexistent with we get a lot of deformation that is not the point compliant mechanisms also have to resist certain output forces.

So, we need it we need them to be stiff also this should be compliant and this should be stiff and that is what this little problem captures very elegantly. So, we need to put a constraint on the stiffness. So, that we get meaningful areas of cross section and this gamma has to be large enough which is actually controlled by S E star, because just like we used this constraint the deflection constraint to find lambda now we need to use a strain energy constraint to find gamma.

You have two unknown constraint to find now, because we have this lambda and also this gamma and we have two active constraint again if it not active, gamma will be 0 right meaning that its strictly less than 0 not equal to or less than or equal to, then will have gamma equal to 0 then you are back to the old problem that is p i pays negative again we are need to this. So, both constraints have to be active so that we can get reasonable or meaningful areas of cross section.

So, while designing for deflection we should pay attention to stiffness and not just for complaints that is the treks of this simple problem that we have, and this what happens to be in all compliant mechanism problems we not just designing them to be excessively flexible they have to be flexible at the same time they have to be adequately stiff in order do the job. So, now how would you find these two things lambda and gamma.

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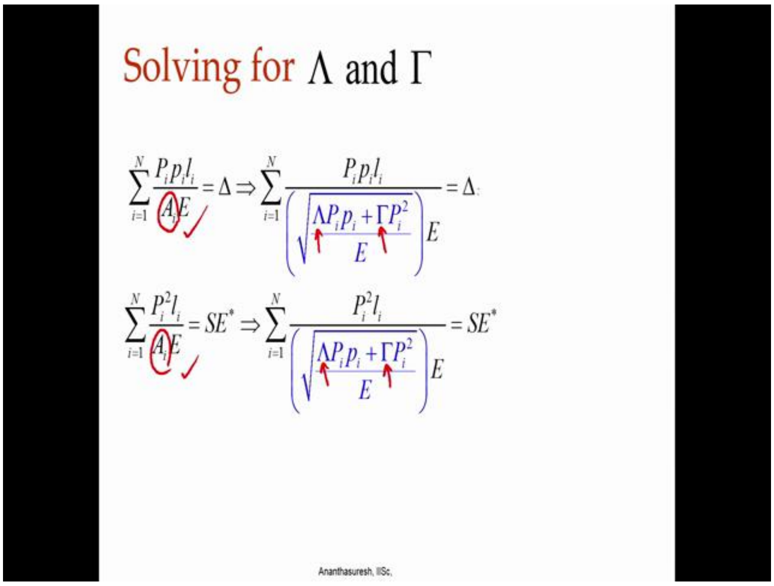

So, for that as we just said we have the deflection constraint and we have the strain energy constraint both are active, like we did before for A i in both of these things we substitute the solution which is still in terms of this lambda and gamma in both of them, but there we have two equations two unknowns we should be able to solve.

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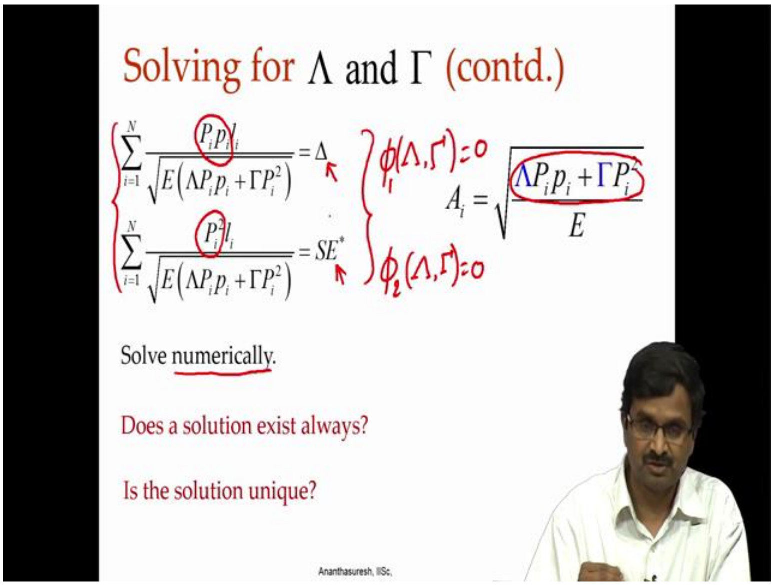

We simplify this a little bit, not a whole lot we get equation that look like this summation capital P i small p i times L i divided by square root of E times lambda times P i p i plus gamma p i square that is equal to that summation equal to delta, likewise the bottom 1 the difference is only now this as p i square instead of capital P i small p i that is equal to S E star, and looking at these equations we see that in terms of lambda and gamma these two scalar equations are non-linear.

So, you cannot get an easy close form solution now solve any problem when you have equation, the first question you ask is like solutions exist always right. In fact, it need not be because even after choosing S E star some value there is a some values that prescribed, we may not be able to make this whole thing positive for all elements right, we have to choose appropriation energy for a given deflection. So, you should know what the strain energy is if which appropriate value then there will be there, but arbitrarily given things it may not exist. We have to solve the problem numerically. So, when you solve numerically or when you

Now, when it exist is the solution unique that is also not true here because, these two are non-linear equations we solve the rest of it analytically now, except that these two equations are not solved analytically, because there non-linear in terms of lambda and gamma, which intern depend on the values of delta and S E star delta prescribed deflection S E star prescribed upper bound and strain energy depending on what those value are these two non-linear equations.

I can call them phi first equation phi 1 lambda gamma, because everything else we know that is one non-linear equation other is phi 2 lambda and gamma equal to 0, when you have non-linear equations they need not have solution are a 1 solution it have many solution also here, it turns out that there can be many solution also when they are there sometimes they may not be there if you do not choose this delta and S E star and applied load and the degree of freedom where you want deflection if that all not properly chosen, which as a desire any would never know because you trying to find the problem.

But whatever requirements you have as designer they may not translate into a realistic solution, that those are the two things we need to take into account that the solution may not exist or in a solution exist they could be multiple solution. In other words the problem is non convex unlike this stiffness optimization problem.

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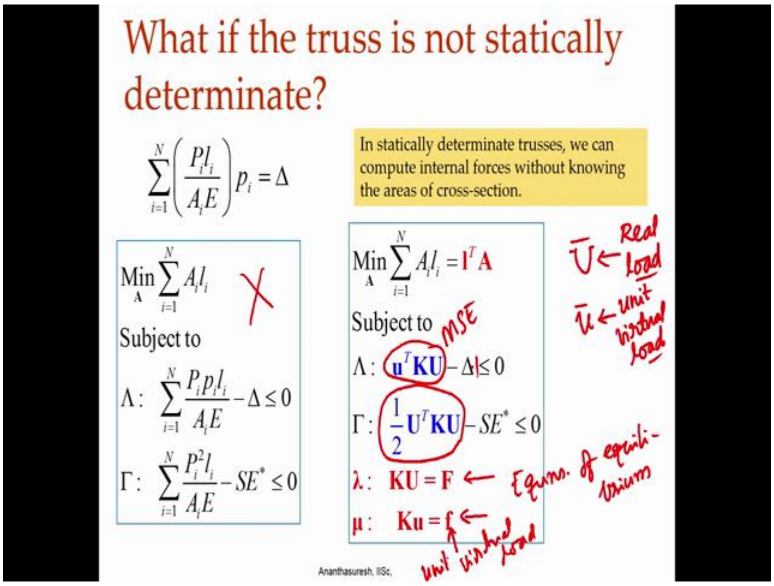

Now, let us look at the other case that we had set aside the beginning that is of statically indeterminate truss, in which case as we already said this capital P i and small p i that is internal force in the truss cannot be calculated without knowing a areas of cross section that is what starting indeterminacy. So, without solving for deflection you cannot compute the internal forces from equation statics alone that is a static indeterminacy, if you want to find deflection you need to area of cross section, which we do not know here right so what do we do in that case.

So, this was the original problems statement now this is not adequate whatever we had done earlier this P i p i we do not know how to do it, because the truss we are constraint now is statically indeterminate, in that case what we do is without having to compute we cannot write p i and p i.

So, we do not write in terms of p i and p i instead we consider the displacements. So, now, we have this capital U and small u both are vectors when we show them in bold we mean they are vectors or arrays, column arrays rows depending on situation is. So, this is the displacement due to applied real load whatever load that is there in the truss with which we want deflection at a point to be some delta constraint there should real load capital U, whereas small u is unit virtual load and this unit virtual load is applied at the degree of freedom at which we want to restrain the deflection.

So, real load virtual load come again that is what we have that we have two governing equations, because now if you see this you recognize as strain energy it is just like half k x square for 1 spring, now we have multiple spring that is what the truss is half u transpose K U is strain energy and small u transpose K U without the half is the mutual strain energy, which is equivalent displacement even there is strain energy we said there is always at multiplied by 1 that is the energy unit.

So, a mutual strain energy there is a small u due to the unit virtual load capital U due to the real load we get this problem pose that way, it became a little bit more complicated see which start of invoicing volume with a constraint at another constraint otherwise compliant mechanism synthesis does make sense and now if it is statically indeterminate we need to explicitly put the equations of equilibrium, these are equations of equilibrium which in the finite element sense at it very easy to do K U equal to F its like K x is equal to f for a spring now matrix k, which is stiffness matrix finds the displacement vector is equal to applied force vector and this is the virtual load unit virtual load, where there is load applied only at the point where we have displacement prescribed, that is what we have now.

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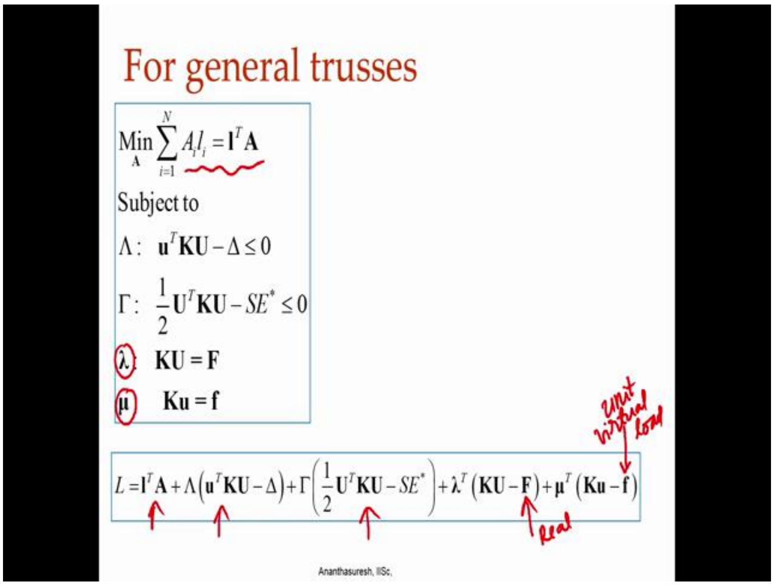

How do we solve exactly the way we had done earlier, we have the problem statement with all the things here. Now we write the lagrangian, which longer, in fact this also we try to do it in matrix vector notation, L is the length vector and then A is the area of cross section vector like in the product, such what it is done. Now we have the lagrangian written with objective function; first constraint on the deflection constraint on the strain energy.

And then we introduce this Lagrange multiplier vectors for these are seek, upper case are bold type lambda nu indicate their vectors again corresponding their (Refer Time: 28:02) vectors for the 2 static equilibrium equations; lambda transpose times K U minus F plus mu transpose K U minus small f again, this is the unit virtual load whereas, this 1 is a real load that is applied on the truss under which we want the deflection prescribed whether constraint satisfied.

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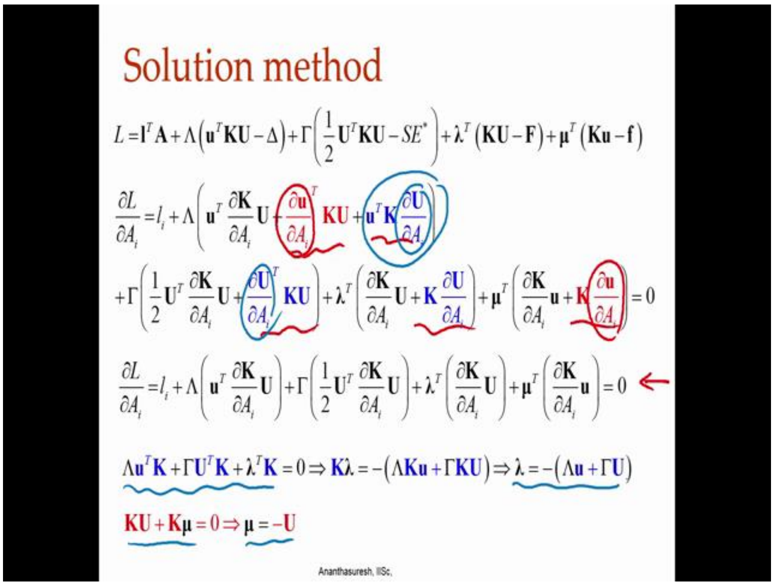

So, when we have this we are had the lagrangian and we have take the derivative with respect to each area of cross section dou L by dou A i dou A i and equate to 0, we can expand noting that k will have areas of cross sections stiffness matrix and then u small u and capital U they also depend on or influence areas of cross section you have to write all these derivatives lengthy things straight forward differentiation.

And then we group the terms now if we see have taken out the things that are in red and blue there missing in what I have written in this equation why is that, because the things that have red and blue the red ones have the derivative of the displacement virtual displacement with respect to areas of cross section, and the blue 1 have let me switch over to blue color here the blue 1 have derivative with respect to that dou u by dou A i which we do not want to compute. So, what we do is we take out those terms separately. So, here I have taken out three terms that are in blue in compact notation so from there we get this lambda this is can be called as a joint equation in such optimization literature. Similarly, with two red terms at here with you see mu. So, lambda and mu are a straight away calculated lambda is minus capital U mu is minus capital U lambda is minus lambda times u plus gamma times capital U.

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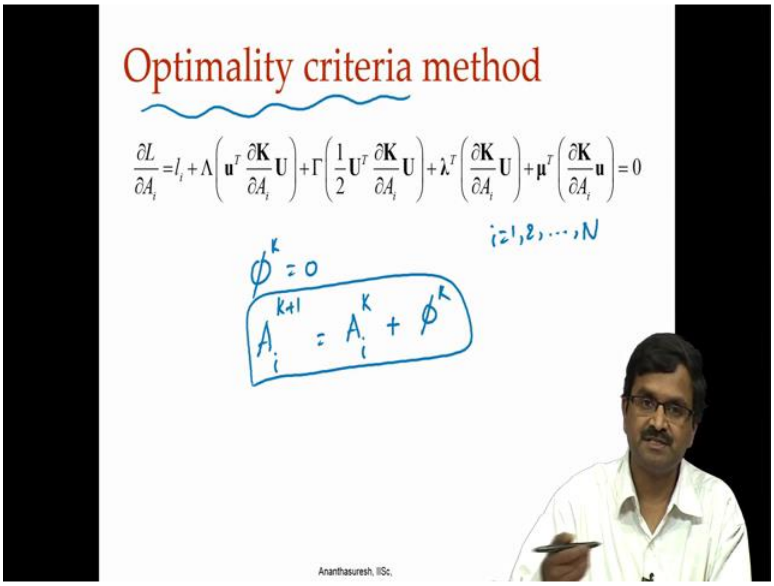

You substitute these back into this equation, again this equation goes from i equal to 1, 2 how many our truss elements you have and you get an equation when you say optimality criteria method what we do is if i have this expression, let us call that whole thing as some phi we want that phi to be 0. So, what we do is area of cross section at i-th element k plus first iteration we say whatever is there at k-th iteration plus phi at k-th iteration. So, this will be changing as iteration number k goes.

So, you update in this manner and there are some more tricks that are there in the optimality criteria method which you can learn from a class are structure optimization here we can update all the areas of cross sections in each iteration.

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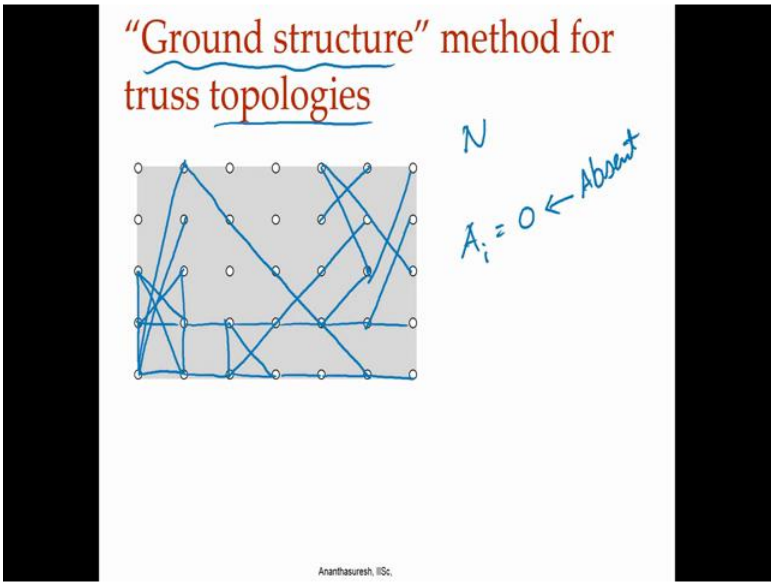

And get the optimal areas of cross section if we do that there is way to connect to what we can call topologies let us say that I have a design domain, now I want to connect them in all possible ways that is all the grid points I will collect, each 1 of them is a truss element there are many, many truss element that you can define all possible ways, and then I can connect this cross ones, I can connect from here to there, to here all quantum just fill everywhere right. So, we can do all kinds of elements everywhere we get a super structure or a ground structure. So, your n now will be all elements we can connect them in all possible ways right so you can do all the elements.

So, that we get a very dense grid of this truss elements and you optimize with the method that we discussed and somehow the areas of cross sections some A, I may go to 0, in which case they become absent in the final solution some other will be there they can never go negative we ensure that by adding appropriate constraints and S E star that is upper limit and strain energy we prescribe. So, we ensure that none of the is will go to negative they will be 0 sometimes depends on what you set you can also set upper bounds and lower bounds area section you get a solution, if you go back to the ground structure some of the elements if this particular element has area of cross sections which is close to 0 that disappears.

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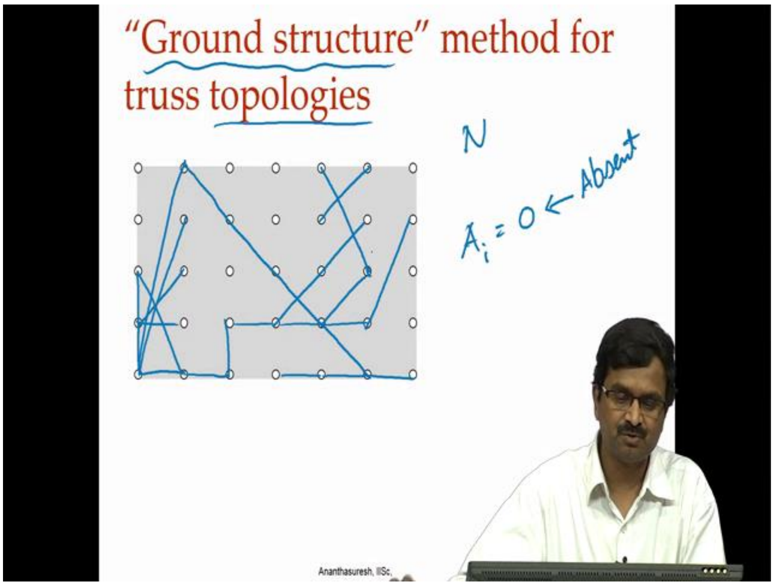

So, let us say this 1 also there that also disappear. So, by take a ground structure and eliminating these things you get the optimum solution. So, that way we can design trusses which also will act like mechanisms or compliant mechanisms, because we have prescribe the deflection delta that we want at a degree of freedom that is how this truss topologies optimization works ok.

We will see some examples later in next lecture we will consider beams to reinforce what we have done in this lecture and also with the continuum elements.

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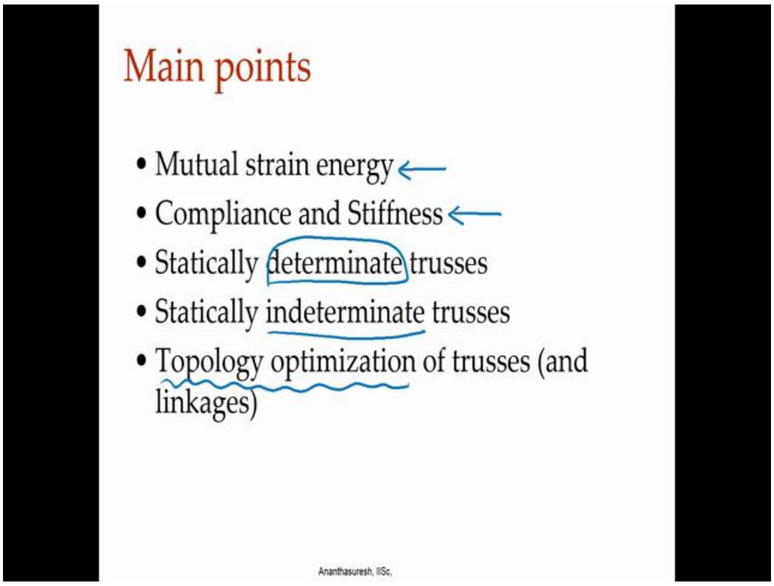

And then see some example. So, todays main points are that we need to understand where this mutual strain energy comes it follows from principle of virtual work, and we must note that when you design for flexibility or compliance as we call it we should also account for adequate stiffness we can handle statically determinate trusses as well as indeterminate truss.

This of course in a determinate trusses is easier indeterminate we have to consider the corresponding equilibrium equations and that joint method, we simply mentioned in passing today and also in passive there mention this topology optimization things can lead to optimize trusses which can be used as compliant mechanisms as well as linkages rigid body linkages also get come out of this method which is being persuade by some people today.

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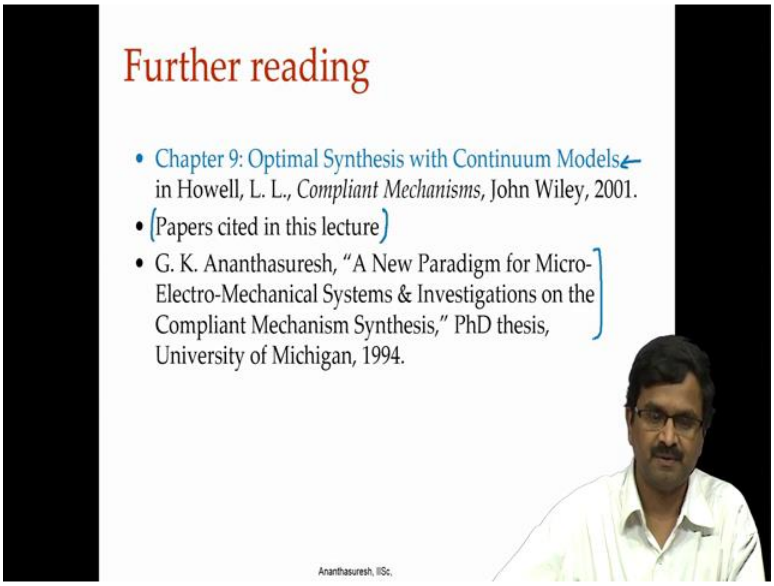

For further reading again like we said you know papers in this early papers are very important and also there is a chapter that gives you some incline and PhD. Thesis also describes all these things in detail in the next lecture we will consider beams in the same manner that we consider trusses.

Thank you.