Compliant Mechanisms: Principles and Design
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

Lecture – 24
PRB - based synthesis Examples

Hello, continuing with yesterday's our last lecture. We are going to discuss synthesis of compliant mechanisms using pseudo rigid body model where our synthesis equations are in two categories; one set is what we call loop closure equations their kinematic equations and the second category is force equilibrium equations. So, we have two sets of equations and we have to see which variables lie in which set and choose the free choices so that we can solve all these equations. So, let us look at the kinematic synthesis and kinetoelastic synthesis using these two sets of equations for function generation using two examples today.

Let us consider the first example of a function-generating mechanism using this concept of pseudo rigid body modeling or what you call PRB modeling.

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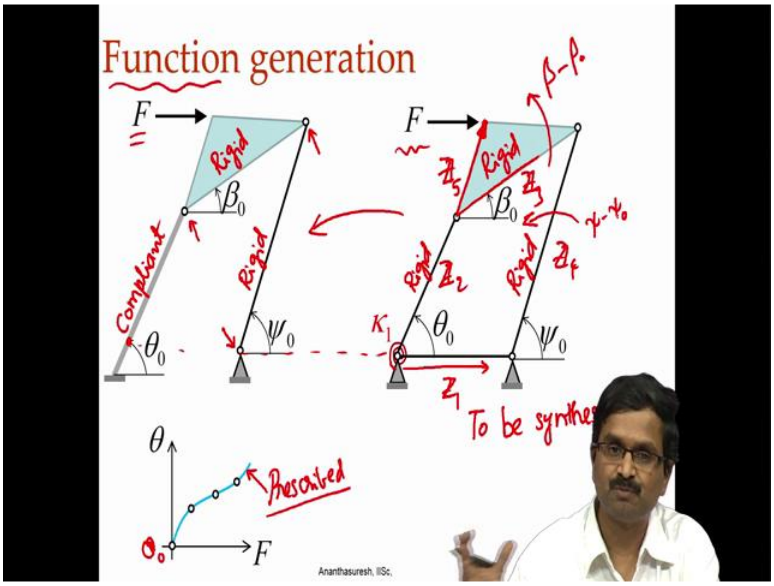

So, here what we have shown is a compliant mechanism where one link here is compliant. This is the one that is shown in grey which is the compliant element and this coupler is a rigid and so is this and this and of course, the fixed frame is rigid. So, this is the compliant mechanism and what we want is we want function generation. So, a function such as this one is prescribed. So, this is prescribed that is this is what is given to us we want to generate this function as we applied the force F at some point on the rigid coupler body and generate this function with thetas and F's are given to us we want to generate it. Theta again need not be here let us call this actually theta 0 it is not like it is 0 here, that theta 0 and other points how many over we want to take.

The first thing you do when a problem such as given is to have it is pseudo rigid body model; for that what we mean? We have this slender beam which we know we can approximate by putting this joint over there so that is what we are showing here. We shorten as per the theory of pseudo rigid body model and put a pin joint there along with a torsion spring; so which we have kappa 1 here. We do not have kappa 2, kappa 3, and kappa 4. In this example because they are normal kinematic Joints, there is no stiffness there; only because a cantilever beam is approximated like a rigid beam now this is also pseudo rigid, because you have torsion spring and this continuous to be rigid and this continuous to be rigid.

Now, this is our model for synthesis. After we synthesized the four bodies that we have; l 1, l 2, l 3, l 4 as well as kappa in the angles then we go back from here to here to construct a real compliant mechanism. First we synthesize this one; this is the one that is to be synthesized.

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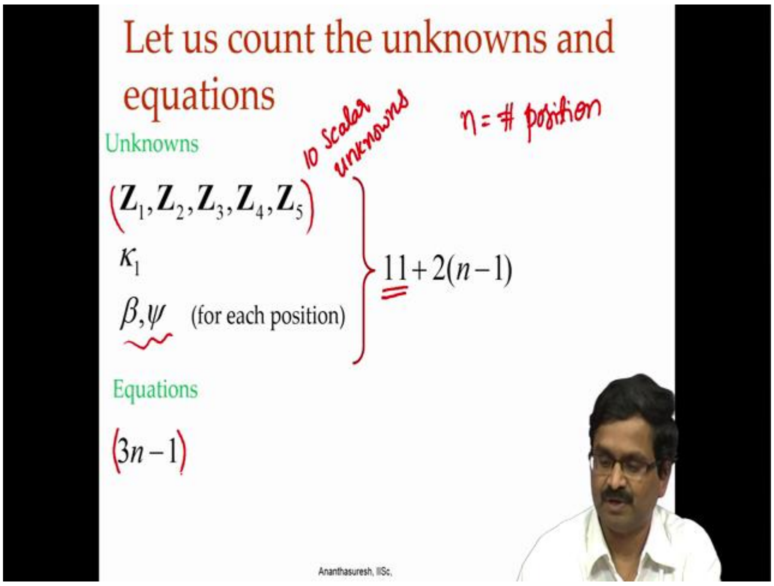

Let us count as we did in the last lecture the number of unknowns and Equations. What are the unknowns here? We have Z 1, Z 2, Z 3, Z 4, Z 5. What are those? If you go the previous 1; this is Z 1, Z 2, Z 3 and Z 4 and this is Z 5 that basically says how relative to Z 3 how Z 5 is located so that we have the point at which the force F is applied. Those are the 5 bodies that (Refer Time: 04:53) lines that we do not know; those five planar vectors represented as complex numbers. And then we have kappa 1 which is the Torsion spring constant in the pseudo rigid model; then we have these angles beta and psi; beta is the rotation of the coupler that is this one; rotation of that is beta minus beta naught and rotation of this is psi minus psi naught. Those are the ones beta and psi for each position of the mechanism we do not know. So, they become unknowns. For each position we have two. So, if there are n positions we have to subtract 1 because related to the first one how much are rotating.

So, 2 times n minus 1 and 11 because there are 10 scalar unknowns are here because there are 5 complex unknowns and that amounts to 12 scalar unknowns. And kappa is the eleventh one that is there to begin with and then we have 2 for each extra position, if number of positions is n including the first one which we denote as theta 0, beta 0. psi 0; we have this many variables here. This is our initial set up for unknowns. Now how many equations do we have? We have 3 n minus 1 equation again n is number of positions that we consider initial position being the first one n equal to 1 and 2, 3 and so forth.

Where do we get this 3 n minus 1? 3 n because each position has loop closure equation which (Refer Time: 06:46) two scalar equations and then there is force equilibrium equation also that is 3 but then why you are subtracting 1 because the initial position does not have any force equilibrium equation; because there is no force, it is just stable the way it is, it is just in equilibrium the way it is without any forces. So, we have 3 n minus 1 equation; 11 plus 2 into n minus 1 unknowns and 3 n minus 1 equation. So, we can make a table.

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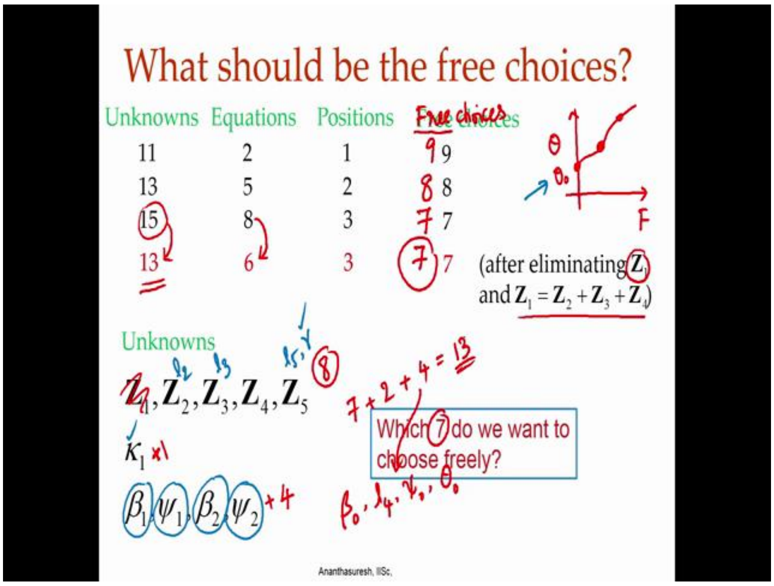

So, we have 11 unknowns, 2 equations. We will have 11 minus 2; 9 free choices. So, if I do this; there will be 9 free choices. Meaning that, we have excess variables so you can assume the value for those variables. That is what it means what is meant by free choices. And if you go to 2 positions then equations increase by 2; 11 becomes 13, unknowns become increased by 2, 11 becomes 13 and equations 3 are added 2 plus 5 3 5 and there 13 minus 5 will have 8 free choices.

If I go to 3 positions that is in the curve where we have theta and F; if there is a curve let us say if I actually show theta 0 here if I show a curve like that I can take this is a first point, second point, third point and so forth where 3 positions for unknowns become 15; 13 plus 2 and equations become 5 plus 3; 8, so a 3 positions 15 minus 8 will have 7 free choices.

And now what is shown here in the red is after eliminating, because we do note that out of these 15 we also have Z 1 as an unknown and there is an equation. So, if I reduce this by 2 and this 2 we are simply setting aside Z 1 and this equation. No loss of (Refer Time: 09:06) there. So, we have now for 3 positions again 13 minus 6 will have 7 free choices; that does not change except that we knock off Z 1 and that equation. This is how we get; now we have 7 free choices.

So, which one we would take them as free choices to solve, because know we have if I set aside Z 1 we have Z 2, Z 3, Z 4, Z 5. There are 8 scalar unknowns there; this is plus 1 and this is plus 4 total we have 5; 5 plus 8 13. Our equations are 6, so we have to solve for these 13 things we can assume Solve for the remaining 6 using the 6 equations. We have to always balance; which 7 of these 13 that we have to be choose; that is the question.

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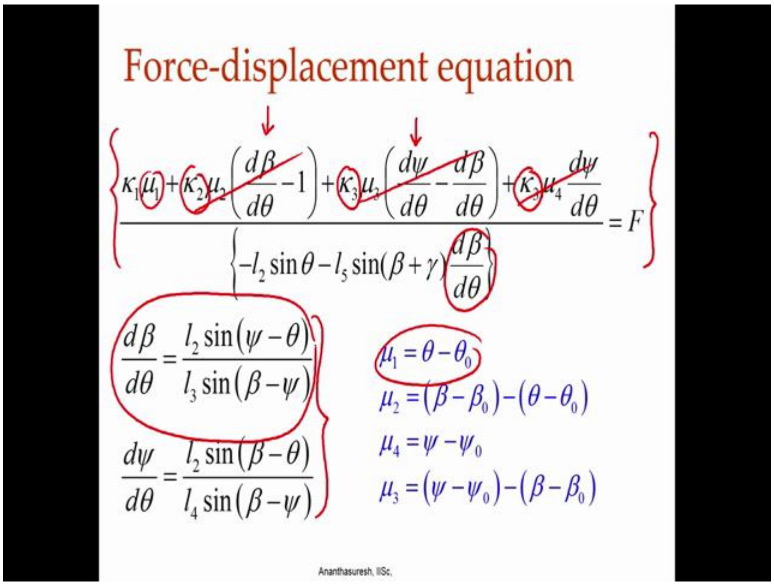

For that we need to look at the equations; the nature of equations. In the last lecture we noted that the kinematic loop closure equations tend to be linear in terms of this complex number Z 1, Z 2, Z 3, Z 4 that indicate the planar vectors along the line joining the two joints of the 4 bar linkage. In terms of that it will be linear provided we know the angles, but force displacement equation is inherently non-linear as you can see here. This is the Force-displacement equation for general position.

If it is the first position then we have to put psi 1, beta 1 and so forth. This is a thing that we taken from an earlier lecture where we used principle of minimum potential energy and gotten this equilibrium equation for static equilibrium. So, here we have this d beta by d theta, d psi by d theta. These are the things that we can determine has shown here. These are the kinematic sensitivity relationships for a 4 bar linkage to see that at any instant at what rate beta and psi would change little little to theta; theta is the input 1 and beta and psi are coupler and output things in terms of lengths of the bodies as well as the angles in that configuration.

We also know this mu 1, mu, 2 mu 3; mu 1 is theta minus theta 0 rotation of the input body; mu 2 is the change of angle between the input body and coupler that is why it is beta minus beta naught which is the rotation of the coupler minus rotation of the input body which is the theta minus theta naught then we have mu 4 which is a third angle that we have between the coupler and output psi minus psi naught minus beta minus beta naught and then mu 4 mu 3 and then mu 4 psi minus psi naught which is the output bodies rotation.

All of them will not be here in this example; kappa 2, kappa 3 and kappa 4 are actually not there because in the pseudo rigid body model if we can go back and look at it quickly we go back to our thing we have only kappa 1 and not these 3; so that gets simplified a little bit for us where we have these. So, these terms will go away because corresponding kappa's are not there; (Refer Time: 12:58) this mu 1 we can substitute this here and then for d beta by d theta we can substitute this one. If we do all that we get our force displacement equation as shown here.

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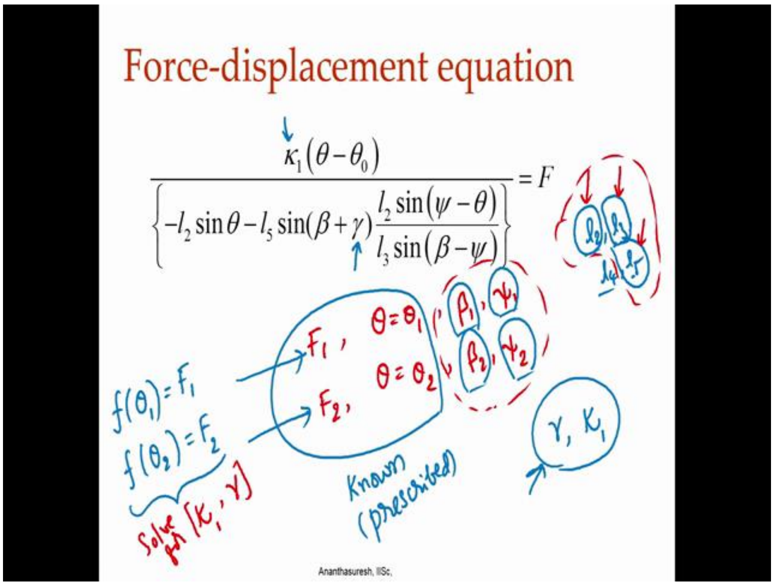

Kappa 1 times theta minus theta naught divided by long expression that involves l 2, l 5 and then the angles and gamma also. Gamma is part of the Z 5 that is coupler point we have to locate we have the force to be applied.

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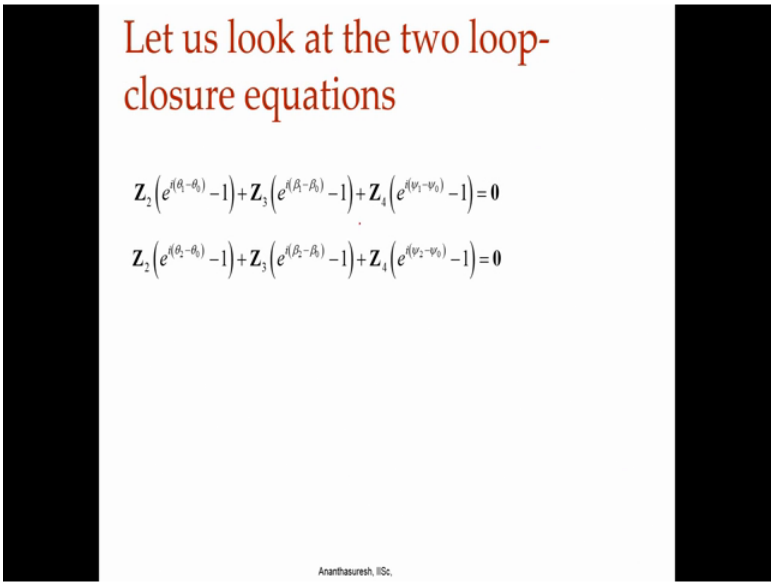

Now, if you look at the loop closure equations; we have two of them. So, Z 2 times something Z 3, Z 4 they all involve angles. So you have to look at the force displacement equation this has to be instantiated for theta equal to theta 1 and then theta equal to theta 2 and corresponding thing it is theta 1 will have F 1 and then F 2 and correspond to theta 1 we will have beta 1 and psi 1 beta 2 psi 2. Anyway thetas these things are known to us.

Let us see what we know; these are known to us because they are prescribed in function generation these things are prescribed. These angles we do not know, we do not know this gamma, we do not know this kappa, and we do not know this l 2, l 3, l 4 and l 5. So, if I want to solve this equation; we need to see how many of these things that we can specify ourselves. Here in this problem we have 7 free choices. So, if I count 1, 2, 3, 4, 5, 6, and 7 of them, I can put in if I do that then what I would not know will be this l 5. If I choose what I would circle, let us I say I choose these 7; 1, 2, 3, 4, 5 let us say actually choose this one let us say 1, 2, 3, 4, 5, 6.

And then we can choose l 2 by l 3 is there. I can choose l 4 is not in this thing here. I can choose l 3 also; see l 4 is missing. Instead of choosing l 4 we will choose l 5. If I take these 7 then what will be left out will be l 4 and then we also will have gamma that we do not know then will have kappa 1 that we do not know.

So, out of the variables that are there we would have chosen 7 that will leave us l 4 which any way fortunately for us is not in this equation. So, we are left with only gamma and kappa 1. This force displacement equation we have 2 equations; one correspond to F 1, another corresponds to F 2. So, this long thing if I were to write f of theta 1, so it will be theta 1, beta 1, psi 1 and F 1 that will be one equation. Another one f theta 2, beta 2, psi 2 and F 2 if I have these equations once we assume what we have circled that is beta 1, psi 1, beta 2, psi 2, l 2, l 3 and l 5; these 7 are free choices we assumed them conveniently. That is our free choice so we can assume whatever values that are suitable for our problem that is the usual thing in kinematic synthesis based on this Burmester Theory.

So, we can choose them once we choose we are left with 2 variables which are here; gamma and kappa 1 we have two equations. So, these two will give us the values of kappa 1 and gamma. So, we solve for them solve for these two. So, here force displacement equations can be solved independent of loop closure equations. Sometimes we get 2 set of equations as we said at the beginning of the lecture that you get kinematic equation loop closure equations one set, and then force displacement equations sometimes they may be coupled; sometime they may be weakly coupled, sometime strongly coupled, sometimes not coupled at all.

So, here we have a case where force displacement equation even though they involve some of the lengths l 2, l 3, l 5; l 4 is missing. So, we have these three lengths and the angles. They are kinematic variables and we are able to solve the force displacement equations by assuming some of these so that we can solve for this kappa 1 and gamma. 2 equations, 2 unknowns we can solve them. There may be little non-linear because this is linear kappa 1 where as this is sine beta plus gamma is there, but we can figure out how to solve these two because 2 equations, 2 unknowns it is not difficult to solve them. We have to solve numerically, but that will be just fine.

You can change the values the 7 parameters that we have according to the convenience when we solve this variables kappa 1 and gamma; if they go too out of range you can always change values of our free choices. Once we do that we will have our kappa 1 and gamma. Once we do that we go for the remaining ones; let us take a look at what are remaining.

If you look at our variables now; we assumed these things whatever we assumed let me circle them in blue. So, we assumed beta 1 and we assumed psi 1 we assumed beta 2, psi

2 and then we also assumed along with these angles in a way l 4 we did not assume; there is l 2 we assumed and anyway theta 0 is known to us this is a function generation problem. So, l 2 we assumed theta 0 if I know Z 2 I know and l 3 we assumed then l 5 we assumed and then use in the force equilibrium equations we solved for gamma that completes Z 5, Z 5 is we have the coupler then we were what locates the point where force is applied, l 5 and gamma give that, l 3 if we have beta 0 then that would be also computed. So, the things that are remaining would be l 3 what we do not have is beta 0 and the psi 0 and we have we have solved for this and that using this. So, our total 13 unknowns we have 7 and then 2 we have solved 9; four are remaining; four remaining ones will be let me write them in red.

The remaining ones; l 3 we do not know that we call it beta 0. So, we do not know beta 0 where as beta 1 we have assumed and then we have Z 4 that contains l 4 as well as psi 0 that is there that is 1, 2, 3, 4 and here assumed l 2 so that as theta 0 still that we do not know that is arbitrary; we can we can have it anywhere, but that also matters in terms of where which portion of the curve that we will get. So, these are the remaining four. So, we have 7 that we assumed, 2 we solved that leaves 4 of these total we will have 13 that we have over here. So, tallying these number of unknowns and number of equations really helps us figure out where we are and the nature of coupling decides which ones we will solve first either we solve the loop closure equations first or we solve the force equilibrium equations first.

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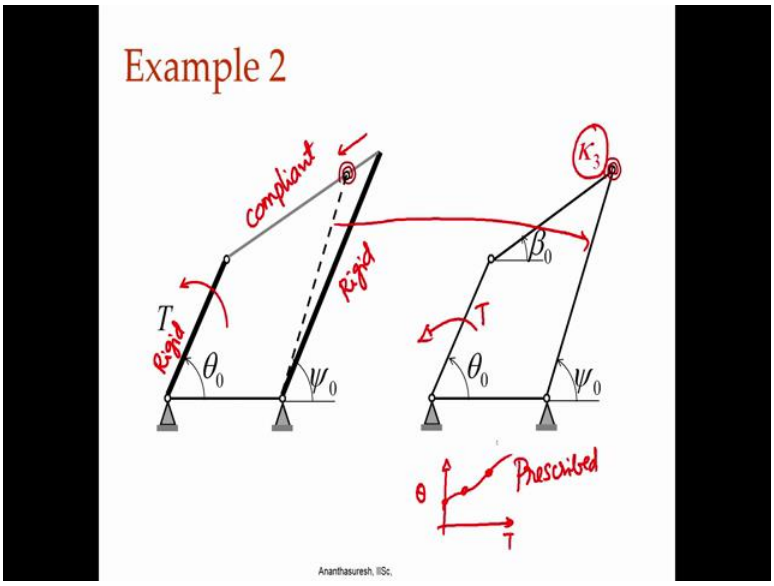

So, in the example 1 that we considered we were able to solve the force displacement equations before we solve the kinematic equations.

Now, let us take another example where we have again a compliant mechanism with one compliant body, compliant Segment. So, we have this is rigid, this is rigid of course, that is rigid. Here apply a torque and we would like to again to function-generation torque versus theta, theta verses torque. Again the pseudo rigid body model we move this back here and put a torsion spring and this becomes the new rigid segment; instead of this one now that is that; with torsion spring which is now kappa 3 earlier we had kappa 1, we have kappa 3. What we have here is a torque that we applied.

So, we want to generate this function theta versus torque so some function like that. That is again prescribed for us and we want to generate it by taking points what we call precision points and any number of that we have free choices will vary. So, we will look at that.

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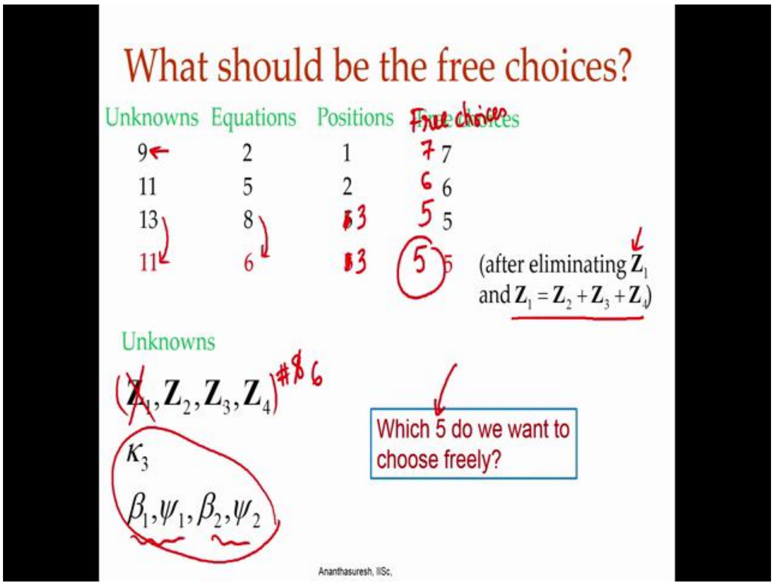

So, what should be our free choice in this problem? Here there is no Z 5 because we have no coupler here; coupler itself is compliant which you have made rigid pseudo rigid with (Refer Time: 23:31) torsion spring there. So, we have 9 unknowns in the first position, these has 8 with 4 complex numbers and kappa 3 that is 9 here 8 plus 1. For 1 position we have 2 equations which is our Z 1, Z 2, Z 3, Z 4 equation loop closure equation the first position and then for (Refer Time: 23:57) other position we will get beta psi 1, beta

2.,psi 2. So, if I go to this not 5 if I go to 3 positions this is not 5; 3 Positions then 11 minus 8 actually free choices are 5 that is I want I meant there.

So, if I write free choices here; initially 9 minus 2; I will have 7, 11 minus 5 that is 6, 13 minus 8 this is 5 and if I eliminate Z 1 and this equation this comes down by 2, this comes down by 2 that does not change in free choices here we have 5 free choices so which 5 would you take among these variables Z 1, Z 2, Z 3, Z 4 has 8, this is 9, 10, 11, 12, 30, 13, but then we eliminated Z 1 so that becomes 11, so this becomes 6. So, would you choose Z 2, Z 3, Z 4 there are 6 of course, you have only 5 here would you choose them or kappa 3 and these 5. So, angle and this is a good fair set 5. We have 5 there 6 here which 5 we would choose? For that we have to examine the equations.

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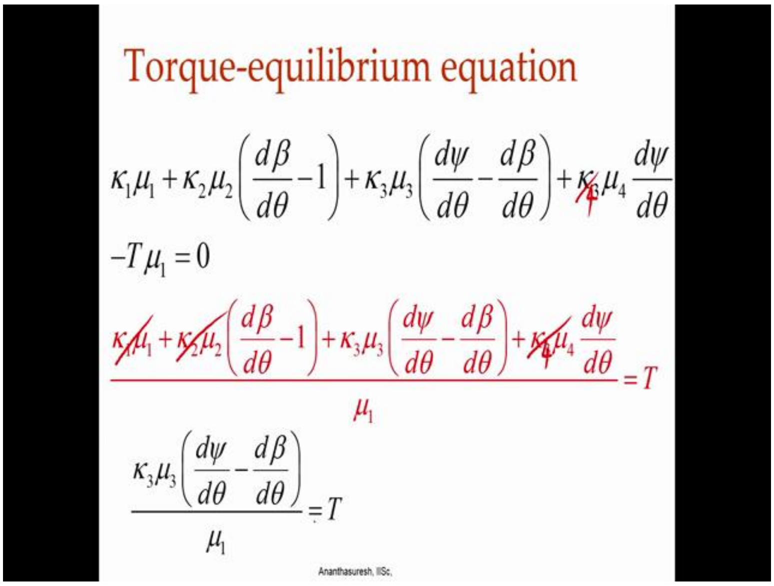

So, the Torque-equilibrium equation which again can be written using the principle of minimum potential energy or principle of virtual work except the force times displacement of that force you have torque times the angle which is rotation input 1 mu 1 is theta minus theta naught and we have all the others and here we can simplify because we do not have this kappa 1 kappa 2 and this should be kappa 4 actually there should be kappa 4 that is also not there.

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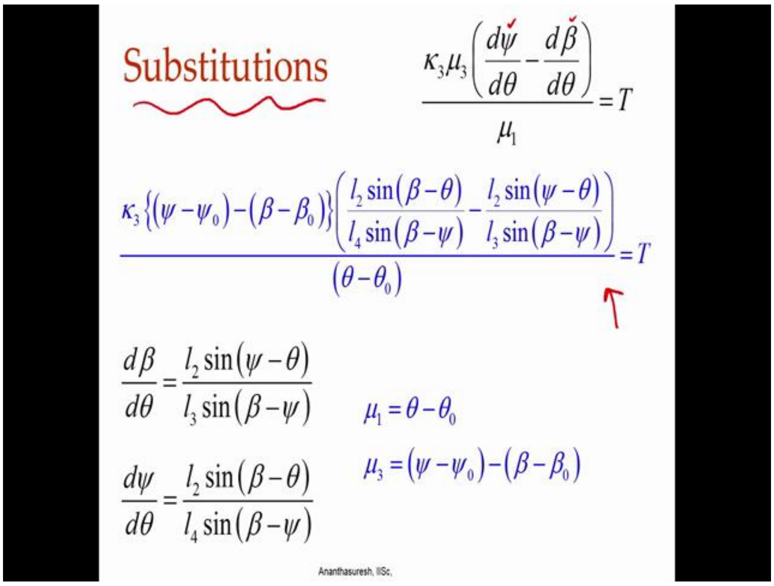

So, if you take out we have only this we put that kappa 3 mu 3 and this. So, once we take this equation. Let us also look at what is d beta by d theta because that is there, d psi by d theta is also there and then these mu's we had mu 1, mu 2 and the other things we had substitute all of them into this then we get equation that is shown here. This is our force equilibrium equation with all these substitutions done.

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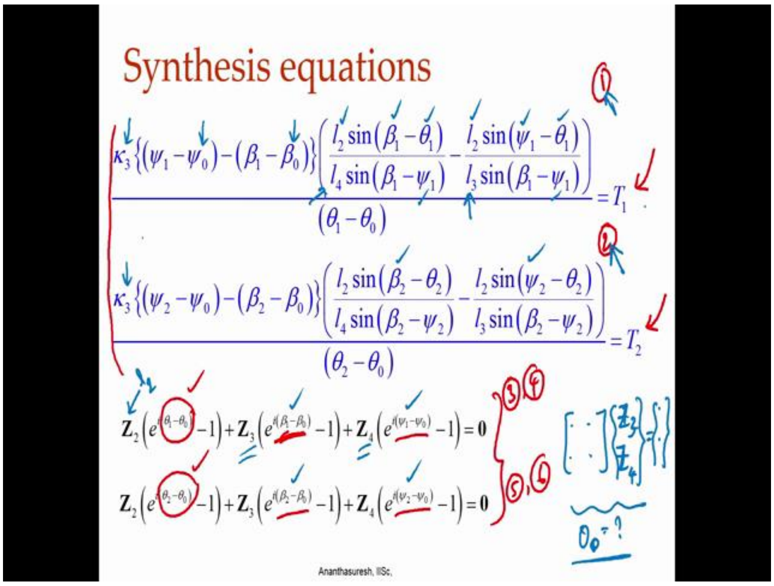

Let us examine this now written for position 2 and then position 3 meaning first position is given position rotate by theta 1 by applying T 1, rotate by theta 2 by applying T 2.

Now if you look at this how many variables are there in the equations? Previous one there was like 7 variables were there we could freely choose and solve for the 2. Let us see how many variables are there here. We have kappa 3, we have psi 1 and then we have psi 0, beta 1, beta 0, beta 1, beta 0 and then we have l 2, l 4 and then theta 1 (Refer Time: 27:22) is known to us; (Refer Time: 27:25) as part of the prescription and then we have l 3, we have psi 1 indicated theta 1 we know, beta 1 we have written; these are there; we have got next equation.

We will have psi 2 and we will have beta 2, how many? 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, so we have 10 variables here out of which have only 5. So, it is a very strongly coupled equation because we have only two equations here but then we have 10 unknowns in these. So, we cannot to work with force displacement relationships in this problem unlike the previous example. So, you have 10 even if I have prescribed 5 of them I would not be able to solve for the remaining ones because I saw many of them.

So, in this case we will look at the; well I have written let me erase this here. So, we have the loop closure equations. These are 4 equations; if we look at what we have here; total we had 6 equations, 4 loop closure and 2 force equilibrium equations. So, we have that; now we have (Refer Time: 28:43) 1, 2 then 4, 5; 1, 2, 3, 4, 5, 6. So, this is equation 1, equation 2, this is equation 3 and 4, equation 5 and 6. Now we have only 5 free choices; how do we solve? We decided that these equations alone we cannot solve because there are too many variables in the force equilibrium equations here than the free choices we have.

So, if we examine the loop closure equations we observe that the beta 1 is there and then actually not beta 1 really let us look at beta 1 minus beta 0 as an unknown and psi 1 minus psi 0 is unknown, beta 2 minus beta 0 as unknown, psi 2 minus psi 0 as unknown. These are the rotations of the coupler body, pseudo rigid body model, coupler body or the output body. If you take these 1, 2, 3, 4 as unknowns; I am taking the rotation not just the beta 1, but beta 1 minus beta 0, beta 0 I would know when I know Z 3 because Z 3 has beta 0 embedded in it. So, I am not taking beta 1 as known, but as a variable I am taking beta 1 minus beta 0 I could freely choose those 4 and thetas we know; theta 1, theta 2, theta 0 they are kind of part of prescribed except theta 0 we can position wherever we want. At initial in theta 1 minus theta 0 is the rotation theta 2 minus theta 0 is also rotation. So, I would know this and this even if I do not know theta 0 because theta 0 is part of Z 2; Z 2 if you take the input planar vector, that has l 2 as well as theta 0 in it.

So, what I can say is that; these are anyway known and these I can prescribe. What I prescribe; I will put it in blue. So, here is one I prescribed, here is 1, here is 1, here is 1 four and then in this I will prescribe l 2 because I have only 5 free choices; then I will be able to solve this system provided actually theta 0 is not needed because the rotation of the input for first position, second position coming from the first going to the second or third whichever way you call it. So, those things are known. So, l 2 is there, but I do not know theta 0, but in terms of that theta 0; I can solve for Z 3 and Z 4 here. I take these right hand side are complex number linear equations. So, Z 3, Z 4. I will set up an equation like this complex number Z 3 and Z 4 equal to these things going other way; we will have some things here, some things here, but in all of these theta 0 is not known; we do not know that, but in terms of that we that symbolically we can get it is a 2 by 2 matrix we can easily invert and get Z 3, Z 4 terms of theta 0.

Once we have it we can go back to Synthesis equations by then we would know this beta 1, theta 1, beta 1, psi 1 all of these things we know inside theta 0 is still an unknown and kappa 3 is still an unknown. The other things we have calculated because by the time we solve we will be able to get these Z 3, Z 4 which will mean that I would know psi 0 and beta 0 along with the lengths l 3 and l 4 and l 2 anyway we have assumed so that is known already and the beta 1 we have assumed and psi 1 we have assumed, theta 1 is prescribed to us. So, those things we already know and anyway psi 1 is known. So, like by the other ones beta 2, psi 2 also we know; what we do not know here is kappa 3 and theta 0.

So, there are 2 unknowns; we have 2 equations; equation 1 and 2 force equations we can solve. It may appear non-linear, but what we should is theta 0 anyway it is input angle we have assumed l 2 where does the first position theta 0 lie; you can (Refer Time: 33:19) a loop and see that this equation, that is this equation and this equation have satisfied while (Refer Time: 33:26) kappa 3 is in this equation then you can easily get that it is not very difficult. So, you can solve these equations like that and synthesis the linkage and then go back to the real one from pseudo rigid body model to real compliant mechanism.

So, here in the supplementary file we would have the (Refer Time: 33:45) where these equations have solved for this example 2. So, you can see and then compare with finite element analysis as well.

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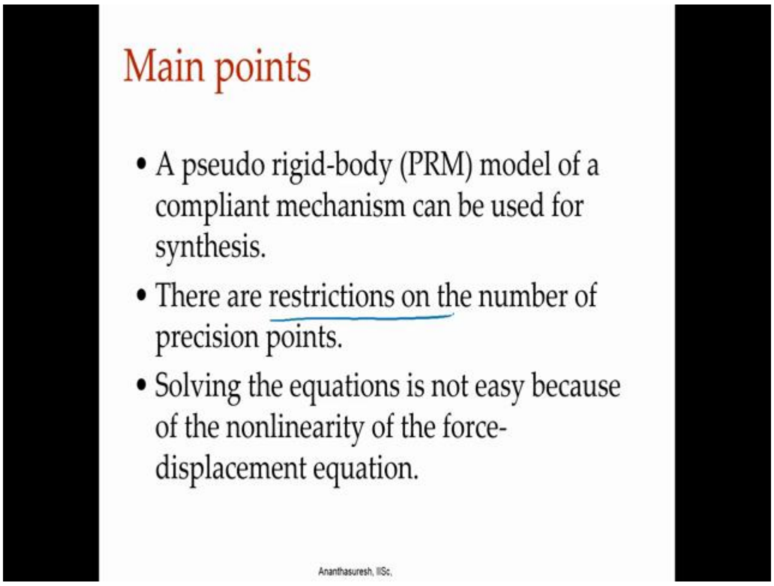

So, to summarize pseudo rigid body model can be used to synthesis as compliant mechanisms, but the trick is here is to identify the number of positions for which we can synthesis the linkage and we will have force displacement relationships as well as loop closure relationships; we have to look at those equations and then see whether they are coupled which one of them can be solved and which one of them should be free choices and based on that we can get this.

So, there will be restrictions on the number of positions is very clear. So, we cannot do for any number of points we want; few of them we can do and solving the equations is not easy; we only took both examples only 3 positions as number of positions increases we will have fewer and fewer choices; so our task of solving them becomes more and more difficult, so we can in the curve 2 2 points if you take a path a few points precision points that is all we can do, but then the free choices gives us a way to explore wide design space and see what will work for us.

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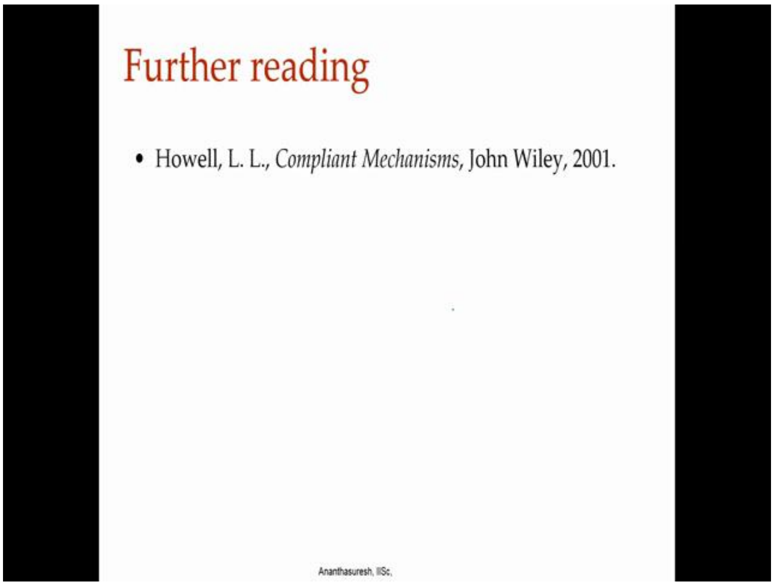

If you want to know more about this you can look at this book by Larry Howell; Compliant Mechanisms where there is a chapter on synthesis, there are couple of examples in supplementary file also, you will find an example (Refer Time: 35:19) solved and compared with finite element analysis for this work.

Thank you.