**Compliant Mechanisms: Principles and Design**
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**Lecture – 23**
**Burmester Theory for Compliant Mechanisms**

Hello, continuing with the last lecture where we used pseudo rigid model to synthesis complaint mechanisms. Today we will discuss a little bit more on that and extend it to what can be called Burmester theory for complaint mechanisms.

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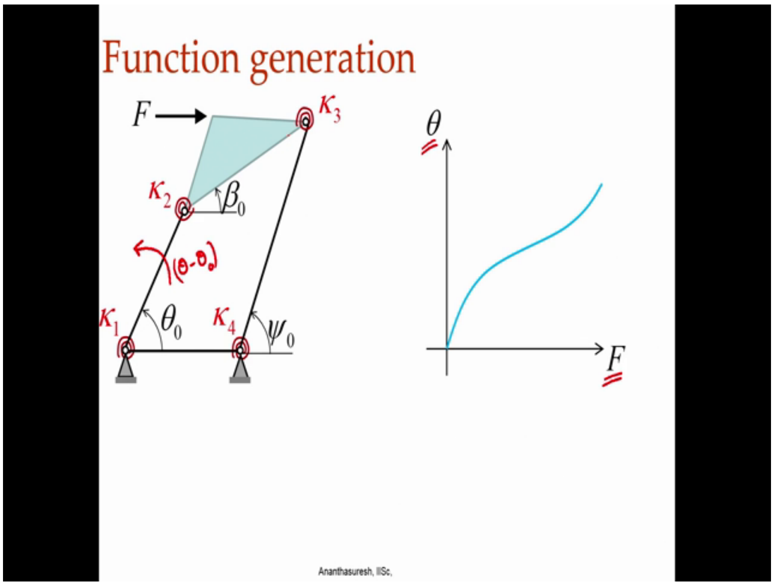


Let us start with that now talking about Burmester theory for that we need to recall a few things that we discussed in the last lecture. The first is this function generation problem. We had discussed function generation, path generation and motion generation. We will discuss at length about function generation, where we have two variables in this case theta and f the force theta is the angle of rotation of the input link here.

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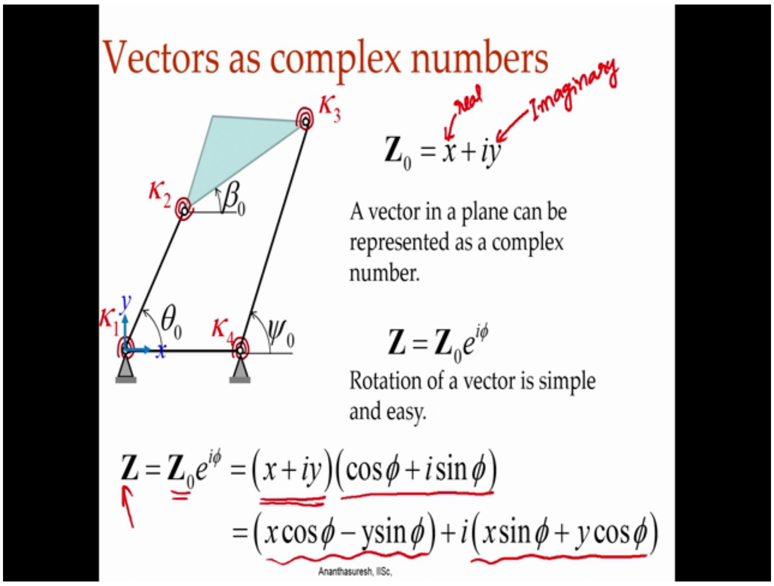


We can call this to be theta let we just go back. This is the rotation of that is now theta naught to be theta, if we rotate by theta the rotation will be theta minus theta naught, right. That is what we have over here and then F is what we are specifying actually is this curve, this function between theta and F. So, we want to get theta is a function of F for different values of F given, we want a complaint mechanism that can give this theta for any specified F. Here we are showing a general 4 bar mechanism with 4 torsion springs meaning that any of the 3 moving bars can be complained, that is why we have torsion springs with constant kappa 1, kappa 2, kappa 3, kappa 4.

In order to do this we also discussed in the last lecture that vectors in a plane can be represented as complex numbers. If we have vector let us I call it Z naught, there is a real path and there is imaginary part, this is imaginary part. Which give us the x and y components; it can be represented like a complex number. The reason we do that what we said in the last lecture was that rotation of the vector becomes really simple and easy to manipulate. If we have this Z naught, if we want to get Z and you want to rotate Z naught by an angle it is a phi, all we to do is multiply Z naught with e raise to i phi where i phi as you all know is given by cosine phi plus i sin phi.

You have x plus i y which is Z naught and e power i phi is this, we just need to multiply them, then you get a new real component, new real imaginary component which will give you the changed vector phi. And that is what we need in linkage, because in linkage the line vectors in a plane keep rotating and of course, they translate as well, that will take care when we write them as the relative vector related to the point about which they move about. We can get this rotation rather easily when you have this complex numbers used to represent vectors in a plane.

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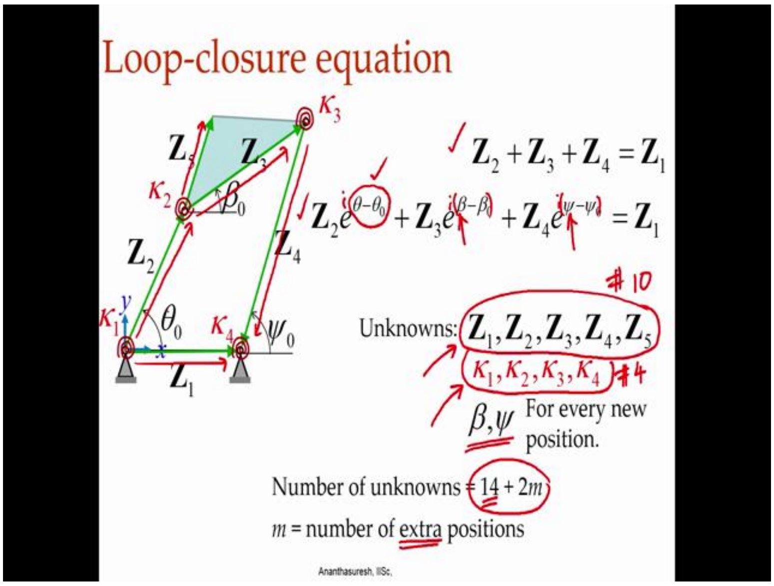


With this we can write what is called a loop closure equation we have all this Z1, Z2, Z3, Z4 and Z5 as indicated in this diagram. There are relative vectors Z1 goes from this point to this point. This is Z1 and this is Z2 and that is Z3 this is Z4 and this is Z5 where force is being applied. Those are our unknowns, where we consider a synthesis problem all this complex numbers right. These are complex numbers so, they have 2 scalars in them x and y components will come to that how to count the unknowns. First we write this loop closure equation Z2 plus Z3 plus Z4 equal to Z1. I start from here I go Z2, Z3, Z4 that would be same as coming from the origin to Z1 that is a loop closure equation. That is actually a complex equation so; we have 2 scalar unknowns in it.

Now imagine a rotated position where Z2 rotates by an angle theta minus theta naught, that is theta naught is angle now when it is rotated it would make an angle theta, rotation is theta minus theta naught. That i should be here the left out i in all of them. That is i i so, these are rotations of Z2 body Z3 body and Z4 body and angles are theta minus theta naught, beta minus beta naught and psi minus psi naught. That is a general equation, that will still be Z1 because Z1 is the fixed bar; it need not be aligned with x axis as I have shown it can be rotated as well. That why put a complex number Z1 it as it is own x component it is own y component this is the loop closure in the rotated position, this is the original or given position this will be rotated position.

Now if we have something like this let us look at how many unknowns are there. Unknowns from the view point of synthesis, synthesis is we do not know what the mechanism is we want to find the mechanism meaning that we want to find all these 5 complex quantities there will be 10 scalars in them. Because each of them will have 2 scalars real and imaginary components or x component and y component and we have also have this torsion spin constant in general for a pseudo rigid model of a complaint mechanism consisting of 4 reasonable segments kappa 1, kappa 2, kappa 3, kappa 4. Total we have 14 unknowns so, 10 here and then we have 4 here. Total we have 14 unknowns, additionally when we rotate another position thetas will be given to us those are known meaning that we want to have as we apply different forces we want certain angle, this of function generation problem I want to generate theta verses F it is like more designing a non-linear spring. Where we have force and theta specified, we want to generate a given function. Thetas are known, what we do not know are this beta and psi those are things that we do not know.

Every new position will have these 2 new unknowns, if there are m extra positions; the given 1 is, if it can be zero or first afterwards you have second, third, fourth positions for every extra positions you will have 2 extra unknowns. The total unknowns are going to be 14 plus 2 times m. These are the number of unknowns considering the kinematics in this problem and the unknown position vectors reality position vectors are all of those as well as the Kineto static parameters, which are the spring constants that is the distortion spring constants this are the unknowns if you know them, then we can solve the problem what is given of the forces and the theta that we want, but you are giving a function.

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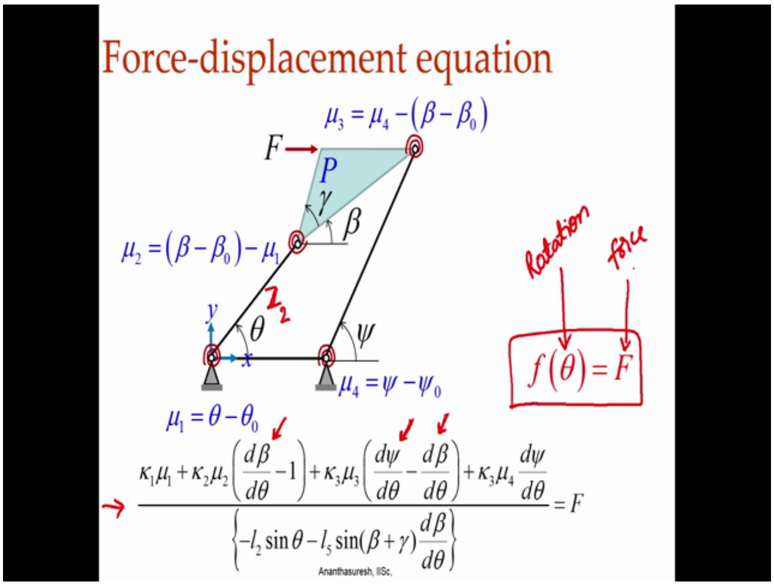


These are the one set of equation that we have loop closure equation, as we already discussed we also have; this force displacement equation which is given by this thing here. Rather complicated it is only near it is real equation not a complex equation because between force and displacement at any point the both are scalar. Scalar equation but it is non-linear because it as sin thetas and cosine thetas and also if you look at this sensitivity coefficient d psi by d theta, d beta by d theta there are also again further non-linear because they also have trigonometric terms in them.

So, in terms of the angles and the link lengths the body lengths that we have, which will be in these quantities? These are going to have the body lengths as well so, which are still unknowns now when we solve synthesis problem unless I know what this Z2 is I would not know what it length is only the magnitude of that is going to be give me the length, which I do not know yet. These quantity is d beta by d theta, d psi by d theta they are going to be involved in these quantities. This non-linear equation is a troublesome equation here. Let us represent this non-linear equation simply as f of theta equal to capital F so, f is the specified force; force is specified and this is the rotation. Someone tells us that, there is a function generates this function between theta and F that is our synthesis problem.

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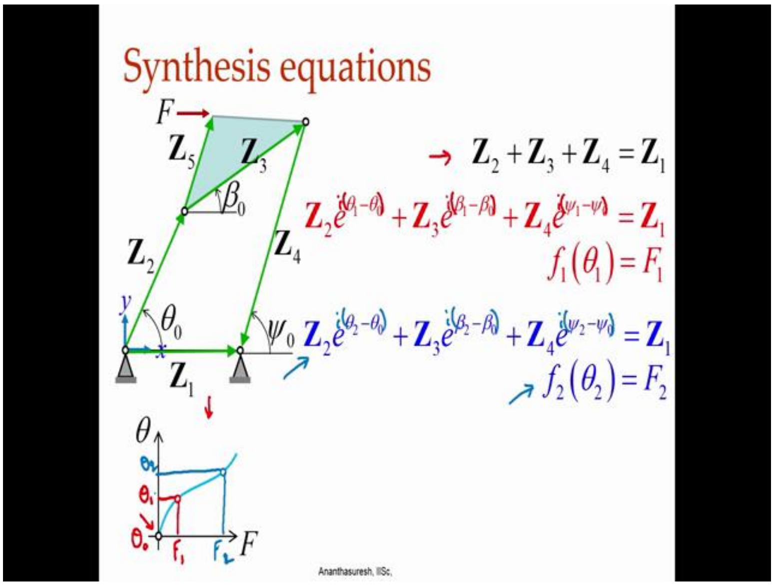


Now let us look at how we can reconcile with this synthesis equations; if I have this four bar linkage model again, I have just now shown the torsion springs but that there. Our focus now is function theta verses f that is what we want to generate that is our focus right. If I start with the position, the given position, that is the first one then I have this equation; Z2 plus Z3 plus Z4 equal to Z1 that is the loop closure equation; force is 0 theta is 0, theta is not 0 rather I should say theta the rotation. Theta minus theta naught does not start from here let say it not the origin this is theta naught, that is given to us. That is provided I know Z2 then I would know that is 0 that is where there is no force no rotation from current position to the any other position.

Now when I take let say a point over here then at this point I will have 10 let us call this F1. For that I can write both the loop closure equation as well as the force equation, once again I forgotten this i is here let us put them otherwise it does not make sense. So, theta 1 minus theta 0, beta 1 minus beta 0 all of that is there and then the force equation F1 of theta 1 equal to F1. I put f 1 because it has some changed values, theta and other things same function we wrote non-linear equations. So, f of theta 1 equal to F1 this is the force equation force displacement equation is the kinematic loop closure equation. That is for this point now if I consider some other point; let us say this blue point let us actually take blue color for that here, I would have F2 and this is theta 2 then I can write this equation the force equation as well as the kinematic loop closure equation again I have to put i times this i times that, i times this.

So, we get again a complex equation and a scalar equation. Complex equation in kinematic loop closure equation and a scalar equation that is the force displacement equation, for each position added will have more unknowns as we said, because thetas are known theta 1, theta 2 are specified F1, F2 are specified rather somebody as given is this function they I wanted to generate this, we are taking point some that meaning that we know both thetas and forces, that is why I am assuming that thetas are known, forces are known in this equations.

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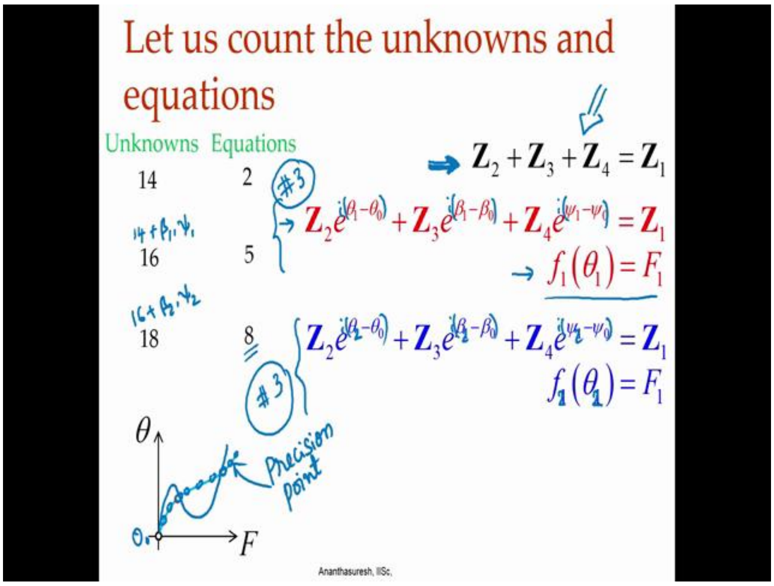


Now if you count the number of unknowns and equations what we see is that at the beginning when I have just the first position that is this circle here where you have theta 0 and F equal to 0. I have 14 unknowns, what are the 14 unknowns Z1, Z2, Z3, Z4, and Z5 and then we had the kappa 1, kappa 2, kappa 2, kappa 4. That is 5 complex unknowns that about to 10 scalar unknowns and then 4 scalar unknowns which are the torsion spring constant (Refer Time: 14:28) 14. How many equations do we have 1 complex equation so, 2 scalar equations right. We have complex equation, you get 2 scalar equations by equating the real parts and equating the imaginary parts that is what we have 14 and 2.

Now, if I were go to the second position that is from original position first if I call second position by rotating by theta 1 minus theta naught again I have to put back this i's are I have forgotten. We have now 5 equations we have the 2 equations that we already have and then 2 equations over here 1 equation over here together we have 3 new equations here, 3 new scalar equations, 1 complex equation that amount 2 scalar equations and 1 scalar equation that is the force displacement relationship right.

Now, we have 5 equations unknowns are 16 what are the new ones, new ones are is beta 1 and psi 1. Whatever we have 14, we had originally 14 plus we have beta 1 and psi 1. Now if I go one more that is next position so, again they should be theta 2; I should see earlier one whether there was psi 2 or theta 2 let us go back a little bit, there theta 2 beta 2 psi 2 are there. So, here miss that. These are all theta 2 beta 2 so, here we have whatever 16 we had their plus new unknowns are beta 2 and psi 2 then new equations are 3 here again 2 from the complex equation and this again should be theta 2. This is the scalar equation complex equation total 3 scalar equation so, we have total 8.

As you can see number of unknowns with each new positions it is going up by 2, 14 into 16 into 18 number of equations is going up by 3; 2 to 5, 5 to 8 where we counting this equation notes because in order to solve for this unknowns we need to count and make sure that we have enough equations to solve. If there are no equations we are not actually including our specification of the problem by saying that we have 2 equations we are constraining this unknowns to do something for us. If I include this equation here that is I am ensuring that this loop closure equation is satisfied meaning that at least when you put this Z1, Z2, Z3, Z4, Z5 I will have this equation satisfied likewise if I satisfy these 3 equations; I satisfy that when we rotate we get this beta and psi relationship from theta. Theta 1, beta 1, psi 1 will be related along with the body lengths that is what is contained in Z2, Z3, Z4 and Z1.

Now, if I also account for this; that means, I applied that much of force it will move by the angle that I want that is theta 1; that is within theta 0 position it will move to theta 1 position by rotating by theta 1 minus theta naught and then if I satisfy these equations they also satisfy the next position right. You have to make sure that as we are doing more and more why do we need more and more because we have continuous curve that we need to generate, if I take more points here I will able to approximates this quite well. If I only take what I had taken earlier, I do not know what I get I may get something like this. It is just passing through this points and in a way it will do whatever you want, but if I say that, I need to go through all this points there are called actually precision points because you would your synthesized mechanism will surely go through the points precisely because, we are trying to solve the equations here that is why these are synthesis equation that is way we need to see how many positions we can go. The more positions we go the better for us, we will be able to have control over the function that is generated by this mechanism; that is the function generation.

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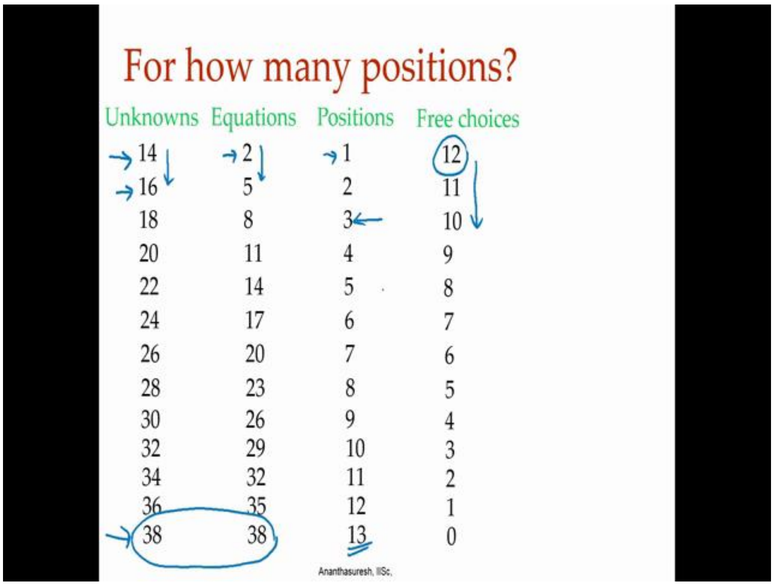


Now how many positions can we do? We started with the first position we had 14 again 10 plus 4, 10 for the 5 complex numbers we have represent the 4 bars and other bar related to the coupler the linked the middle one, where the force is applied and then 4 are the torsion spring constant. 10 plus 4 and we had 2 equations and that is what one position that is the first position itself. It is not so, useful does not move anything it does not move by itself right when we move it, we have welcome back to this. When we move to 2 positions that become 16 and 5, this increases by 2, this increases by 3. So, as increase number of positions 16 5, 18 8, 20 11, 22 14, 24 17 and so forth., finally; we come to a stage here we have 38 unknowns and 38 scalar equations, scalar unknowns scalar equations that is a formidable set of equations 38 and they are all non-linear because we have complex number which have e power sub angle so, we have all the non-linearity's they are non-linear equations. For first configuration and then 12 more additional positions because it is 13, we can actually solve the function (Refer Time: 20:52) compliant mechanism right. We have the curve we can actually chose the starting point and 12 more points, we can get an approximation into most of the curves actually if you are able to solve this 38 equation 38 unknowns.

Here when the number of equations is less than the number of unknowns, why did I stop at the point where number of unknowns number of equations are the same because if I have the next one will be 41 equations because it increases by 3, where unknowns as 40, 41 equations, 40 unknowns that is over constraint system. There is no guarantee that there will be a solution. In fact, even when the number of equation, number of unknowns is equal also there is no guarantee. For non-linear equations there will be a solution, but if number of equation is more than number of unknowns we can be absolutely sure that there is no solution unless somebody gives you very special data which is not the case when you are trying to synthesis a new mechanism now if you count the number of free choices at the beginning you have 12 free choices, you can assume lot of things because that for you to decide because you have fewer equations than unknowns; that is your freedom now as you go to more and more positions the number decreases 11, 10, 9, 8 like that finally goes to 0.

And this free choices we have to chose judiciously verses how many positions; obviously, going to extreme case of 38 unknown, 38 equations is not a wise choice because you would not be able to solve them. We settle for fewer points may be just 3 or may be 5. We can do that and try to see what equation we can solve easily.

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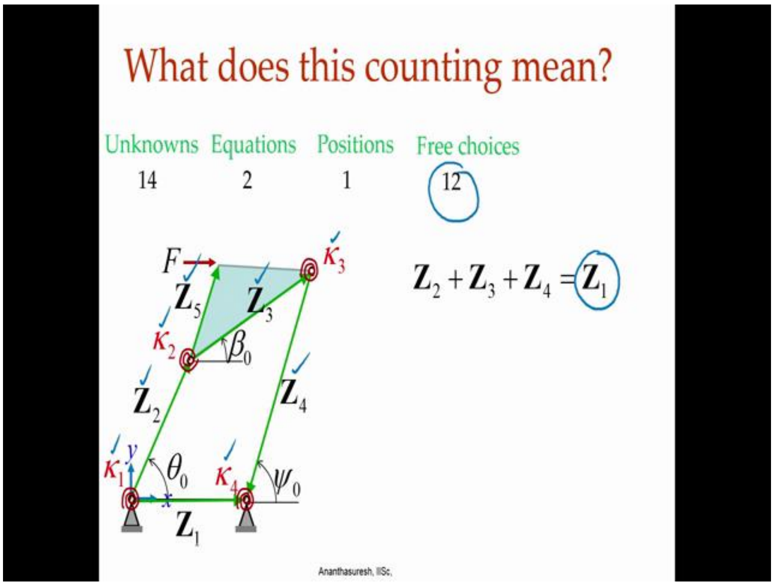


Let us see what this counting means? If you have the first one we have 14 unknowns, 2 equations one position free choices are 12. Let us see what are the 12 free choices here we have 2 equations, 12 free choices what are they. Well one equation is Z2, Z3, Z4 and Z1; let us say that Z1 we do not know, if I assume the rest of them I have 12 of them that mean that I will assume 2 things there, 2 things there, 2 things there, 2 things are that is 8. 4 complex quantities 8 and then 9, 10, 11, 12 that mean I am assuming it is 12; I can solve this equation and get my remaining unknown Z1. That is what it means when if you count the number of equation here it is slightly very straight forward that if you assume 12 quantity that you are designing the entire linkage or complaint mechanism including the 4 torsion spring constants to get one unknown which is Z1. That is not very interesting one position is not of course, is interesting.

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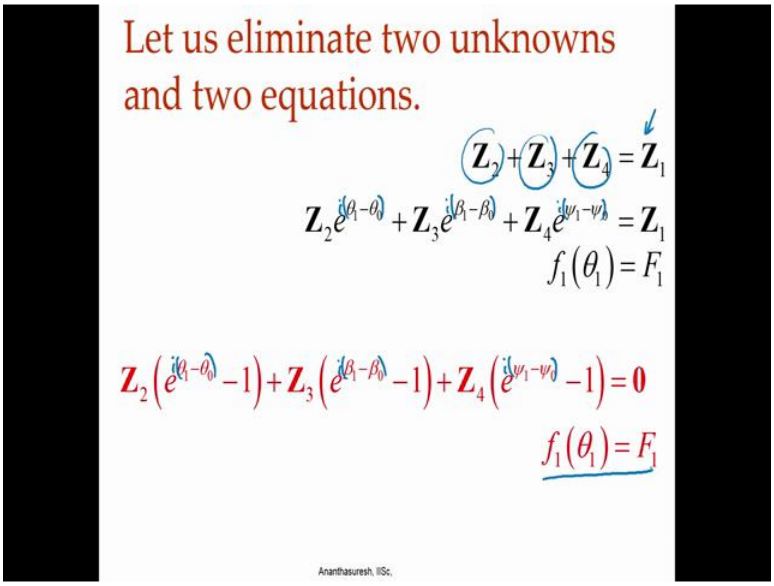


Now, if I want to go for more positions let us actually get rid of the trivial thing about the Z1. Z1 if assume that Z1 is going to be computed any way using this equation, we can eliminate this equation as well as that unknown. In order to do that what we will do is will take this general position again and I missing this i this is just cut and paste, if you miss one place you miss everywhere. We are subtracting this general position equation and from it we are subtracting the first position. I have Z2 e power i theta 1 minus theta naught minus 1 because of that Z1 same thing here, same thing for this, same thing for this.

Now what we have done is we are eliminated Z1 and one complex equation complex variable and one complex equations just for ease of looking at what are the things that we can assume as free choices and what we can actually solve in a synthesis problem of course, the force displacement will remains the same that does not change.

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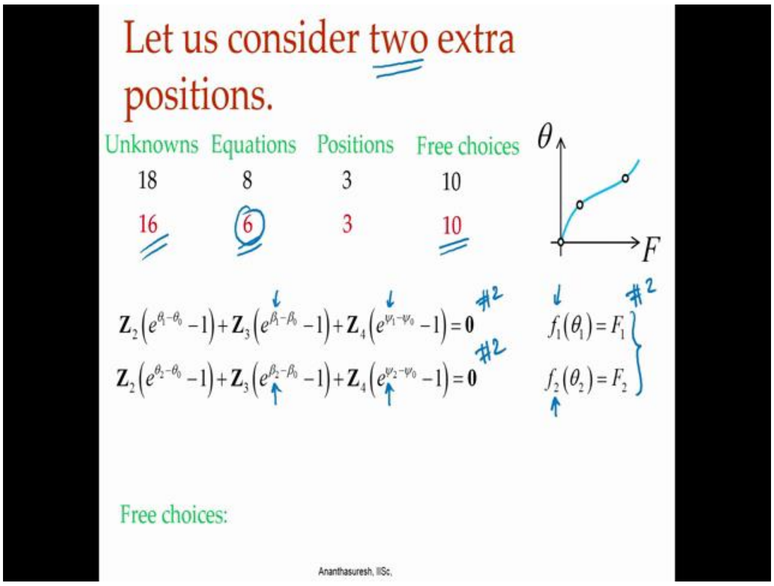


Now, let us consider 2 extra positions. We have 18 unknowns and 8 equations, 3 positions. If I take 2 extra means that I have first and then second and third 3 positions; we have 10 free choices we have 12, but the next position if you consider that becomes 11 now ten. So, 10 free choices with this reduction where Z1 and the first loop position equation eliminated will have 16 unknowns and 6 equations and free choices are still 10 because we have reduced here reduced 2. What are those if you look at the equations I have written our 6 equations that are 2 scalar equations here, there are 2 scalar equations here, there are 2 both of them together we have 2 force displacement equation total 6 that we know and the unknowns. Unknowns are Z2 we do not know Z3, Z4 that is 6 unknowns and then where are kappa’s? Kappas will be in this equations right. Here we will have the kappa’s torsion spring constants those are there and also we do not know this beta 1, psi 1, beta 2 and psi 2. What can we assume as 10 free choices?

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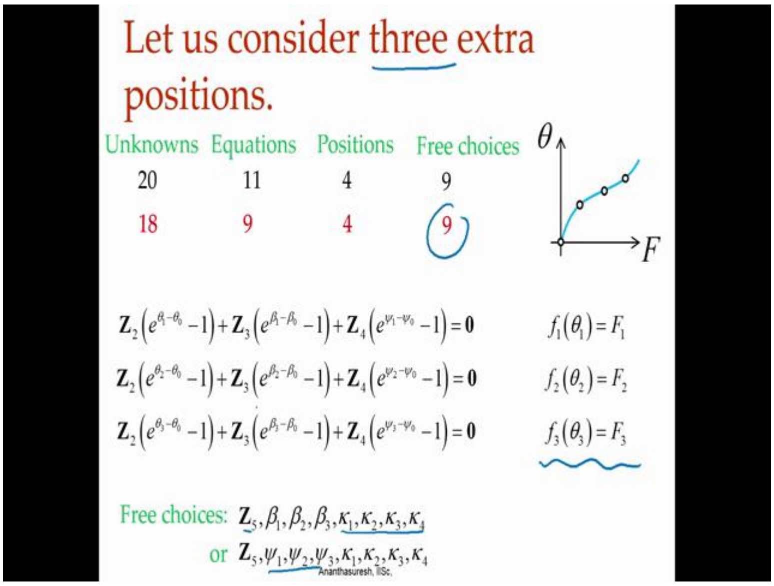


One possibility is we can assume it is Z5 which is not in this equation, that we can assume; that is again will be there is in this force equation right and then beta 1, beta 2 beta 3 and so forth. That is 2 positions now we have 3 we can write the equations in this manner and then see if I have 9 free choices what will they be right, 9 free choices I can put Z5 and then I have these and these angles that we have. We can see now actually I have 10. This is 4 plus 7 2 yeah it is 9. Now we have 9 free choices for 3 extra positions I can have 9 these or these angles either beta or psi. We know one I can get the other these equations.

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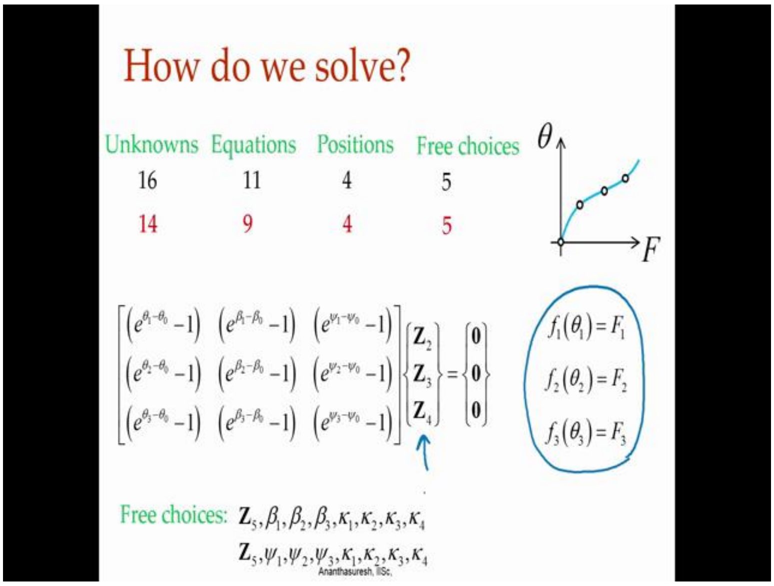


Now this complex set of equations there is one little trick that is why the Burmester thing comes. We can arrange the equations whatever we have them; we can arrange them in a matrix form, is a complex number matrix anyway these are unknowns. If I assume that these angles are known I will be able to compute this complex numbers. Very straight forwardly because they are linear equations, when you represent in a matrix form they become easy to solve. So, with these free choices I can solve, but the difficulty here lies that these are non-linear and they involve Z2, Z3 and Z4 because their lengths will be involved there Z5 also will be there because the force equation will have that and of course, this angles and everything in principle we can solve, but not really in practice unless we are little bit careful about which one to choose as free choices.

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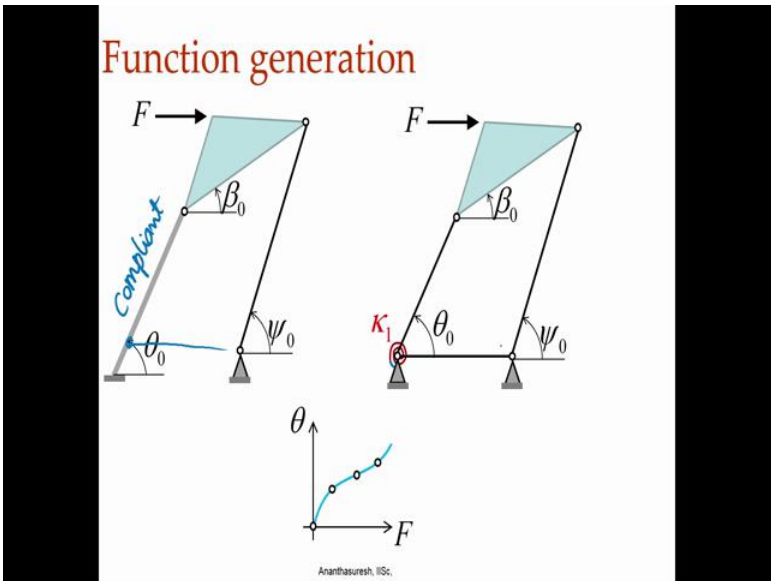


What we will do is in the next lecture we will consider this function generation problem, where we take one complaint segment and the other one is rigid, as applied force let generate this function and this the complaint equations. In fact, what we have done is we took the pseudo rigid body model so, here is you are the pin joint will come as shown here and only one torsion spring right not the other ones and that means, the other torsion spring constants are 0. They are not there they are Kinematic joints we can write this equations.

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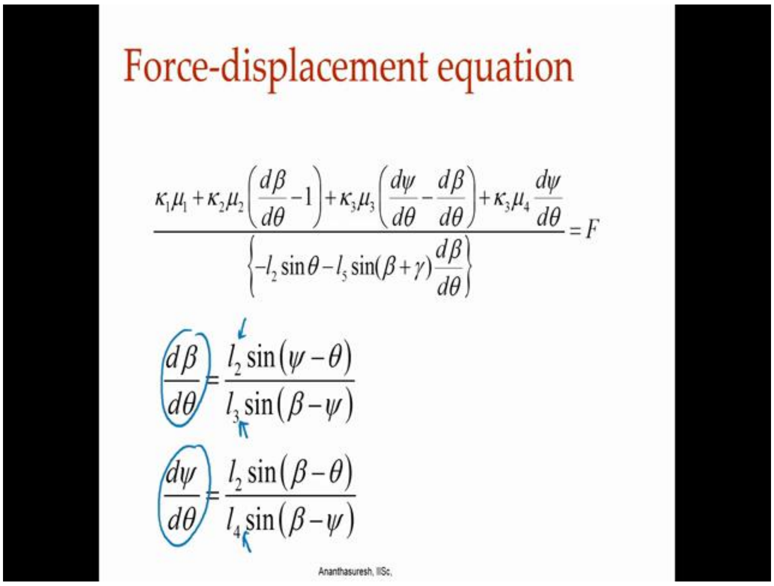


We are looking at this Force-displacement equations and recalling what d b by d theta, d psi by d theta are see this l 2, l 3 are there and l 4 there are the magnitudes of the unknowns that we have Z2, Z3, Z4. We have to see how to manipulate this equation so, that we will be able to get that in a fashion which is the easiest to have them linear then we can solve them happily.

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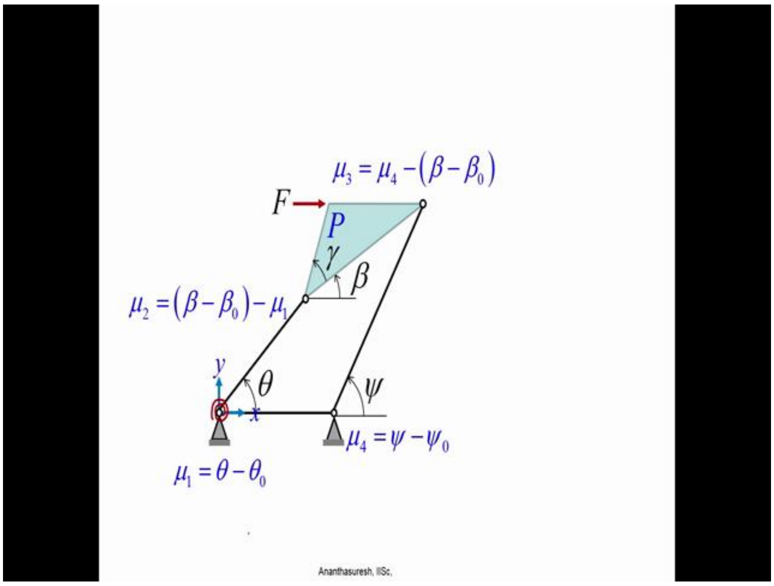

That is what will consider in the next lecture. For this lecture were we had taken this and try to see how to formulate synthesis equations.

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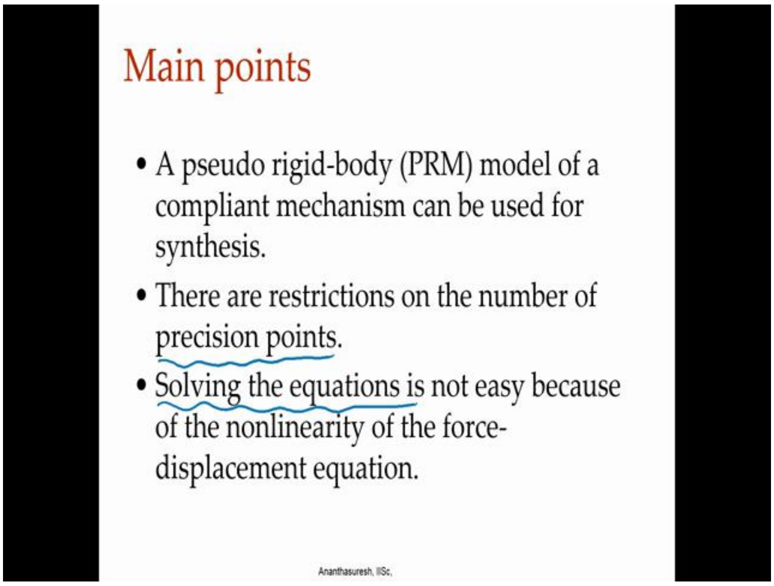


What we have learned now is a pseudo rigid-body model is going to help you solve or at least formulate the equations and know how many positions are possible this precision points as we had said, those points we can go up to 1 plus 12 more or function generation similar thing we can done for patch generation motion generation also, but solving them is an entire different matter. Because they are highly non-linear and solving them just saying that I have 38 equation unknowns not going to help, we need to see practically what number of precision points can be taken to solve them which we do with an example in the next lecture.

Thank you.