Compliant Mechanisms: Principles and Design
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Lecture – 22
Loop-closure Equations for PRB models of compliant mechanisms

Hello, this week in the first three lectures we had talked about analysis using pseudo rigid body modelling; in the next three lectures this week we will discuss synthesis. By synthesis what we mean is designing a compliant mechanism consisting of one or more elastic beam segments and trying to get the functional specifications met; we will discuss what those functional specifications are and how to set up the equation and how we will solve them. Today we will focus on understanding the equations involved in synthesis and in the next two lectures we will consider how to implement them and synthesize by wave examples a few compliant mechanisms. So, let us look at the Synthesis using Pseudo Rigid-body Modelling.

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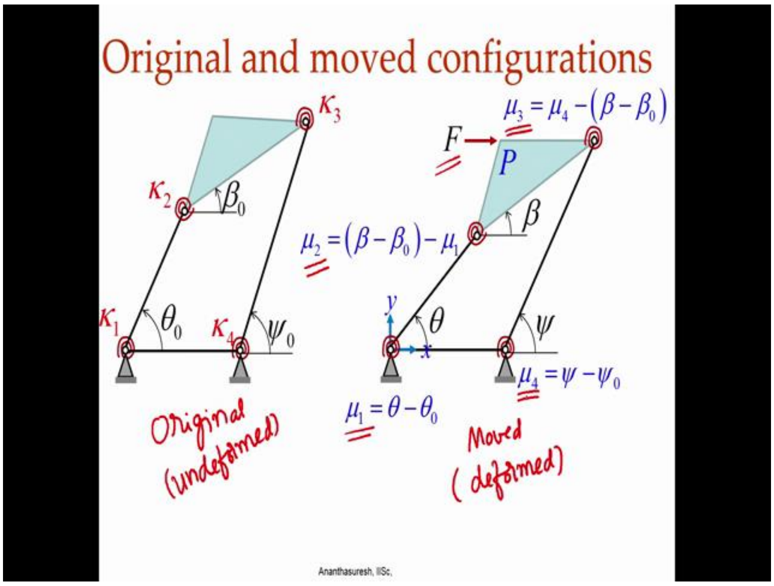

So, let us recall that when we have a compliant mechanism; let us say in this case we have taken a 4 bar linkage, which has torsion springs at it has 4 joints and we want to see if we can synthesis a mechanism such as this for giving specifications of some functionality. So, when we have a 4 bar linkage with 4 torsion springs we already understand that it is a pseudo rigid body model of a compliant mechanism; a compliant mechanism may have only one elastic segment or 2 or all 3 as well which every that is as long as the validity of the pseudo rigid body model is true; we can create a model such as this where as this 4 bar linkage where we have a torsion spring at each of the 4 joints it can be more than a 4 bar linkage also, but we are taken example of a 4 bar linkage.

When a force is applied such as this F here; which is going to move from this original configuration; this is our original or un-deformed configuration by original what we mean is that it is and un-deformed to the moved configuration; see Moved are what we call deformed which is the correct word here not just displaced, because there will be an elastic segments are (Refer Time: 02:56) which will be deforming elastically and that is when we have this mood configuration with application of force and at each joint we have the rotations which are given by mu 1, mu 2, mu 3 and mu 4 which we can write; when you have something like this we discussed how to analyse them, but now let us talk about actually synthesizing them.

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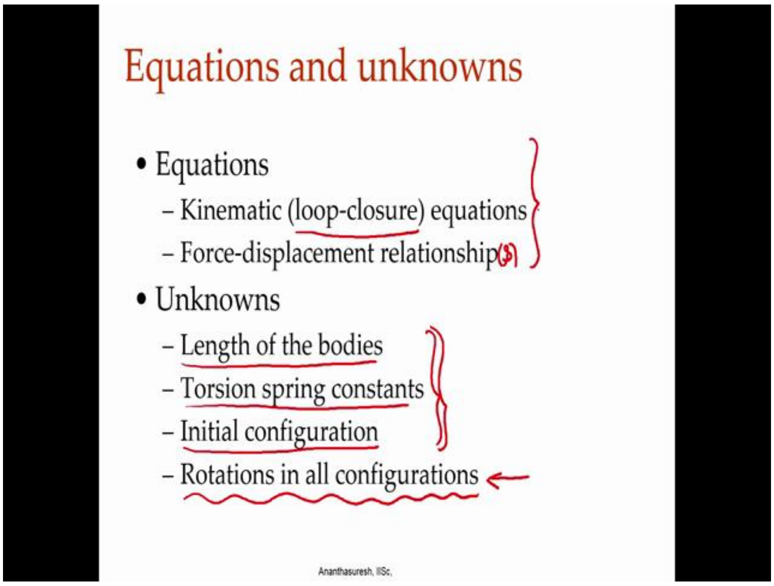

Whenever we synthesize a mechanism be it a rigid body mechanism or a compliant mechanism, we should always worry about equations and unknowns; unknowns are the ones that are the design parameters here those are the ones that we need to find as opposed to in analysis most of these are already known and we just need to find the remaining one that is analysis; in the case of synthesis we have to determine for example, the length of the bodies and torsion spring constants initial configuration, because that is also an unknown and then rotations in all configurations normally analysis given the first three you can find rotations in all configurations that is what you discuss an analysis. In the case of synthesis these are the first 3 are also unknowns and as you will see some of the rotations in all configurations will also be unknowns.

Once we know what the unknowns are they become our variables and parameters in our synthesis problem. They are the unknowns; in order to solve for them we need equations; there are 2 kinds of equations as we already know from analysis that we discussed in the last 3 lectures there are kinematic equations or what we call loop-closure equations; we close the loop as will see now and also the force-displacement relationship or relationships, because if there are more than two or three positions that we want to synthesis for there will be a number of force displacement relationship that we need to take into account that gives an equation. So, the first one is kinematic equation, the second one is elasto static equation.

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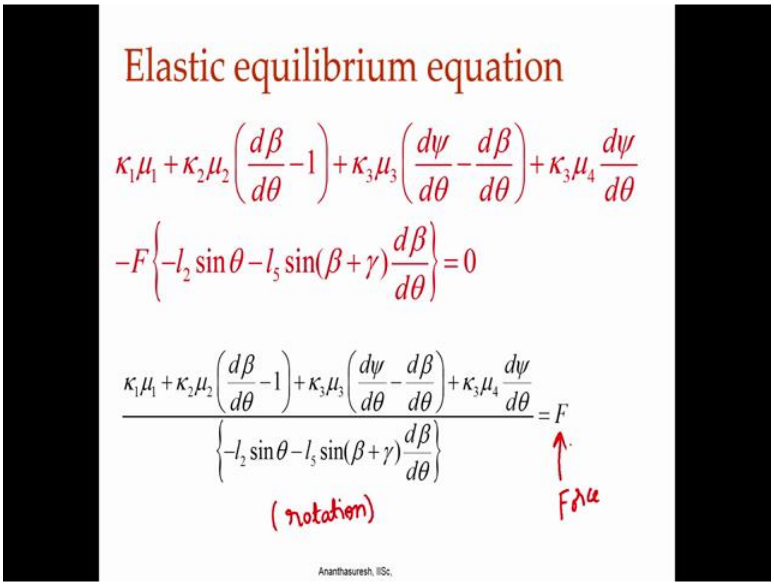

Elasto static equation which is equilibrium equation or force displacement relationship on the right hand side we have the force and on the left hand side we have rotation which is the displacement here. So, we need to find this rotation theta on which the psi and beta also depend. So, we have force and rotation equilibrium equation for the example we had considered in the analysis case you will have this elasto static equilibrium equations.

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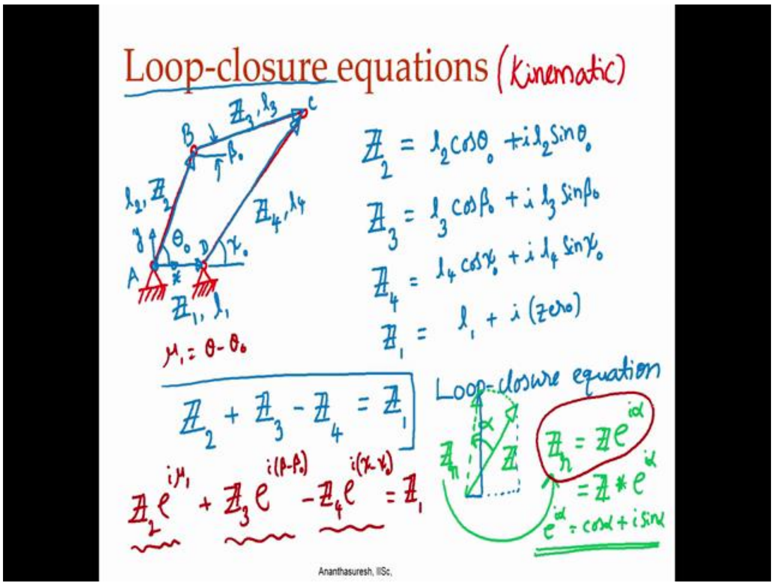

Now, let us look at the loop-closure equations or what we call kinematic equations, so these are kinematic equations. We had used that in analysis in the context of computing the sensitivity coefficients that is d beta by d theta d, psi by d theta that is coupler angle and output angle; at what rate do they vary with respect to the input crank angle which was theta; for that we had used to loop closure equations. Now we will understand and use these loop closure equations in the context of synthesis for that we need to introduce a concept that uses complex numbers which makes it easy. So, they are complex numbers which makes our job easy as we will see. So, let us consider without springs first because kinematics does not need to include that let us take a 4 bar linkage with 2 fixed hinges and 2 floating hinges.

Now, what we do is to represent this using complex numbers. So, let us take the coordinate system here you can take it anywhere you want, but right now will take it at the pivot that is given. So, we have x axis, y axis; now this one is a vector. Let us call that Z 2; when I write Z 2 in this manner in a script Z fashion what I mean is that it is a complex number which will have real and imaginary parts.

So, let us indicate that this angle is let us say theta 0 and the length of this is let us say l 2 as we had done earlier then Z 2 to if I take the origin at the point where we have indicated that is in this point I can write Z 2 as a vector or as a complex number I can write it as l 2 cosine theta 0 that is the real part and then l 2 plus l 2 sine theta 0 that will be the complex part I will put i.

So, we have written the input crank as a vector and represented that using a complex number there is a nice reason for why we want to use complex numbers, but let us first write all the vectors as complex number. So, Z 2 we have similarly I can have from here to here Z 3. So, Z 3 is basically the complex number that represents the vector indicating the coupler of between the input and output we have the coupler that is Z 3.

And we can also write Z 4 it is the output crank vector that has again real and imaginary parts and we can include the bottom one let us call that Z 1 that l 1 we calling Z 1. So, each of them if we know the angles let us say I call this angle psi 0; I call this angle beta 0; we can write Z 3 if I indicate this length by l 3, this is l 4, and this is l 1, I can write Z 3 as l 3 cosine beta naught plus i l 3 sine beta naught and Z 4 as l 4 cosine psi naught plus i l 4 sine psi naught and Z 1 is easy because it is just l 1 and there is no imaginary part because there is no y component for Z 1 vector. So, it will be i times 0. So, that is basically not there.

Now, we have the 4 vectors represented like complex numbers. So, now, I can write what we call loop closure equation that we can see that is I start with Z 2; I am going from let me indicate them as A, B, C and D. I move from A to B that is Z 2 plus, I move from B to C that is Z 3 and then I move from C to D with D to C is Z 4; I am moving from C to D that will become minus Z 4 and that will be equal to Z 1 or minus Z 1 equal to 0; we are closing the loop and then I am going from A to B, B to C, C to D; I could have move from A to D directly that would be Z 1. So, this is our loop closure equation. So, we just simply closing the loop this is the loop-closure equation.

At this point you may wonder why we are using complex numbers to indicate vectors; we could have just used vectors; I could have called v 2, v 3 and I could have called v 2 plus v 3 minus v 4 is equal to v 1, but we are using complex numbers; there is a significant advantage for using complex numbers when we consider this one in moved configuration by that what I mean is that if I have let us say if I have let us say a vector like that which is represented with some complex number Z between 2 points is a relative position vector between 2 points. Now, if this were to rotate by some angle let us just take it as alpha here, what is this rotated angle let us call that Z r which is to indicate rotated it turns out that Z r can be simply written as Z multiplied as a complex number multiplied by another complex number which is e raise to i alpha; it is as simple as that.

So, when we have this Z rotate by an angle alpha about Z axis the rotated complex number indicating that rotated vector can be simply given as Z time e raise to i alpha. We follow the complex number multiplication rules here. So, Z star e alpha, but by that what I mean is that Z star e raised to i alpha; e raise to i alpha as you know is I will just write here e raise to i alpha is cosine alpha plus i sine alpha. So, that is also a complex number.

So, when I multiplied these two; I get the rotated vector that is why we use complex numbers to represent vectors and it is very useful in synthesis as we will see shortly. So, if you want you can work it out if I have vector; I rotate it by an angle alpha then you can see what these components are so let us let me draw those components. So, this will be the y component there will be x component what will they be relative to the original one where that would have been the x component, that would have been the y component how are they related with alpha it will be related in this manner we can work it out and check.

Now, what is advantage for us if I consider a rotated configuration for this; let us say this thing has moved to another configuration all I need to write is Z 2 into e raise to i whatever angle that is which we had said rotation is we had called it mu 1 plus Z 3 e raise to i; we had called it let us say I think the mu 1, mu 2 was rotations at this point here rotation would be just beta.

So, we can just write it as beta minus beta 0 rotation angle because here mu 1; if we recall mu 1 was theta minus theta naught. So, that is the rotation. But here we have to write the rotation of the coupler body; coupler body that we have that it will be i raise to beta minus beta 0 and then minus Z 4 times e raise to i that would be psi minus psi naught; that is the rotation of the output one psi minus psi naught; that will still remain Z 1. So, when we have rotated configuration. So, I do not want to draw here because it will becomes very clumsy; rotated configuration each vector we can write simply by using this complex numbers Z 2, Z 3 and Z 4 multiplied by e raise to whatever the corresponding rotation is we get the equation that we can do for any position.

So, getting loop closure equations become very easy you may still not see the advantage other than saying that we have this easy way of writing rotated configurations we can also do this use vectors. So, where do we again gain; we gain because all the operation involved here are complex number operations which can be easily implemented in any programming language, but there is more to it that we will see as we go along.

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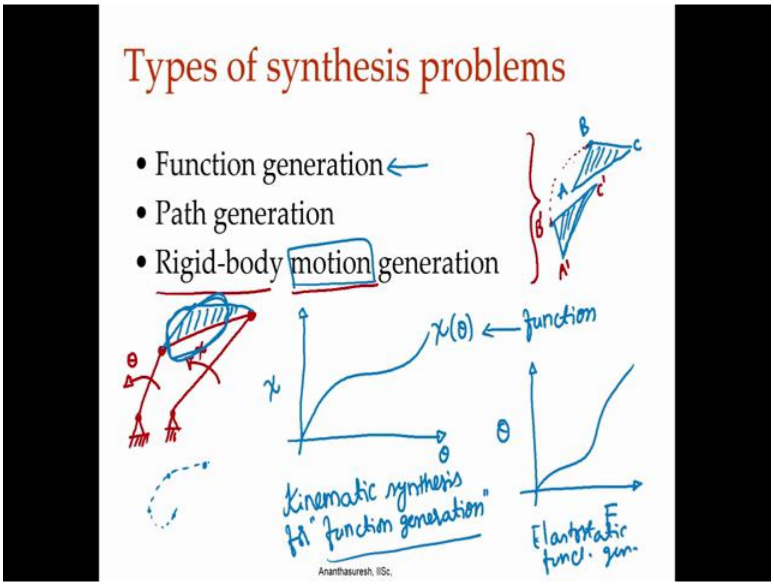

So, now before we use this loop closure equation that we just wrote; let us talk about what (Refer Time: 17:58) of synthesis problems we want to solve. In kinematics; in the kinematics literature there are three kinds of synthesis problems known as function generation, Path generation, rigid-body motion generation; sometimes people leave out this rigid body, but it makes it clear if you say that in this manner what is function generation?

Function generation is if I have a 4 bar linkage like that into fix pivots where are this moving towards here; out of them the rotation if I indicate that as theta; if you indicate this as psi in function generation what I want is that somebody would prescribe or of the problem on hand they will say psi and theta have to follow a certain function this were the function. So, here psi is a function of theta. So, this is the function.

So, somebody wants a function of this kind then you need to synthesize the lengths of the bodies on the initial configuration. So, that you get psi theta in the case of compliant mechanisms because they cannot move unless you apply a force we can say this as a function generation where we can say if force is the input here I can say theta here has to be in some form; this is the normal kinematics problems kinematic synthesis of function generation kinematic synthesis for function generation.

We are basically generating a function in fact, before digital computers came if you want to generate a function it had be done analogue sometimes mechanically. So, that is where this function generation term comes. So, somebody wants a function of certain kind sine cosine. In fact, there are beautiful mechanisms that can do even Fourier expansion of a given function. So, they can synthesize or analyse a function for Fourier expansion. So, in general it is known as kinematic function generation now if you have theta versus F that becomes our elasto static function generation. So, we will be looking at that type of problems today.

The third is the motion generation where if I have a triangle that we had (Refer Time: 21:03) coupler if let us say original one is A, B, C like that let us say it moves to a new configuration which B comes here let us call it B prime, A prime here and C prime. So, what we see is that the B point has moved along a path at the same time the rigid body is rotating also. So, if you do that we call that a motion generation problem rigid body motion generation problem we have to guide a body from position to position as prescribed by an application sometimes we have the planks in a chair in a classroom you take it out and you know you flip it. So, sometimes there is a hint sometimes a whole thing comes like a rigid body and from folded configuration to a deployed configuration. So, such things we call motion generation problem. So, there are 3 types of synthesis problems. So, what we will do now is just begin writing the equations for a function generation problem.

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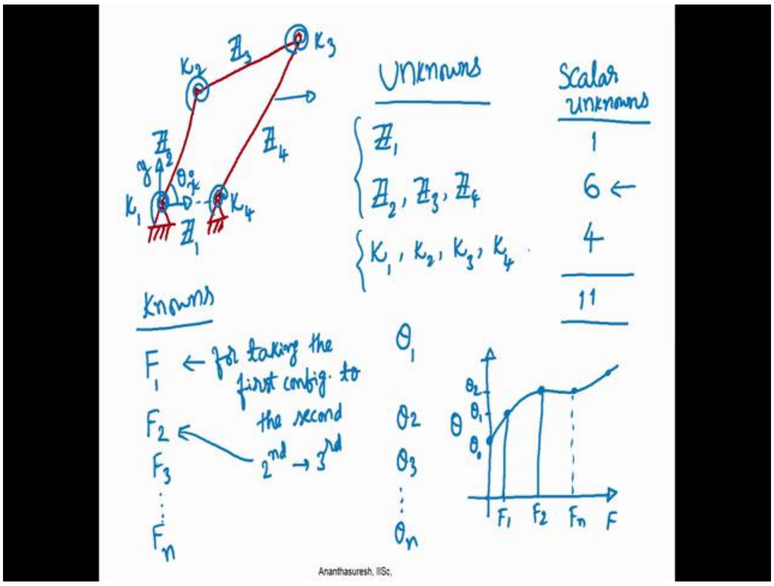

So, let us say if I have 4 bar linkage and let us not forget that we have springs. So, we have the 2 pivots and we have the springs, but generality let us say that there are the torsion springs at all 4 joints. So, let us also indicate our unknowns because we do not we have not synthesize it. So, Z 2 will be unknown and so will be Z 3 and so will be Z 4 that we have and then this also which we called Z 1.

The initial angle is embedded in Z 2 as we saw; Z 2 already has its theta 0. So, is theta 0 and. So, is size 0; and then we having this torsion spring constants kappa 1, kappa 2, kappa 3 and kappa 4 So, let us count our unknowns when I say I need to synthesize this 4 bar linkage pseudo rigid body model for unknown are Z 1 and Z 2, Z 3, Z 4 and these kappa's; kappa 1, kappa 2, kappa 3, kappa 4 how many are there; let us say the Scalar unknowns because when you write complex number when you say that is not know there will be (Refer Time: 24:12) unknowns. But here the way we are taking the x y coordinate system; Z 1 will have only one scalar unknown because it is already known to be are on the x axis y component is not there. In general you can orient whichever way that does not matter for function generation because we are only worried about the angle theta and angle psi whichever configuration it is it does not matter.

So, let us why we count it as 1 and then here we have 6 unknowns because each of these Z 2, Z 3 and Z 4 have 2 unknowns; scalar unknowns in them there are six and then here we have 4 scalar. So, total we have 11 scalar unknowns; for function generation what is given what is known that also we should write; what is known what do; we think that we know. So, first of all the forces we start from the given configuration we go to the first configuration which I can say there is F 1 force that takes you from original configuration to the first configuration and there will be F 2, F 3 and so on so F n configuration.

So, this is the force the other unknowns because that is what people will say I want to apply this much force it should go to some angle; so, for taking the first configuration to the second and this is from second configuration to third and so forth; we have the known’s and then we may say it has to happen that is these are the force is given, but then when you say function generation we say that there should be this angle which we can call theta or mu it is probably better if you call it mu; this is mu 1 we had called let us say an input angle I would like to have this mu 1 from the first configuration second configuration and so forth. In fact, mu 1 other number becomes problem let us call it theta itself.

There is a theta 1, there is a theta 2, theta 3 and so forth theta n; n configurations theta 0 is our original one that goes to theta 1 that goes to theta 2 theta 3 and so forth in other words what we are asking is a function theta as a function of F which is some curve we are saying if I apply a force F 1 in fact, I should not make it 0 so let me change this curve let us say I start from somewhere. So, this becomes our theta 0 when there is no force when apply F 1 I want theta 1 when apply F 2; I want theta 2 and so forth So, I am taking this points original point and some more points. So, that this could be let us saying F n this is F 2. So, this is a function I am specifying I may not give a function as an expression, but I may give you for sure what thetas I want for different F values. So, we are we know forces and theta those are know and in the initial configuration and the spring constants are our unknowns. So, we have to count the unknowns and try to get as many equations as needed to solve for these things.

So, equations if you look at we will have; now, let us (Refer Time: 28:46) unknowns; unknowns will be this initial configurations Z 1, Z 2, Z 3, Z 4 and therefore, spring constants Z 1 will have only one scalar others will have 2 each that become 6 and 4 total 11 and the known’s are forces and these rotations that we want; there is a corresponding between the input force input force can be anywhere. So, it can be a force here there does not matter a force and the corresponding rotation of the input crank; so, theta versus F.

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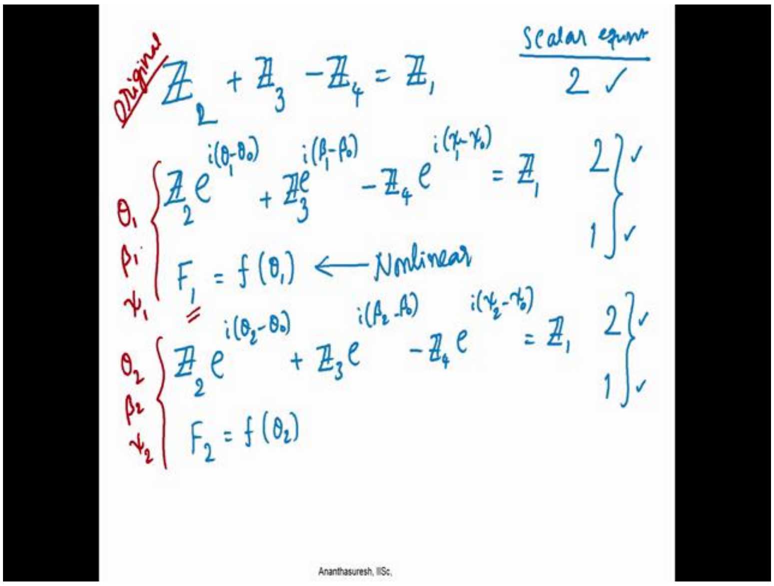

So, those are known's to us; for finding unknowns we need equations what are our equations our equations are of course, loop closure equation we have we had what we had written we had written Z 2 plus Z 3 minus Z 4 is equal to Z 1; how many equations are there; I will say the Scalar equations how many are there because is a complex equation you can equate real parts and imaginary parts there are 2 equations. Now, we write in the moved configuration Z 2 e raise to i theta minus theta 0 plus Z 3 i e raise to i beta minus beta 0 plus sorry, minus (Refer Time: 30:22) after way we wrote that vector minus is Z 4 e raise to i psi minus psi naught that will still be equal to Z 1 now how many questions are there?

It has also 2 equations real parts imaginary parts. So, this one let me call it actually the first position theta 1, beta 1, psi 1 rotated by theta 1, beta 1, and psi 1 to the 2 equations for us, but then we also have the force equation F 1 equal to f of theta 1 this f is given to us theta 1 is also given to us, but this function which was the force displacement relationship will involve all the body lengths and angles, torsion spring constants and so forth.

So, much equation is this? It is one. So, we now have from the first configuration there is no force equation because there is no force. So, is a 0 and angles are not moving. So, there are 2 kinematic equations alone, but for any other position for the first position we have two kinetic equations and one force equation; we had counted that we have 11 unknowns now we have only 5 equations; that is good actually if we have more unknowns than equations it is a happy situation because you can assume whatever is the excess number of unknowns.

So, if you go back to the previous slide out of these unknown 11 of them now we have only 5 equations, 5 scalar equations 11 scalar unknowns I can assume 6 of them. I might as well assume these 6 that is I choose Z 2, Z 3, Z 4 meaning a configuration of the 4 bar to begin with and then I will use those 5 equations that I have to determine kappa 1, kappa 2, kappa 3, kappa 4, and Z 1 which has 1 scalar unknown.

In fact, if you do that that will be bit of a problem because the kappa 1, kappa 2, kappa 3 are going to be in this non-linear equation these are also non-linear because angles are there, but you can see the advantage it became compliant complex numbers why I would call them linear because they are linear in complex numbers when you arrange them in a particular form whereas, this one remains the scalar equation not a complex equation and this is non-linear the force displacement (Refer Time: 33:05) non-linear.

Any case if you just count the number of unknowns in equations here we have 5 equations and we have 11 unknowns. So, we can assume any 6 of them freely, but now let us say I go to the next position Z 2 e raise to i theta 2 minus theta 0 and Z 3 e raise to i beta 2 minus beta 0 minus Z 4 e raise to i psi 2 minus psi 0 that will still be equal to Z 1 that will give me 2 more equations and then I will also have F 2 as a function of theta 2 that is one equation. So, we have totalled now 5 plus 3 8 equations.

Now, I have 11 unknowns and can count how many are there, but we should note that when I wrote these unknowns I did not include a few extra unknowns that are coming. So, let me use a different colour to say that over here extra unknowns are there which are this theta 1 beta 1 and psi 1, because those are where not in the initial configuration. So, where unknown also increasing and then here also we have this theta 2 beta 2 psi 2. So, we have to see the number of equations and unknowns and try to balance the equations are unknown to see how many equations we get based on how many positions we are considering how many configurations.

So, this is the first configuration and then we have; this is 0-th configuration do not be confused by this using one let me call this original configuration that is 0-th configuration. So, this is the original configuration and then it is the next one and next one we have to see how many unknowns and equations are there and try to come up with the appropriate number in situations unknowns and try to balance the number of equations unknowns if unknowns are more than the number of equations then you can assume some of the unknowns, but if the unknowns are dominated by the number of equations there will be over constraint system and that we cannot solve.

We will see this by way of an example problem we will consider. So, that we can first count number of equation unknowns and then work through it that will do it in the next lecture which is a continuation of this lecture.

Thank you.