Complaint Mechanisms: Principles and design
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Lecture - 20
Kinematic coefficients of a four-bar linkage with and without springs

Hello, in today lecture we are going to look at an example of modeling a complaint mechanism that has two rigid bodies and one elastic body of uniform cross section using the pseudo rigid body analysis. So, let us look at that example and go through it is details. So, we are we are talking about analysis of complaint mechanisms using pseudo rigid body modeling or PRB for short we will illustrate that concept with an example today.

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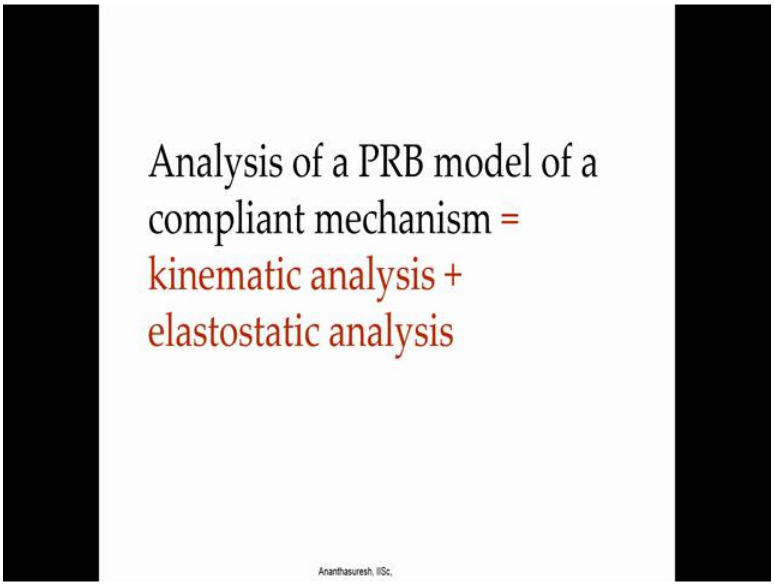

So, let us understand that from the last lecture analysis of a PRB model of a complaint mechanism contains two steps one is kinematic analysis other is elastostatic analysis.

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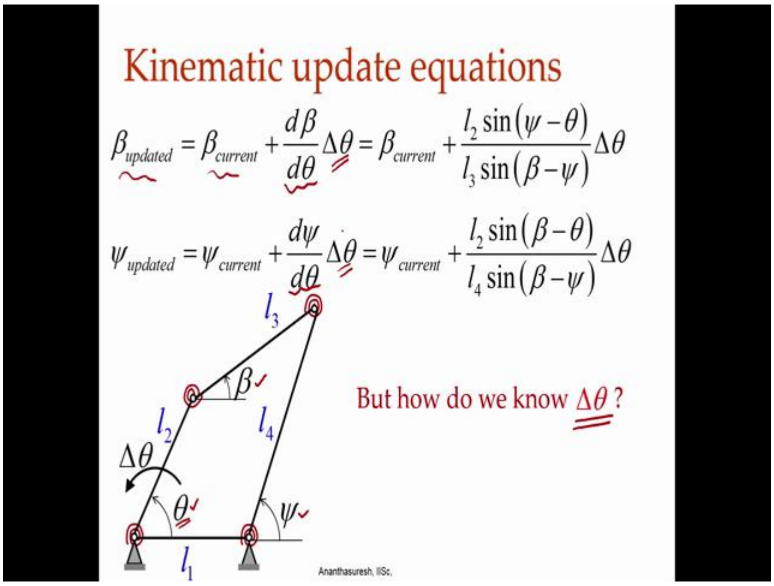

Kinematic analysis the important thing is the update equations. So, how do you update the angle beta and angle psi here. So, we have the angle beta here and angle psi here for given values of angle theta we want to compute beta and psi for those we have this kinematic coefficients d b d beta by d theta d psi by d theta once we know we can if you take small angles delta theta then from current configuration we can get the updated configuration for the beta and psi, this is kinematically updating the things, but who will give us delta theta that is the question.

So, if I have a pseudo rigid body model of a complaint mechanism with the torsion springs that are shown the force springs at each of the four joints we have to know how delta theta is to be obtained for usually a given force when there is force the angle delta theta will change by a small amount because, we need to keep it small for updating the kinematic equations in this manner you can also solve them exactly, but we will come to that how to do it exactly as well as numerically small increments that is a kinematic analysis.

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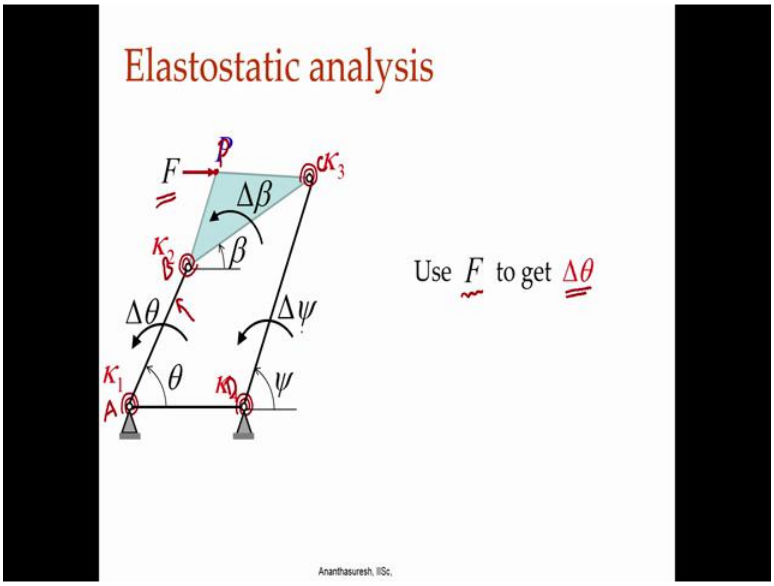

Now moving on to elastostatic analysis, if a force F is given in this particular case we are showing deliberately the force acting at a point away from the two joints. So, if we call this A B C D let us say this is a this B C and d force is not applied at any other joints it is applied at the coupler where there is a point which is under coupler link and at that point you are applying force F we can call that point p. Now we have to compute delta theta for given f when I say delta theta I, if I call this theta 0 if it moves to another position by amount to delta theta it will go to a new position which is theta naught plus delta theta that is rotation of this body is this particular body is what we are calling delta theta, we need to see with the force applied how much does it rotate, when we compute delta theta we know how much delta beta and delta psi then we determine the entire configuration of the pseudo rigid body model 4 bar linkage here four bar linkages with springs. So, we have this torsion springs kappa 1 kappa 2 kappa 3 and kappa 4 and that p that I already have written there, Ok.

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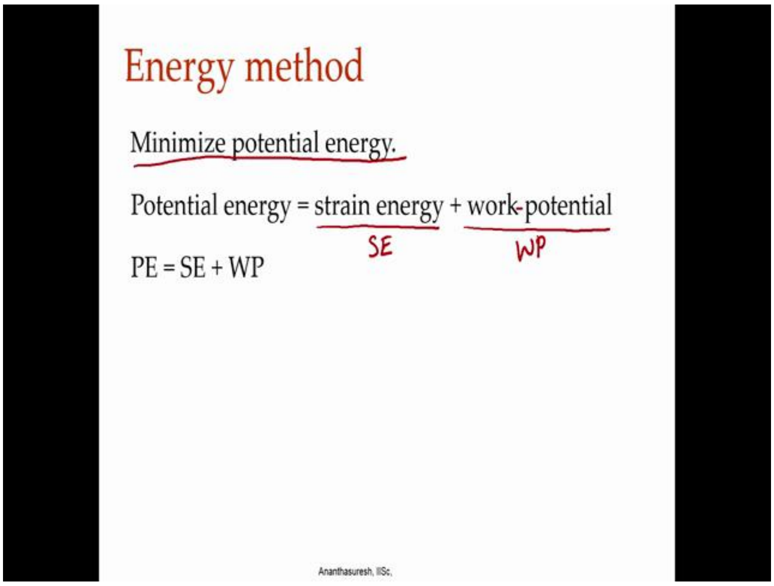

Now, we will use the energy method in order to obtain an equilibrium equation with which we can solve for theta for given f or f given theta basically, a relation between the force and displacement, here displacement is rotation we are using energy method and the principle of minimum potential energy. So, when we write potential energy that consists of two parts strain energy and then works potential which is the potential energy due to the external work. So, strain energy is denoted by SE work potential by WP and PE stands for potential energy.

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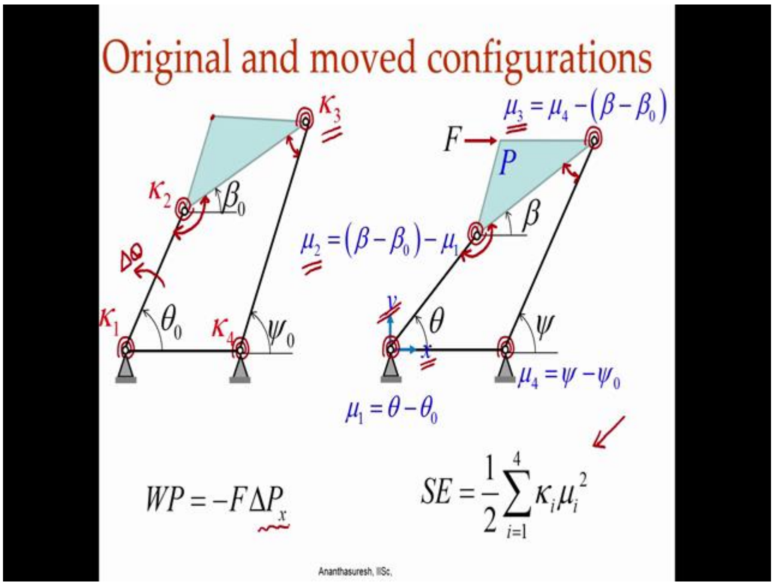

So, if I take the pseudo rigid body models of a complaint mechanism the most general one where we have this spring's everywhere these are the original configuration which is on the left side and this is the deformed configuration. This particular case the rotation here which I had said delta theta that actually has gone the other way that is it is negative if this is my positive convention for delta theta then it has gone the other way over here does not matter just the negative value where I have that then we need to write the strain energy form for it let us say we have this four spring constants kappa 1 kappa 2 kappa 3 and kappa 4 and we need to see the rotation of the each of the joints.

So, and the value of p the point p where there is a force f applied when you apply the force it has deformed to configuration on the right hand side, the rotation at the first joint let us denote it as mu 1 that is our delta theta, theta minus theta naught originally it will be theta naught the deformed configuration if it is theta all measured with the horizontal axis here that is mu 1 and then mu 2 will be delta beta that is beta minus beta naught minus mu 1 that is delta theta which is the first joint, that is what will be the change in this angle here originally there will be some angle over here now that becomes a new angle the difference of these two angles is mu 2, that is what this torsion spring will be experiencing that change in rotation that is delta beta that is beta minus beta naught minus mu 1 which is delta theta, the similar manner we can write mu 3 and mu 4, mu 4 is psi naught here the psi minus psi naught and the change in this angle.

So, there is some angle here that is in the original configuration and the deform configuration deference of these two we call it mu 3 that is rotation of the third joint which is kappa 3 joint the torsion spring and when we get mus and kappas we know then we can at the strain energy as shown here, half k mu square for each of the four springs. For work potential it is a negative or the work done by external forces strain energy we know like how it has k mu square and work potential or the negative of the work done minus f that is negative f times delta P x that is this point somewhere here original P has moved to another point how much has it moved in the x direction, that is what we want if I take the coordinate system like shown here x and y in the x direction how much ever it has moved there will be delta P x. So, we can write the work potential let us do that.

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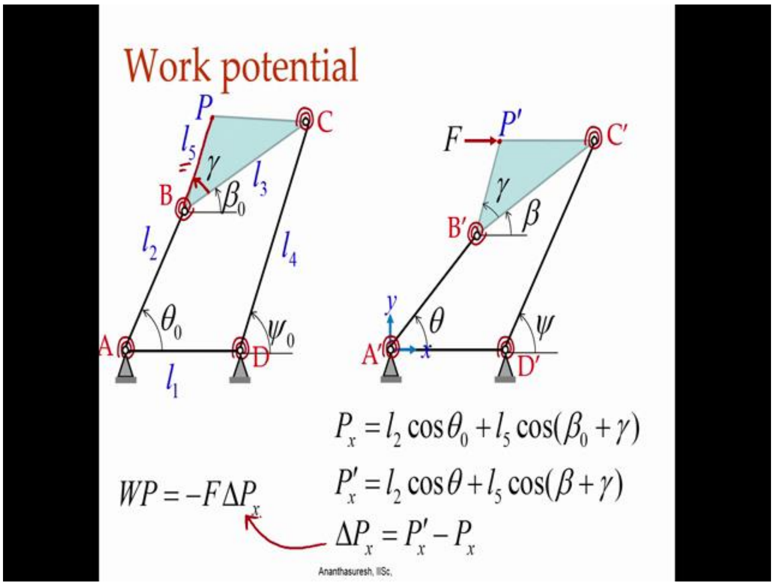

So, if I original point is p does move to p prime, because of the force then P x if I write the original one the point P x coordinate of that P x is l 2 cosine theta naught plus l 5 cosine beta naught plus gamma. So, gamma is this angle over here. In fact, to avoid confusion we should probably call alpha because, the gamma is pseudo rigid body model is reserved for the characteristics length factor, but does not matter it is an angle we have indicated the figure gamma. So, P x is l 2 cosine theta naught plus l 5, l 5 is here this length to denote that p cosine beta naught plus gamma beta naught is plus gamma and then P x prime when p prime is that position vector of that then P x prime is l 2 cosine theta plus l 5 cosine beta plus gamma. So, now, beta naught becomes beta theta naught becomes theta you get the new P x prime the difference of these two will give us the delta P x delta P x is P x prime minus P x and you substitute that into here we get the work potential.

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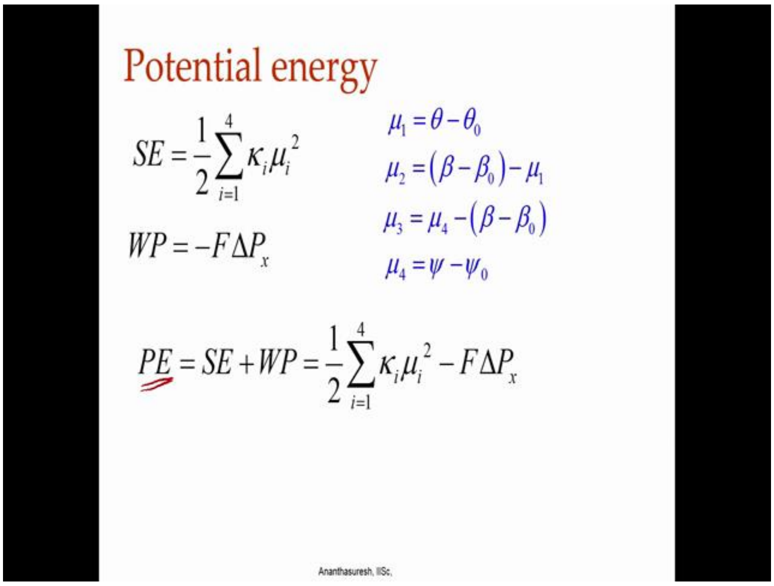

So, we have strain energy now and we have work potential and both of these we substitute to the potential energy expression we get a long expression and that potential energy expression we take derivative of with respect to theta and equate to 0.

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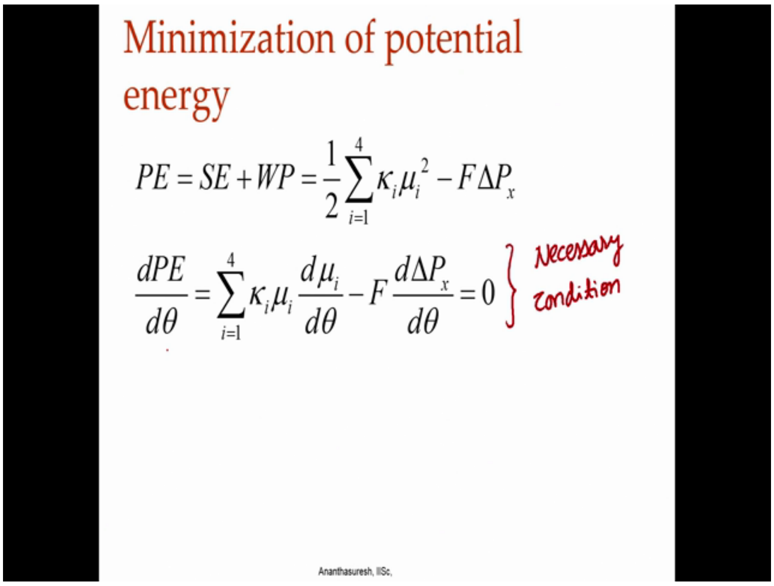

Because this is the necessary condition for the minimum of potential energy with respect to our parameter here it is only 1 parameter that is the input crank rotation, that theta we take derivative respect to that equate to 0. Now, we have an expression for the potential energy we take the derivative.

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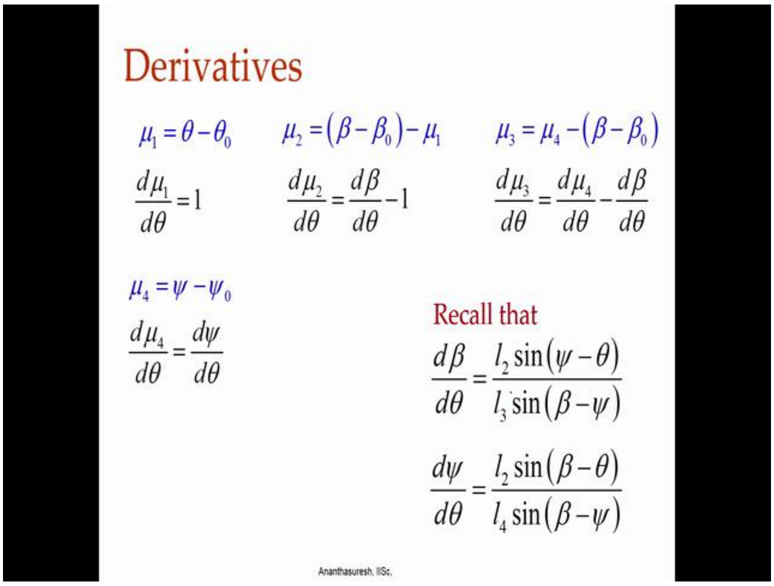

Let us do that we already know derivatives we need to know the derivative quantities of the quantities that are there in the potential energy we have mu 1 mu 2 mu 3 and mu 4 and we take derivative each one of that d mu 1 by d theta is 1, because theta, theta naught is a constant mu 2 d mu by d theta is d beta by d theta minus 1 because, the mu 1 is again we have here as 1. So, d beta by d theta is a kinematic co-efficient for which you already know expression we can do d mu 3 by d theta that involves d theta by d theta again d mu 4 by d theta we can take derivative of that there will be d psi by d theta which we already know because, that is again kinematic co efficient we know the derivatives of all the mus that are there is the strain energy expression. So, recall that these are the expressions for the kinematic co efficiency d beta by d theta d psi by d theta it just depend on the current configuration values and of course, the lengths of the bodies in this 4 bar pseudo rigid body model.

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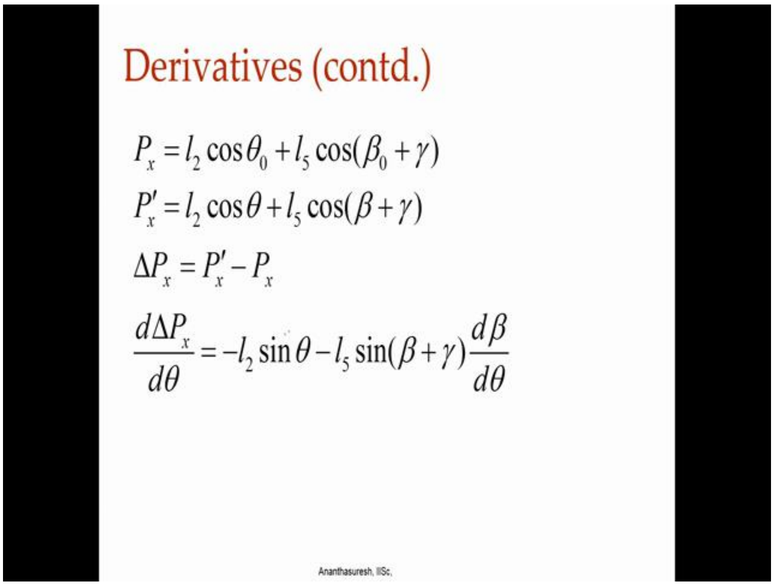

So, with these things we also have P x and P x prime. So, we take delta P x and then we take derivative of that, that terms it to be this again it depends on the current configuration beta and the kinematic coefficient and the current theta value.

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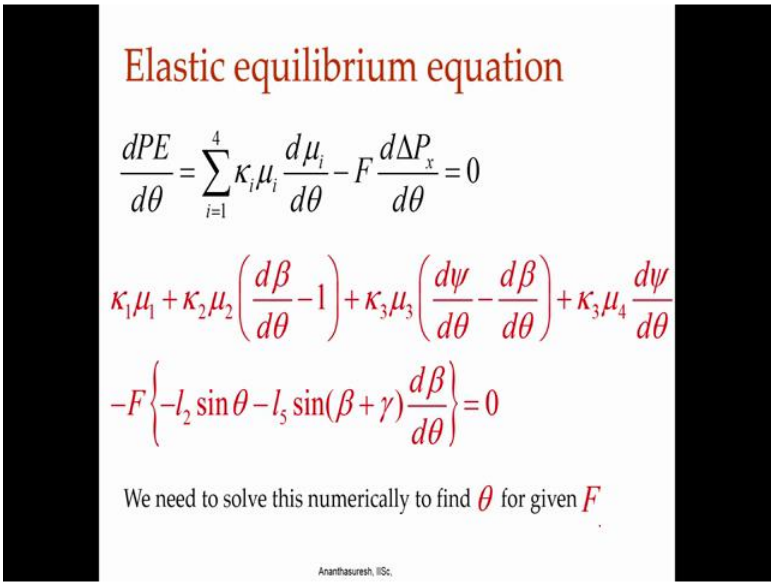

So, with all of these things we can now write elastic equilibrium equation, which is d PE by d theta equal to 0 which substitutes for all of these d mus and then there is mu here and delta P x derivative respect to theta if you substitute all of them we get an expression like this. So, this one if you look at because, the kinematic co-efficient d beta by d theta d psi by d theta they all depend on theta beta and psi the current configuration we have everything expressed in terms of theta we get current configuration, if we know theta we get beta and psi. So, everything here is going to be in terms of one variable theta. So, if I know F I should be able to find theta if I know theta I should be able to find F. So, this is the equilibrium equation we are looking for force displacements equation exact thing we need to solve it analytically solving this will be very difficult or even impossible, but we can solve it numerically to find theta for given value of F or find F for given value of theta whichever is easier we will see which is easier.

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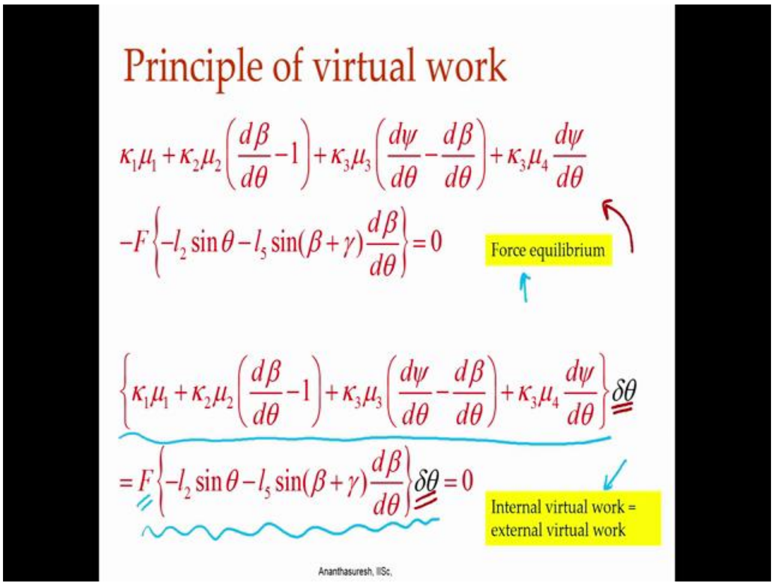

We also can look at this equation in terms of principles of virtual work we used principle of minimum potential energy and got the force displacement equation or force rotation equation here by equating potential energy derivative with respect to theta to 0 alternately we can also use principle of virtual work. In fact, if I start from the force equilibrium equation which is this if I multiply by virtual displacement or virtual rotation delta theta what you have on the left hand sides that is, let me change the color what I am going to underline here is nothing, but internal virtual work and what I have here is external virtual work that is if the system is in equilibrium if you were to imagine a virtual rotation virtual displacement in general virtual rotation there will be work done by the internal forces that is wherever the torsion springs are there will be internal torques and there will be external virtual work because we have the external force F.

So, if we look if we equate internal virtual and external virtual work then you get the equilibrium equation again because, delta theta that is virtual rotation is arbitrary which is nothing, but the force equilibrium you can do either way we can also do this principle of virtual work by equating internal virtual work with external virtual work for arbitrary virtual rotation delta theta that is the two ways of looking at the same thing they lead to the force balance in the as we have got in the here.

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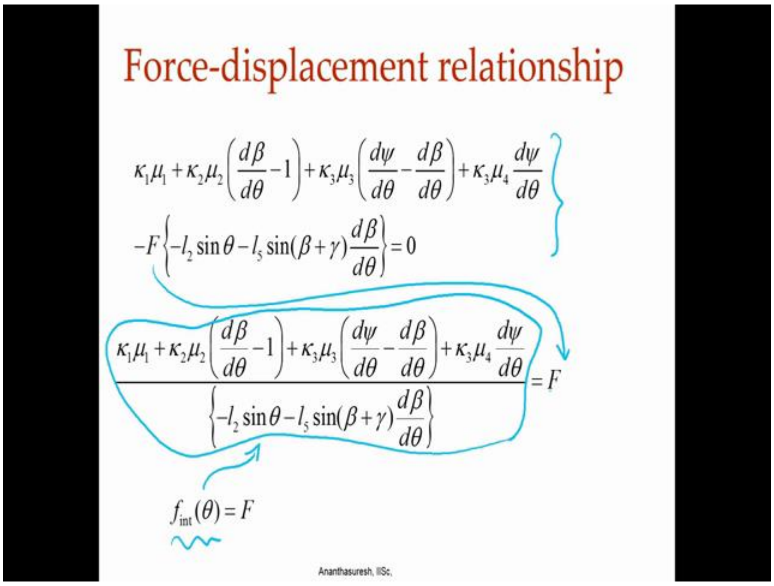

Now, we have the force displacement relationship. So, this equation that is equilibrium equation from there I will I have taken F to the other side. So, this F is left alone here and the rest of it you have is basically internal force that is the force balance also as we have seen minimum potential energy principle, principle of virtual work force balance there are all same things the same things said in three different ways all of them are powerful. So, we are use principle of minimum potential energy here and got an internal force equal to external force internal force is this long expression that we have this entire expression is our internal force which is equated to external force which is our F here.

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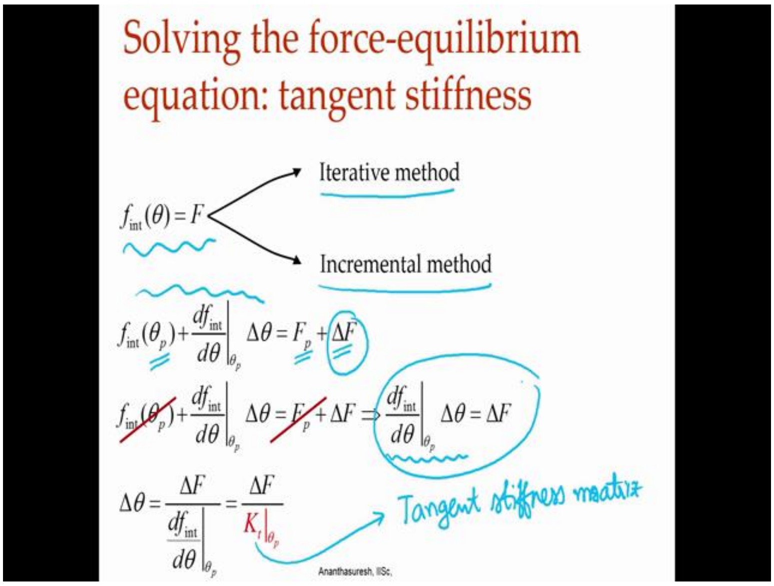

Obviously say non-linear relationship F internal how do you solve it, in general when you have this non-linear equations especially in elastic mechanics you can solve it using what's called iterative method or incremental method or the combination of two all of this both methods that you are deal with they work by linearization or non-linear equation we know that this is a non-linear equation, because f is internal which is our examples non-linear function of theta, which we do not know theta 0 configuration we have when we apply force going to move another configuration there equilibrium should be valid and that is we got this expression, but we do not know theta right. So, what we do is we go the previous one this iterative method that we want linearize around some theta p it could be theta 0 itself that is current configuration when you take the first order term d f internal by d theta a validate theta p times delta theta.

So, then you have F p times delta F, because F also we splitted into whatever was there at theta p and additional delta F, if we do that in the previous thing if we had balanced that is you can cancel f internal at the previous one and f p in the previous then what we have is this d f internal by d theta a validate at the previous one times delta theta equal to delta, when I say previous that is why we are linearizing and moving on to next configuration. So, this particular quantity d f internal by d theta is called tangent stiffness matrix denoted by k t here why it is called tangent will see that we graphically look at it, it is called the tangent stiffness matrix.

So, this is the one that we need to know at every point where we want to update to the new configuration when we apply a force delta F it linearization in non-linear equation linearize, but of course, this linearization is going to be valid only for a small changes that is why we are taking delta f that is the increment also iterative as we see delta theta can be obtained delta F by tangent stiffness matrix that basically comes from this, force balance then in a linearized fashion non-linear finite element analysis all of those pretty much work in a same way linearize and correct and do that and then there predictor corrector kind of methods.

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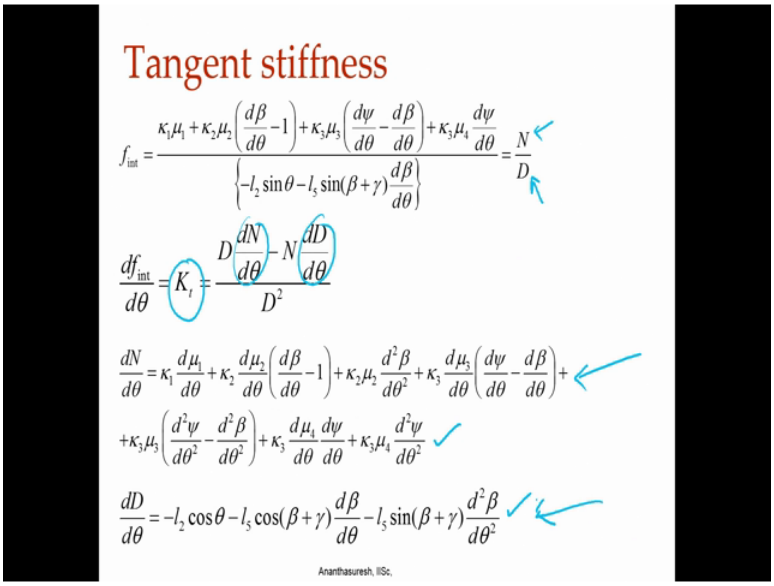

Let us look at this expression one more time f internal we have this let me denoted as numerator n denominator d. So, I can take derivative of d f internal by d theta in terms of n d then that will involve d n by d theta d d by d theta that we can do is longish way we have all the expressions just t d s what we can do it d n by d theta d theta by d theta d d by d theta. So, basically we get our tangent stiffness matrix k t at any given angle theta because, this expression all of them depend only on theta because with theta we can get beta we can get psi and everything else that is there here.

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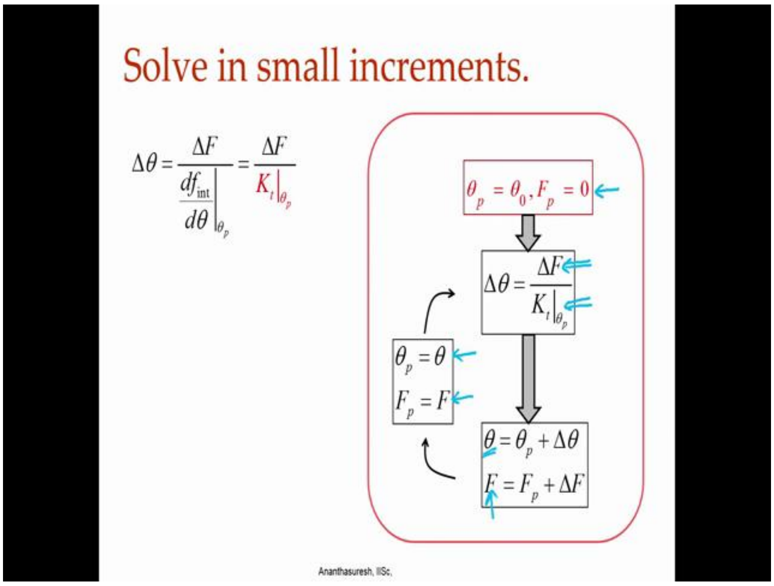

If you want to solve in small increments by linearizing we will go small increments we start with this theta p equal to theta 0 current configuration F p equal to 0 there is no force in the original configuration theta 0 configuration when before we applied the force, then delta theta we apply a small increment delta F and compute this tangent stiffness matrix get delta theta and update your theta by adding theta p and delta theta and then F F p plus delta F and make them your mu theta p and mu F p and go back here compute tangent stiffness matrix again and delta additional increment delta F get delta theta update this and keep on doing it until force reaches that force you intent to apply in small increments what is small we would figure out in your problem what is small.

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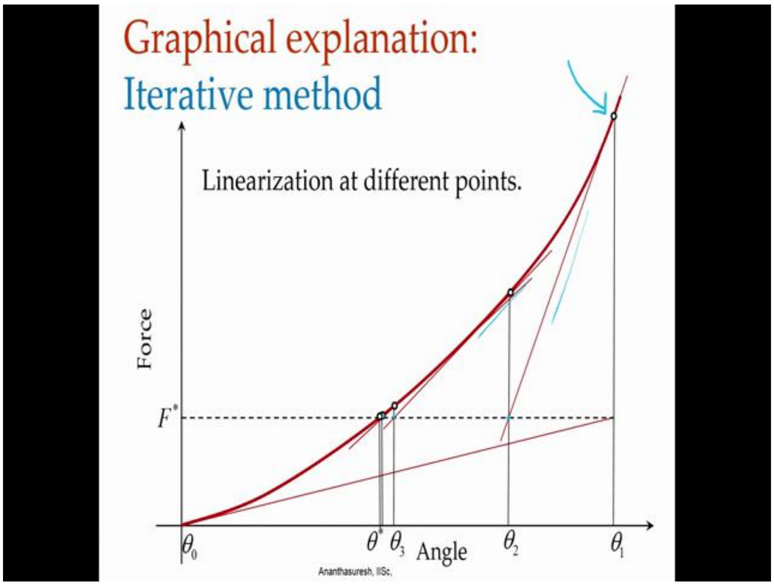

So, let us look at that graphical way of visualizing this iterative method. So, if I have an unknown function this red curve which is not known we do not know that we need to find it. So, respect angle we have the force right. So, what we do is for some F star I want to find theta star. So, given this I want to find theta star given F star if I use this iterative method not increment first iterative. So, what I do know is this slope here I can compute tangent stiffness matrix is called tangent stiffness matrix because, it is a tangent to the force displacement curve. I draw this tangent there and I think then theta 1 is my answer; obviously, it is wrong because non-linear, but F star where it touches. So, you get theta 1. So, we get theta 1, but it is wrong.

So, we go over there and draw the tangent again. So, this tangent is drawn again right this again need is F star over here, but that is also not correct because when I compute F now that will be wrong right. So, I get theta 2 which is a better update then theta 1 there again I draw a tangent here again it needs over here and that is theta 3, but when I compute that is not F star again. So, I get a different value I go that point and do a tangent again there and that eventually theta 3 theta 4 I did not had the room write it theta star it come. So, about four it rations this particular curve where able to find for F star the exact point to our tolerance it may be more if you keep a very tight tolerance this is iterative method it will take in a non-linear finite element code, when we are doing the actually have to do a few iterations like this before the convergent to the solution.

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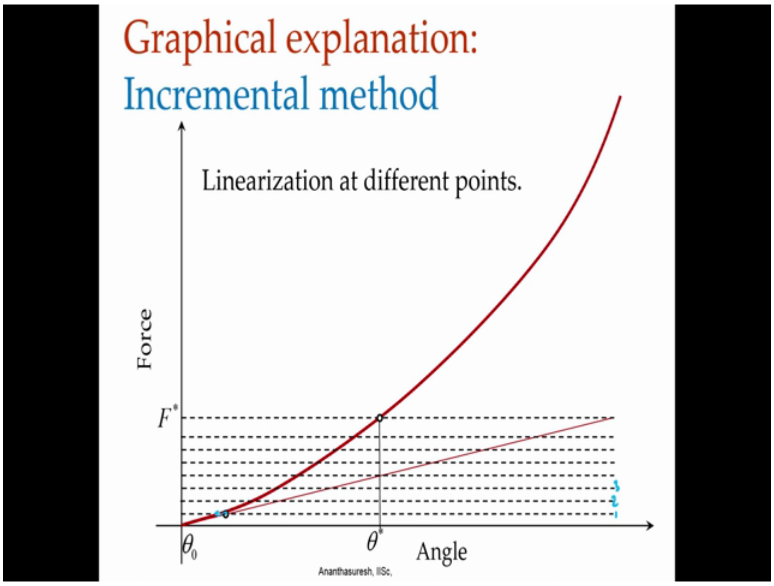

Instead we can also do increment instead of taking F star all at once take it in several steps. So, we have 1 2 3 and so forth, then the first one that we have shown already there is very close to the answer, the answer is here we are over there one more time if I go and tangent I am going to come back drawing tangent nothing, but linearization of the non-linear equation.

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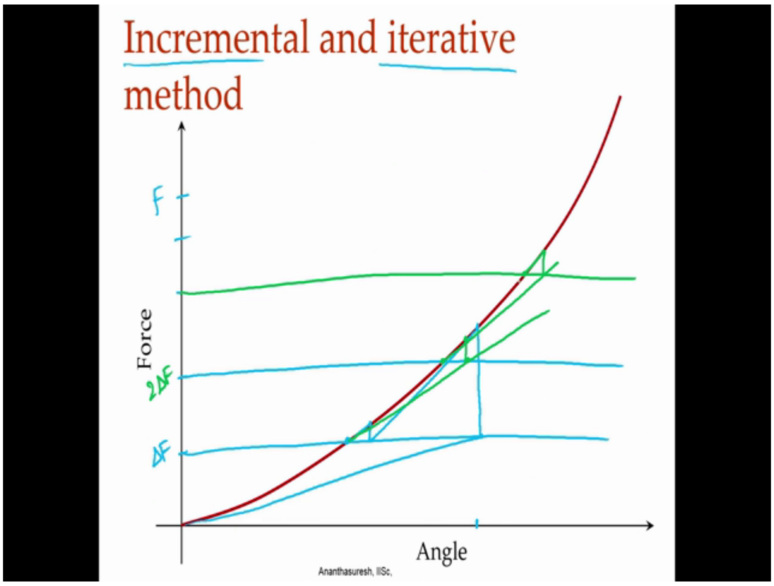

We can do this incremental method, but in general what people do is a combination of the incremental and iterative method that is you take let us say your force value is this much do not apply all of it at once, because you may not able to go there in few iterations. So, it divides up into small increment. Now, you have this which we had looked at already. So, first we do this and we think this is the estimate of the angle. There if you do that, that is not incremental that we have here the first delta F then we draw a tangent there, we come over here then we see this still not have a our delta F we come back here we got it. So, in just three iterations will got there, from here we go to the next level.

We are here now let me change the ink color, we can see better if we take green. Now, we go the next one it happens to touch here this is your 2 delta F. Here the force is not there so I draw a tangent from there; since when lucky this two iterations I got the curve right, then you go to the next one I draw a tangent the force difference is there I do that I get back to the answer. So, increment iterative we do in each increment you can get away with only a few iterations that is what finite element software also does it is a way of solving non-linear equations.

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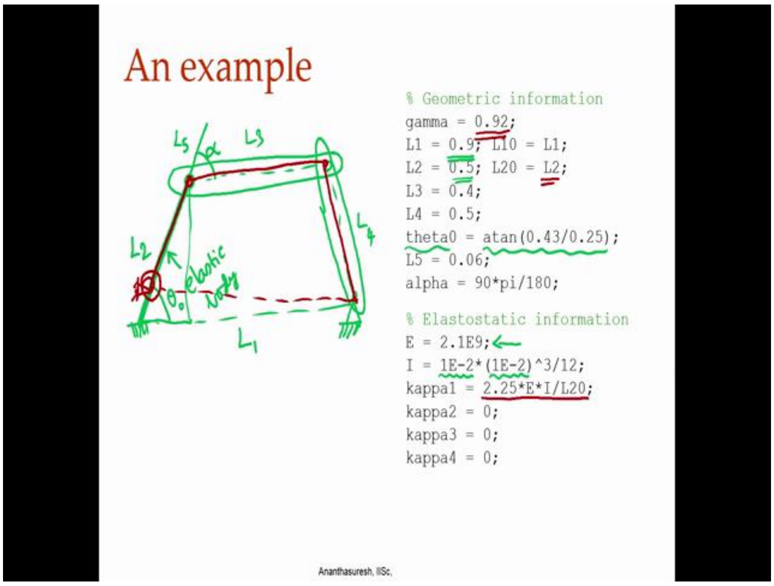

Now, let us take an example so example we have taken is an elastic beam this is an elastic beam a cantilever beam and we put a joint there and there is a rigid body and another rigid body which is fixed here, it is a compliant mechanism because this is an elastic body. Here the length between this these two L 1 which is taken as 0.9 meters this is L 2 which is 0.5 meters this is L 3 that is the line joining these two line joining these two is L 4 and theta 0 this particular one theta 0 here is given as tangent inverse 0.43 bar 0.25 that is something here is 0.25 this is 0.43 and then we also have L 5 something like in general this is L 5 as we had another thing and the angle here is shown as alpha not gamma that we had in the slide because, gamma is reserved for the characteristic factor and then that is given elastostatic information youngs modulus is 2.1 giga pascel and moment of second moment of area of the elastic segments is rectangular, which is 1 centimeter another 1 centimeter is square cross section.

So, we have b d cube by 12 and then since this elastic segment we can replace that with a pseudo rigid body model where we shift this over here and then make torsions spring. So, we have taken kappa which 2.25 e i and then L 2 L 2 0 is same as L 2 here, and gamma we have taken 0.92 I will tell you why 0.92 little later. So, we have the pseudo rigid body model where will have a rigid body model from here to here to here, and this becomes a new L 1 all that as to be calculated, there are all done in math lab code that will have in supplementary file.

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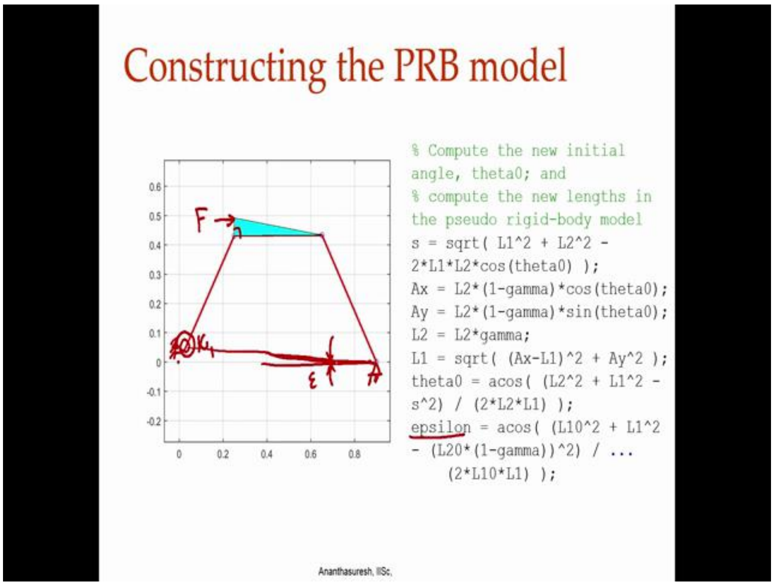

For this lecture it looks like this, the original one was here we have moved it a far here because, that is what it requires it move it by 1 minus gamma times that length and then this stays where it is and here is where we have the force applied in this problem, that point is given with alpha equal to 90 degrees L 5 is whatever that was given in the previous slide. So, that is our construction pseudo rigid model and of course, there is a torsion spring there with torsion spring kappa 1 kappa 2 kappa 3 kappas 4 the other thing is 0 there is torsion spring only here with our PRB model and this epsilon that is shown here is this angle. If I join this here this angle is epsilon, because the originally it was here, but this was moved that is epsilon that is also needed in the matrix calculations if you see that code will understand what it is.

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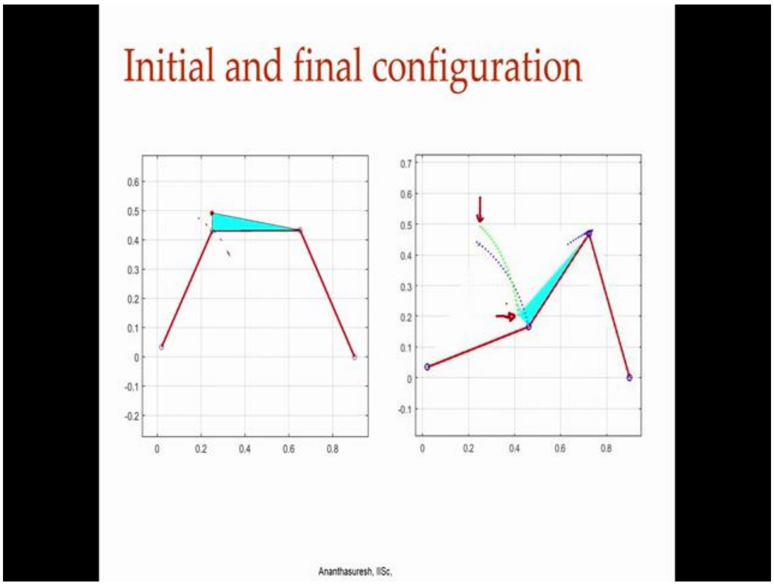

Now, this is the initial configuration this is the change configuration. So, this point here, we shown with is green thing here and the blue one is just the crank that is the blue one and this is a rocker it goes there and comes back that is this; this triangular is a coupler link when we apply the force over here how this as moved to shown here.

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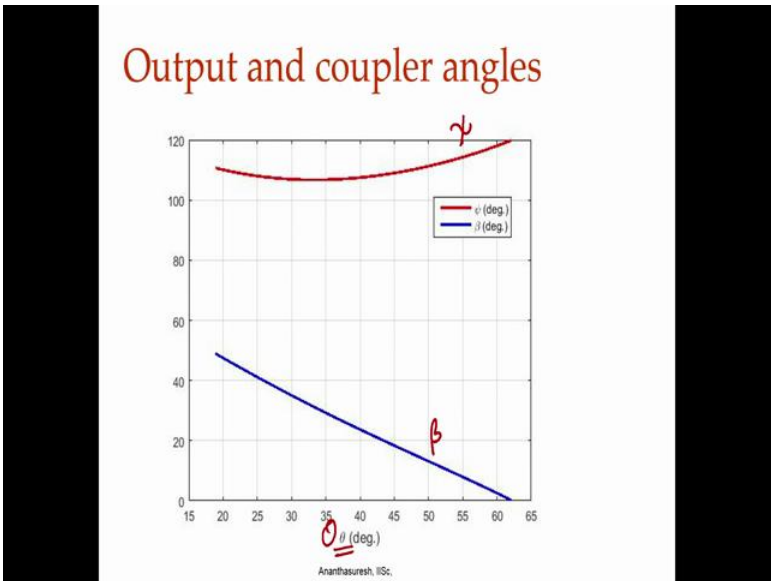

And these are the angles red one is psi blue one is beta with respect to the theta that we have here, how they vary as we apply the force.

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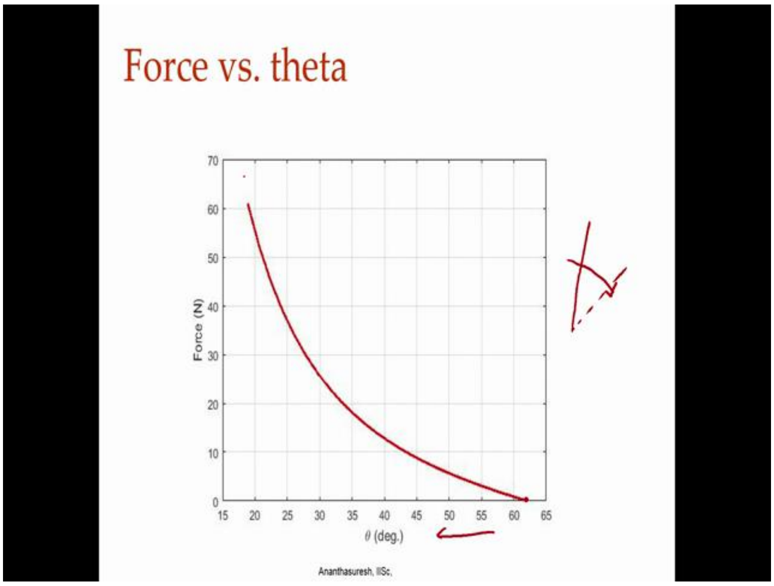

And this is the force verses theta relationship you see this it initially it is some 62 or 63 degrees and then as you applied the force the angle is decreasing because, our crank is like this it actually moving the clock wise direction that is why the angle is decreasing here as force is increasing and the force is gone up to 60 about 61, 61 Newtons here.

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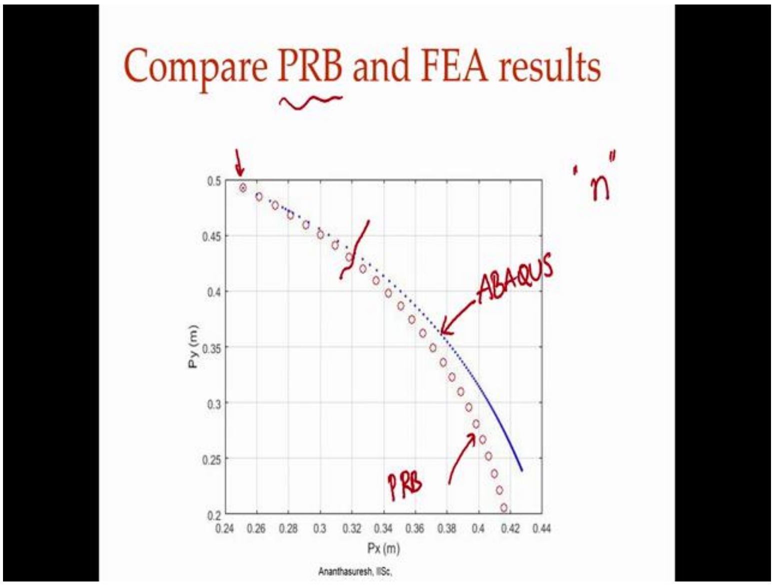

And here is a comparison of the path of the point p p x verses p y verse P x the red ones are the PRB and this blue one came from a finite element analysis software in this particular case we use ABAQUS as you can see they are not in excellent agreement initially these are starting point, that is where we are and it is moving initially up to this point looks like agreement is good after there is deviation, why is there deviation to begin with PRB modeling is an approximation, right. Where is approximation function failed here it is because, when the elastic beam is there attached to the other two bodies and force is applied at the point p the force direction that comes from cantilever beam is not always transverse to it. It keeps changing rather what we called n that is the ratio of the axial force of the transverse force keeps changing configuration, if we take one of them it may not actually work.

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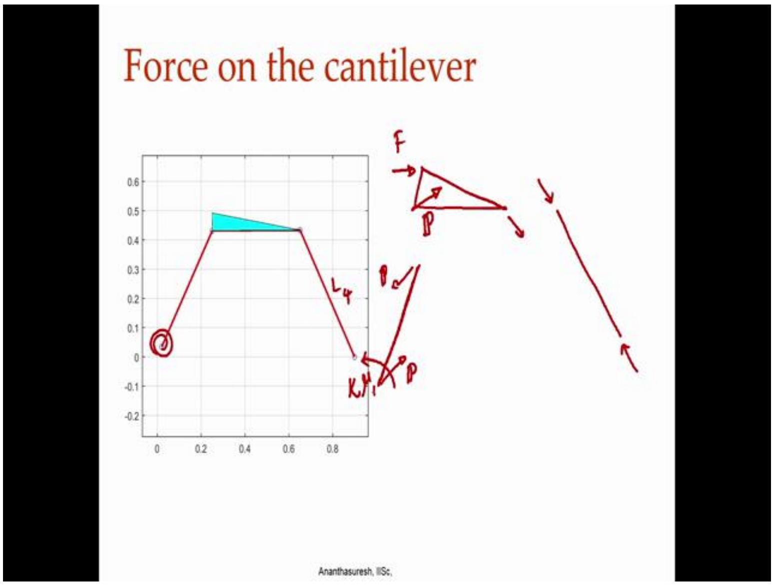

So, for that let us recall that the force on the cantilever that comes here that you want to get that also we can compute here, because if I draw the force balance for this, is a two force member this particular one that is L 4 and this triangle we have that force this way and then we have f applied here and then be a third force let us say that let put it that way let us call that force P like that that is the one comes on to this cantilever, that is not necessarily perfectly transverse to this particular elastic beam that we have and there is of course, because of torsion spring is there. So, there will be a torque also which is kappa times what we called mu 1 or delta theta right, that is there we can do this force balance and find out what this P here is, Ok.

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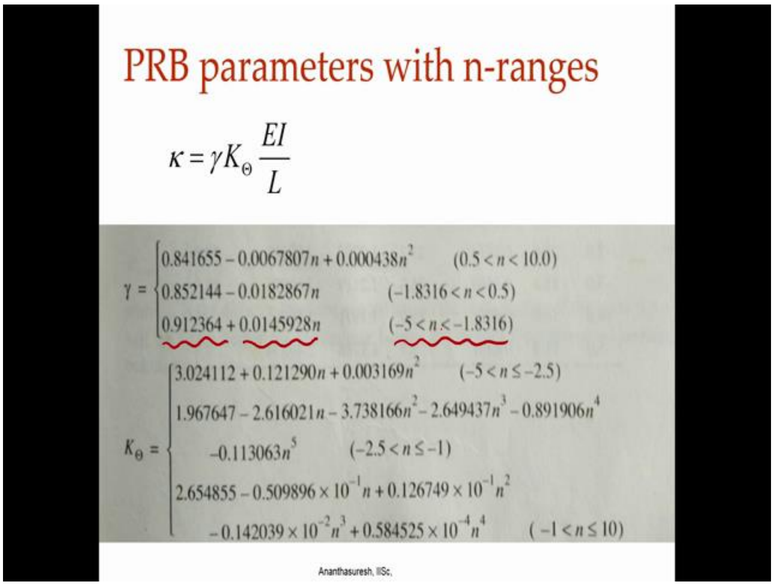

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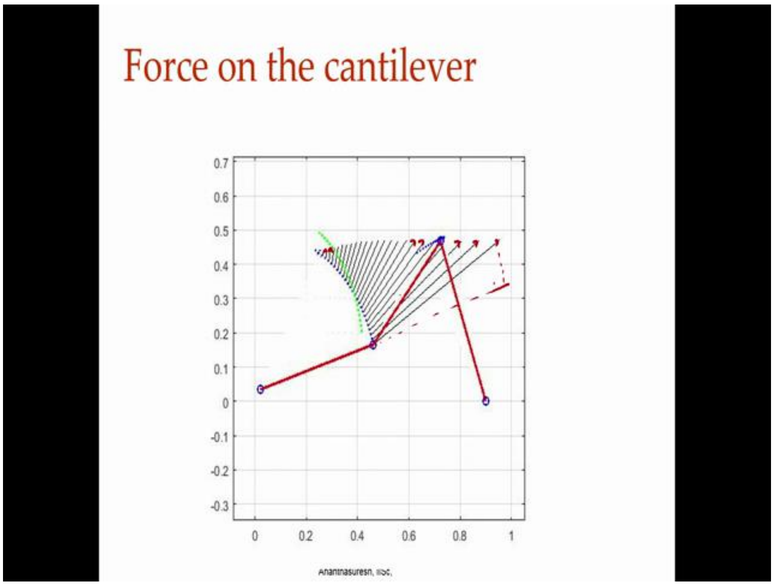

If we do that, here this black arrows represents forces I can actually put arrows here this is the force is varying at this point at tip of the cantilever beam and that is clearly not transverse let us if we take this most of this axial little bit is transverse. So, n here is negative in a large value. So, if we go back and look at our PRB expression for different n values especially here right, it is minus 52 minus 1.83 where take 0.9 times this one, but n keeps changing from position to position as this crank is rotating as we applied the force f. So, I have taken 0.92 I computing average n here that we found that is why the gamma in this problem 0.92 not 5 by 6 not 0.85 because of the way the, the force keep changing at that joint.

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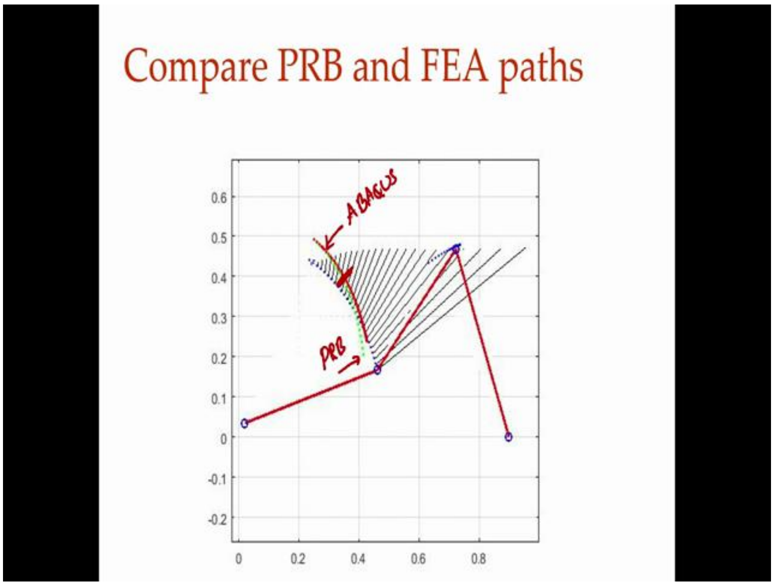

And if we do that then you get a approximation it may look like a large deviation if we look at 4 bar linkage the green one is what f e a I gave here and sorry that is given by PRB the green one this is green dot are the PRB this is PRB and the red one is ABAQUS. They are pretty good up to this point and then they deviate, that is to be expect because n is changing in position it is actually changing quite a lot it is actually not even that minus 5 range for some values it is even minus 15 is there sometimes it goes almost axial force compare transverse force and that is why it is not really tallying, but pretty much it is there.

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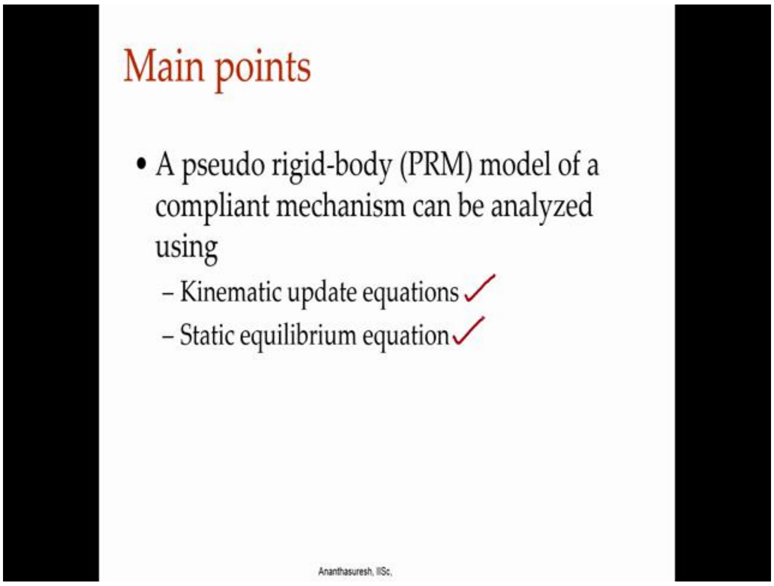

It is good enough for 1 degree of freedom model. So, to conclude we illustrate an example where we have shown how kinematic update equations can be integrated into static equilibrium equation. So, we get a force displacement relationship where we can get by solving one variable problem we can get force displacement equation and solve it. So, that the result compare with finite element analysis quite well it is not exact agreement, but then finite element analysis several degrees of freedom take an long time where as this one is a 1 degree of free modeled and gives you a reasonable result we look at a math lab code for couple of minutes after this.

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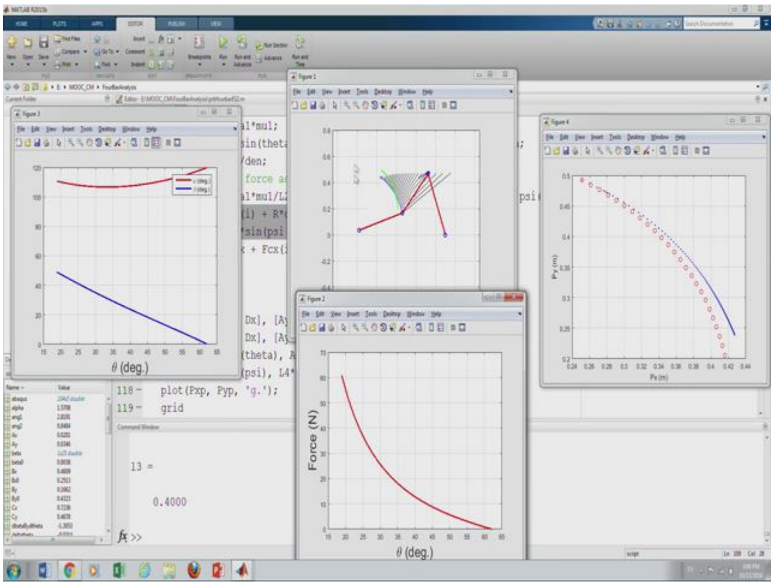

Here is the MATHLAB code program for the example that we consider today we have the gamma as I explained there we have taken this 0.92 this is the characteristic radius factor to move the fixed end to the point where it is perverted with a torsion spring and L 1 the fixed bar, to peak is 0.9 meters this is 0.5 0.4 0.5 the program is written general whether you can put any numbers it would work this is the initial orientation theta 0 we are storing that as another variable, because we use it somewhere else in the program later on L 5 that is 6 centimeters where that couple of points are located and alpha, which in the slides was gamma that (Refer Time: 35:27) gamma there we have made it alpha here that is 90 degrees to that beta 0 position and then we compute the new initial angle because, a way move the fixed by 1 minus gamma times L 2.

So, all that calculation is over here and we compute the other initial angles psi 0 and beta 0 and compute the coordinate point. So, we can actually plot it in the initial configuration lengths all will change L 1 will change L 2 will change because, the one of the piots 6 piots is moved because of PRB model, and then we have the youngs modulus here which is 2.1 giga pascel second moment of area and kappa 1 is taken as 2.25 this can also be adjusted based on the PRB model if we know that n how it varies from position to position we can do that, but here we have taken average one given here which is 2.25 e i by L 20 other angles other torsion spring constants are 0 because, there is no torsion spring there we plot the initial configuration and then we compute the other configuration in steps we are using in steps. So, we go the 24 steps each step we taken delta theta. So, here the program is written for given theta how to compute the force that is the easier way of doing if you want to know delta theta for given delta F then you have to do the iterative increment method that we discussed, but instead here we have taken the easy way out that is knowing delta theta we compute F.

So, delta theta is minus pi by 100, 100th of pi and we are doing 24 steps that come about the force about 60 Newtons in this exam 60 about 61 Newtons. So, we compute the new configuration delta theta when you have you can use kinematic update equations or even in exact one here this is done exactly to get the new angles psi and new angle beta here and the kinematic coefficient which are need for the derivatives that for the tangent stiffness matrix. We can plot the moved configuration and also we calculate the force that is important part. So, we have mu 1 mu 2 mu 3 and mu 4 we compute mu 4 mu 3 because, mu 3 as mu 4 in it and then the numerator in the force displacement relationship this kappa 1 times mu 1 other kappa's are 0.

Only that is there denominator from the expression from the slide if you look at that is this and the force is numerator divided by denominator and then we also find the force acting on the cantilever beam to that force equilibrium for the three things PRB model we can get that magnitude of the reaction force in the two force member that is the output crank and we can also get the these two are the forces acting on the cantilever beam f c x and f c y and then we are plotting that. So, I will run it now, we can see so we just see how this is moving, it is done already finite element analysis would have taken 5 minutes or 10 minutes the 1 degree of freedom model solve them we have all the plots that we needed we see the animation. So, how it moves that you can see the slides and then here we have force and displacement and here the red one the psi is blue one is beta we have those things.

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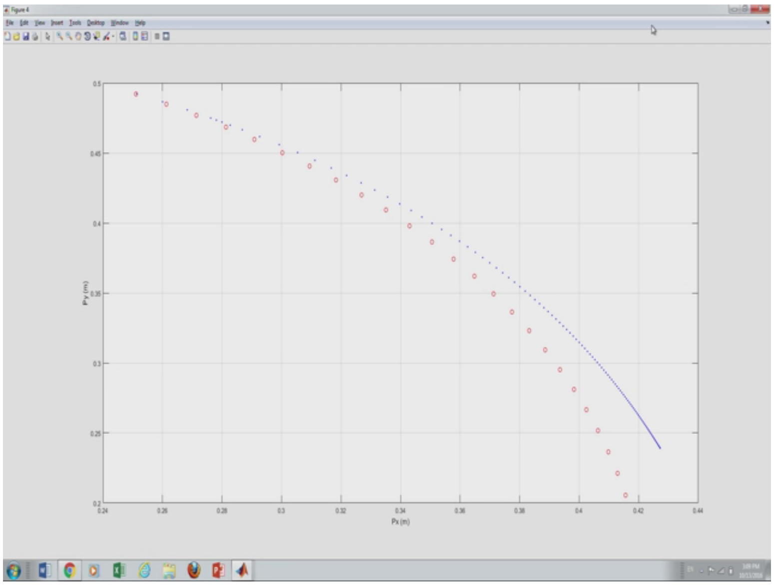

And finally, this one over here where we have the PRB the dots and ABAQUS finite element result is the blue dots and the red dots we have that comparison which you have already discussed in the slides.

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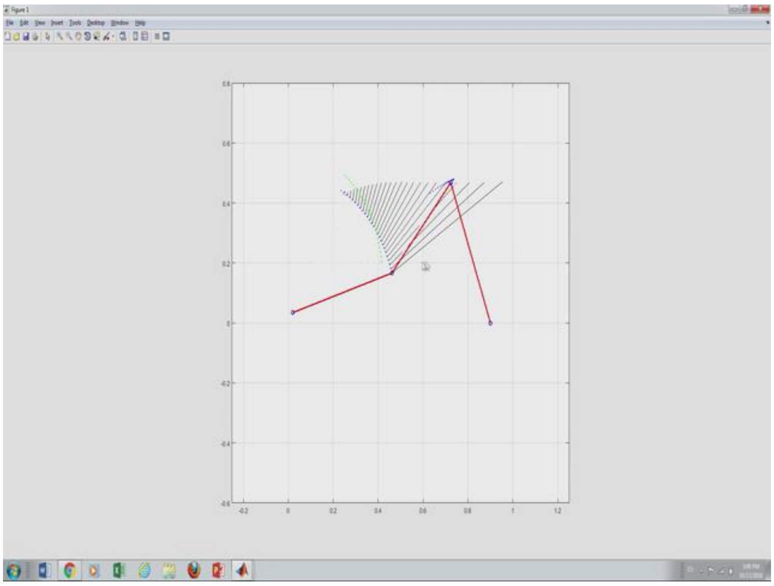

And over here if you look at and then we can see barely this green dots are the locus of the point this black lines are the forces at each position starts from 0 and you can see compare to the elastic beam which b at 0 0 here at there it will be here, there if you see it is almost axial right, it is not transverse in this case and that is why our gamma became 0.92 and we also need to adjust the kappa here torsion spring constant then we get, but then for position to position to it is actually varying if we take that every where the exact result, but we do not need to worry about that much detail because, here itself you are getting a approximate relationship between the path of the point as well as the force from the pseudo body model. So, you can change anything you want here and run the program, so that you can compare this result with a finite element analysis if you want to solve some other example.