Compliant Mechanisms: Principles and Design
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Lecture – 19
Analysis of Complaint Mechanisms using Pseudo Rigid-body Modeling

Hello, this week we are going to discuss Analysis and Synthesis of Compliant Mechanisms of certain kind using Pseudo Rigid Body Modeling. Last week we had discussed at length about the pseudo rigid body model as well as the elliptic integrals and how elliptic integral solution give rise to the pseudo rigid body model using which we can bring complaint mechanisms to the ambit of rigid body linkages. This week we are going to look at that both in terms of analysis as well as in synthesis, this lecture today will focus on the analysis of complaint mechanisms using pseudo rigid body modeling or PRB modeling.

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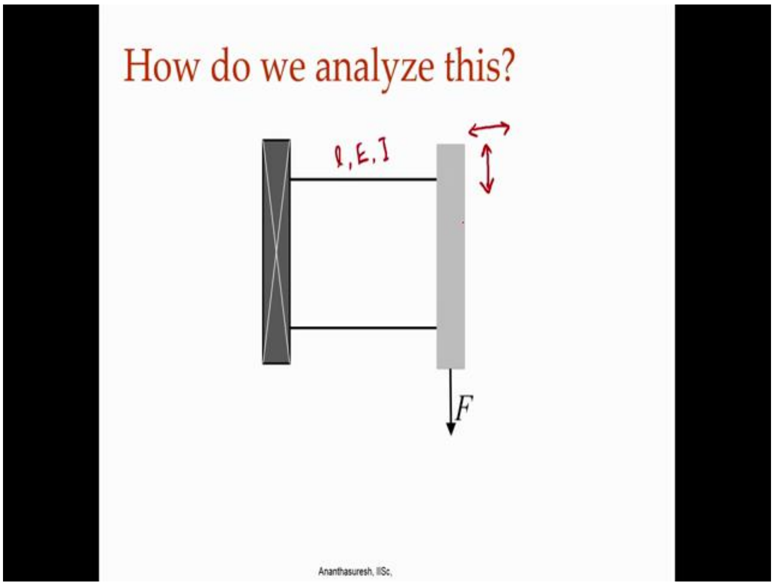

Let us look at this simple parallel guided motion mechanism where when we apply force as shown here this block here is going to I have to get the pen. So, this block here is going to only translate parallel to this it will have motion in that direction depends on which we apply the force as well as this direction that is what it is right. So, when we are given length young's modulus second movement of area and this force value we want to know how this block moves is what we mean by analyzing this how do we do that.

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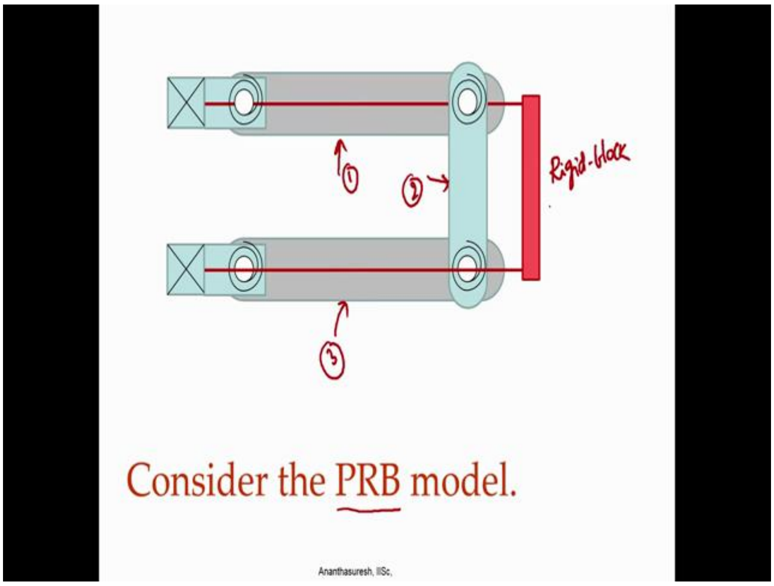

The idea is to use the equivalent model of it with only 1 parameter which is this pseudo rigid body model. So, PRB modeling here refers to pseudo rigid body model which is shown here. So, where the red one is the compliant mechanism there are long beams slender beams and there is a rigid block. So, this is the rigid block and we have constructed a model that demonstrates the same behavior that the real complaint would do using rigid bodies here is 1, and here is 2, and here is 3, with 4 joints and at each joint we have a torsional spring as it is shown to model the elasticity of the complaint mechanism. So, here as we have seen last week that.

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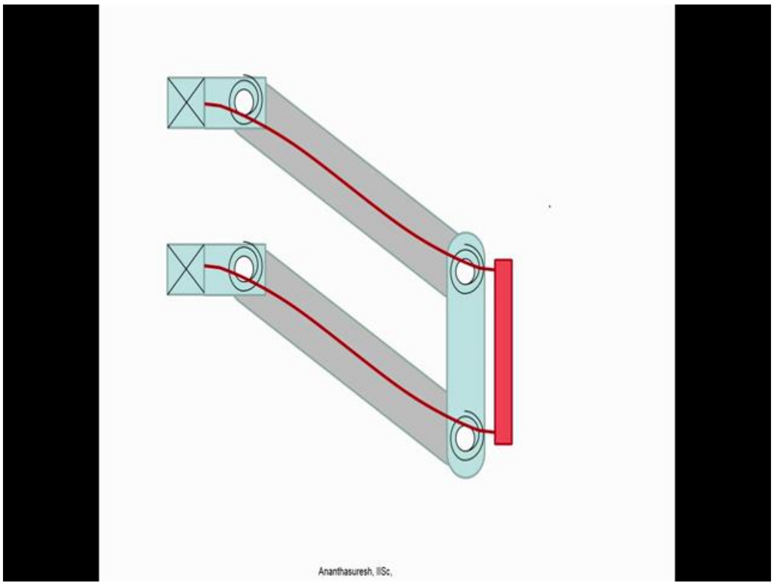

When we apply the force it us going to deflect like that, so, what is like this becomes like this and it just rotates and moves right. So, using this concept where a rigid body linkage with torsion spring is going to demonstrate the same behavior with complaint mechanism we proceed to do the analysis of the compliant mechanism using pseudo rigid body models.

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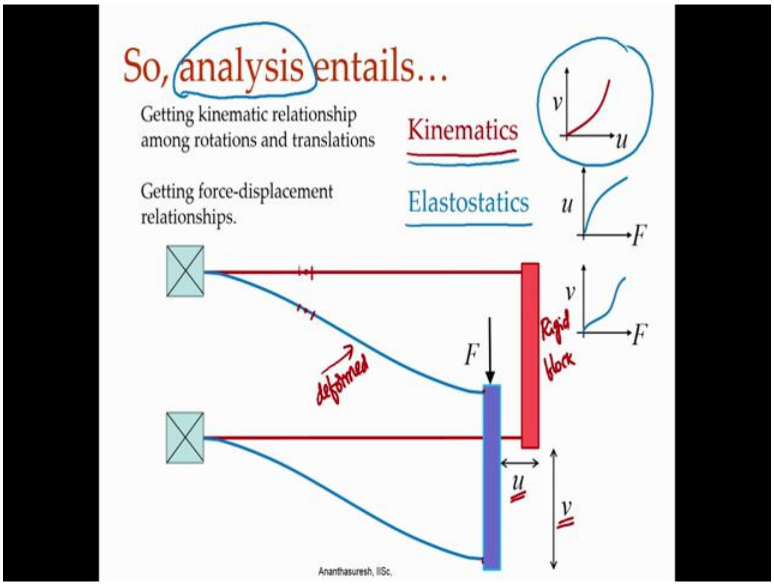

So, analysis here entails two things one is kinematics other is electrostatics, kinematics means here that we want to find the kinematic relationship among rotations and translations or the bodies inside here, we have this rigid block over here this rigid block which is attached to 2 beams slender beams even you apply the force it is going to deform as shown here, this is the blue is the deformed configuration red is the un deformed configuration and what are the translation rotates involved 2 translations are shown the u and the v. So, these two are the displacements of the rigid block the 2 translations it has.

So, when I say kinematic what I mean is that can I get this relation between v and u as u changes how is v changing there is definitely coordination between these 2 translational displacements similarly, if I want to take rotation somewhere which is let us say over here compared to here if I take a small element there that has definitely rotated, I can also relate this rotation to let us say u or v that will be the kinematic relationship and that is what you want to find in this rigid body model our interest is only the end point that is this point and this point we do not care about red points.

So, we are showing only 2 translational displacements as far as this end points are concerned pseudo rigid body model gives you as we have discussed in the last week fairly accurately representation of what is happening in the real compliant mechanism that is kinematics. Now, let us switch to the electrostatics, right electrostatics what do we get there we get the force displacement relationships meaning, that we will be getting u verses f and v verses f force is f here and displacement there are two there is u and there is v. So, when we say electrostatics we want to know how displacement degrees of freedom they can be translations or rotations how they change with the applied force this is electrostatics.

So, when we say we are doing analysis of compliant mechanism we mean two things kinematics as well as electrostatics, as you remember kinematics is something that does to take into account the forces that cost the displacements or the motion here, also when we do kinematics we do not have to worry about the force if there is the 1 displacement there will be another displacement which will be controlled by a function such as this. There will be a relationship kinematically among the variables and we do not worry about the force of course, in a compliance mechanism the force is not there, there is not going to be displacement or motion, but when we study kinematic behavior we would not worry about a force that is the highlight of kinematic analysis. So, we will first do kinematic analysis and then go to electrostatics.

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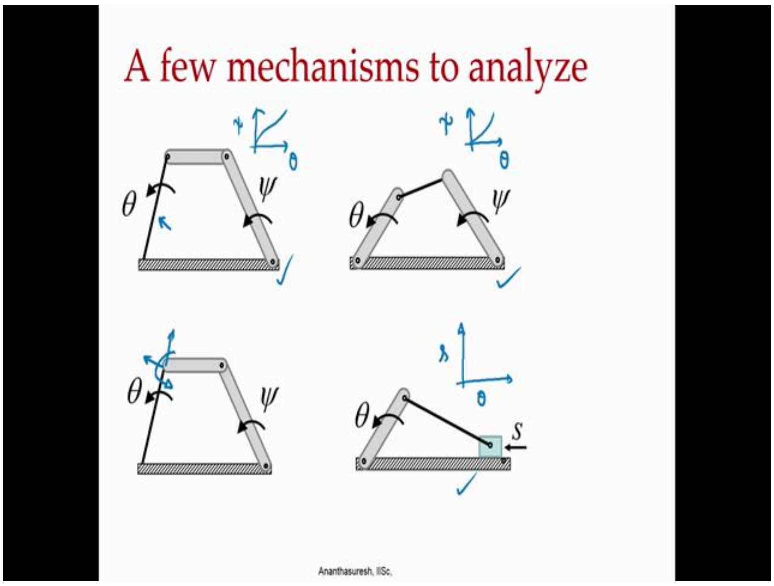

Let us look at first a few mechanisms that we can analyze using the method that we are going to discuss now, the first one there is an elastic segment. So, this is the elastic segment there and the other two are rigid segments and there is a frame. So, can I find the relationship between this psi and theta that is can I get this kinematic relationships psi verses theta given some geometry of the thing and material properties and so forth. In fact, we do not need material properties also work in kinematic analysis because forces are not accommodated for, but then we go into elastic analysis we do need that likewise here, again I can say I want to get this psi verses theta I can get that curve that way at this way whatever also this and also this here, I have shown a variant where instead of having another rotating segment we are showing a slider here, also can I get a relation between let us say s and theta here, also because there is a elastic segments I can put torsion spring and tie to model the behavior that is pseudo body model.

So, we can do this we can do this and we can do this using PRB approach what about this as we have said in the last lecture of the last week, that when there is a fixed connection at this point let us say this is fixed over here fixed to the frame and at this point it will experience axial force transfers force as well as movement that keep changing from configuration and that is not that easy to handle using pseudo rigid body modeling. So, that will not be able to do accurately, but other three you should be able to do and whenever there is a valid PRB with its range that is satisfied mechanism then we can do this analysis by valid you recall that the locus of the loaded tip of the cantilever as well as that has to be apartment like a circle for to extent to what range of that end angle does it satisfies certain approximation and when we look at the spring constant, how well or how long that is going to be constant for what range of the rotation of the cantilever does it satisfy that limits the range of motion and when we are within that our analysis is going to be very accurate when compared to real mechanism or fire and element analysis.

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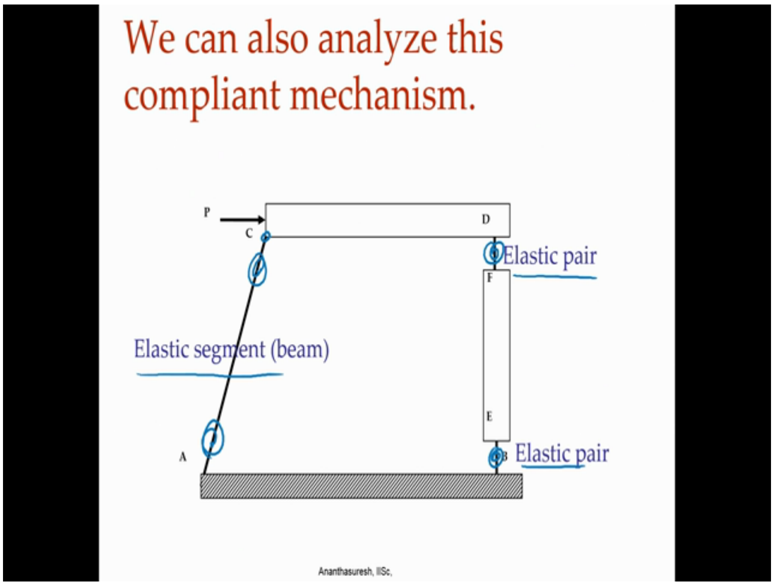

If you recall we also discussed this elastic pairs earlier. So, now, that we know how to deal with elastic segments as well as elastic pairs we can analyze mechanisms like this one also there is elastic segment here and there are 2 elastic pairs. So, just as we can put well of course, it is fixed-fixed. So, it is not going to be for a very long range, but it would be if there would be a joint here then I can put a torsion spring here, a torsion spring there, and then a torsion spring for that, torsion spring for that, that becomes a 4 bar linkage and we can analyze it in the presence of the torsion springs, that is how general procedure are analyzing compliant mechanisms using pseudo rigid body models we had to identify equivalent rigid body models and also the spring constants wherever there need to be added.

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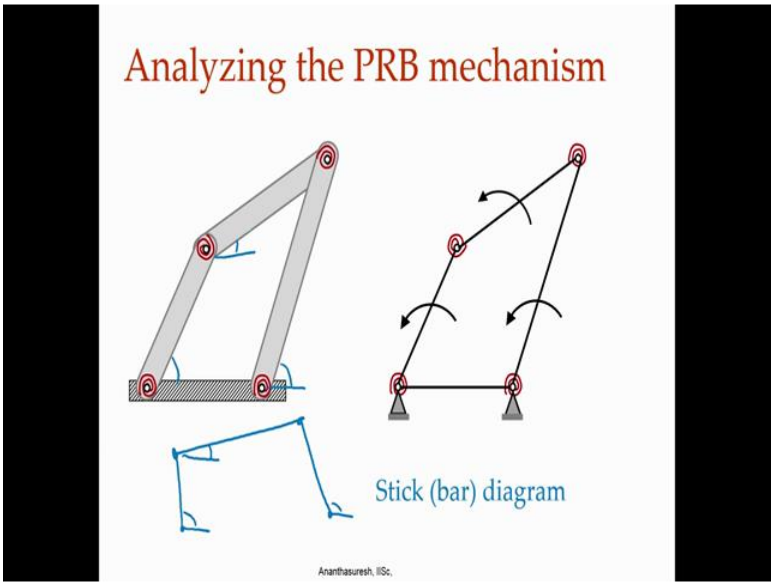

So, in general a PRB model 4 bar like compliant mechanisms, that is compliant mechanisms that can be represented by equivalent 4 bar linking such as the one that is shown here with torsion springs. Torsion spring may not be there at all 4 joints it depends on where the elastic segment is and they may not have the same spring constant everywhere there could be kappa 1, kappa 2, kappa 3, kappa 4, there are 4 of them this will be the model that we want to analyze.

So, when you want to analyze it normally we go for the stick diagram or 4 bars will see you do not worry about the actual shape of the instead representative with a line joining the joints the pin joints that it is connecting or a pin joint under a slider and so forth. So, we have something like this as a stick diagram for the one we have you may notice that I have chosen the mechanism in a particular way in a way that these angles are all in the first quadrant, that is convenient in the equation if I what to take a 4 bar that looks like this lets say then this is obtuse, this is obtuse these this is acute if you make all of them acute then it will be easier for us to write the equation. So, that we do not miss out under science in the tuna metric relationships always remember when analyzing a 4 bar make sure that all bars obtained acute angles that is there in the first quadrate and in the confirmation that you are taking to write down the equations, after that it will be wherever and tuna metric will take care of the science itself there is something that you know we have to note.

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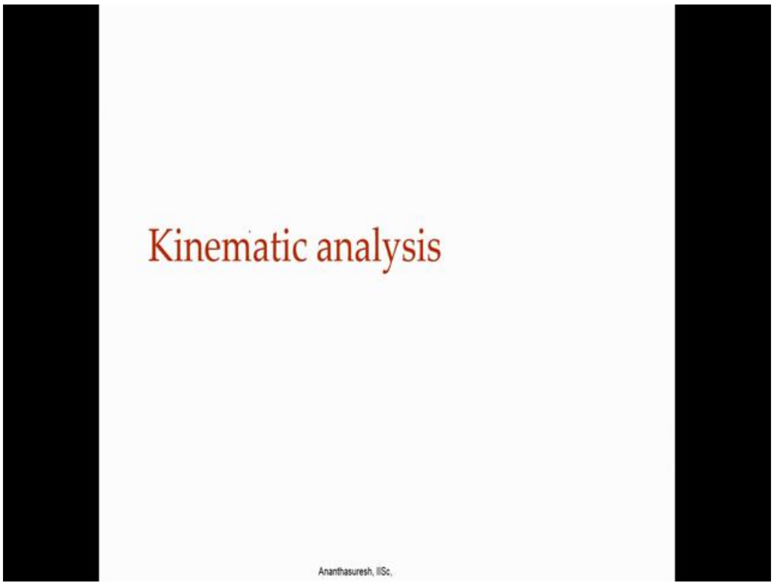

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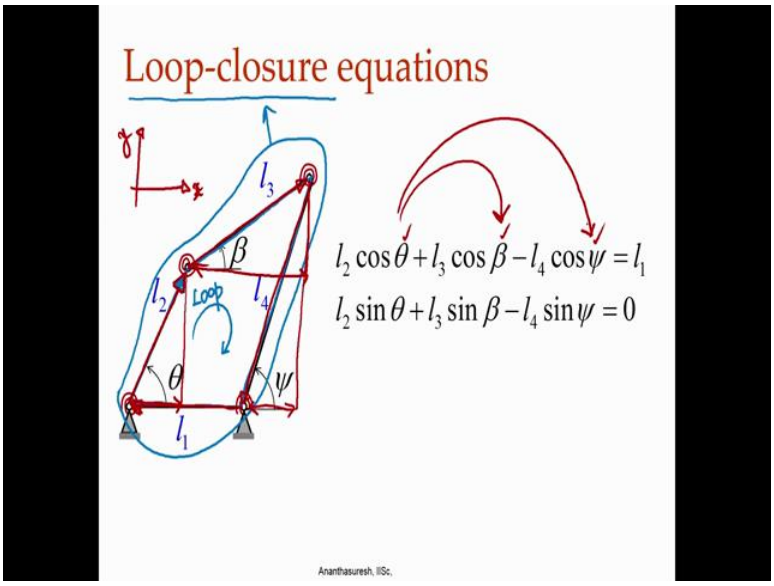

Now, let us look at the kinematic analysis of this bar representation of this pseudo body model of a compliant mechanism. So, we are indicating 3 angles theta for 1 crank beta for a coupler this is for the middle and psi for the other crank. So, input can be let us said, theta then beta and psi depend on the theta, because theta is our input is a single degree of freedom mechanism. So, once theta is specified beta and psi automatically gets specified let us take here the 4 lengths here, L 1 the length of the body joining or the bar joining the 2 pin joints l 2 and l 3 and l 4 these are the 4 lengths we need to know and we also need to know the spring constants kappa 1, kappa 2, kappa 3, kappa 4, but the kinematic analysis that will not be necessary.

So, we have these 3 angles and 4 lengths and let us denote for the sake of analysis the position of each of these bars l 2, l 3, l 4 theta, beta and psi. So, that is our first starting point to get these things and again theta is our input things. So, if I know theta then I should be able to get beta and psi as I move theta I want to know how beta and psi change that is our kinematics we want to relate the displacements or rotations, translations or rotations among them there will be relations kinematic behavior next we will do with forces or the stiffness. So, here our tool is what is known as loop closure equation we close a loop meaning I start here at this point and I go to this and I go that, I go to that and come back here that is the loop closure. So, we start from here go there, go there, go here and you can actually go back then you have closed a loop. So, there is a loop here, there is a loop we are closing the loop we get 2 equations because in 2 d when you close a loop you have 2 scalar equations because there is a vector equation. So, if you think of this as 1 vector.

So, let me change the color we can see it better let us say from here to here I have 1 vector, and then I have another vector from here to there, and I have a third vector from there to here, and a forth vector that is here to here that vector loop gives rise to 2 scalar equations. So, as you can see if I look at the horizontal here that is this axis we can call it x if you want call it x and this is y or in the x axis when we look at how the loop closes l 2 cosine theta plus l 3 cosine beta minus l 4 cosine psi is equal to l 1. So, l 2 cosine theta will be from here to here this distance and then this 1 over here is that distance that is l 3 cosine theta and then we have now this particular 1 and this particular 1 if we add we get distance from here to here, but what is l 1 is only till here we need to subtract this 1 that is l 4 cosine psi it l 1 that is a loop closure.

Similarly, y axis when I take l 2 sin theta plus l 3 sin beta should be equal to l 4 sin psi or rather minus l 4 sin psi is equal to 0. So, loop closure means exactly what it indicates closing the loop and trying to get a vector equation closure loop closure equation from which we get 2 scalar loop closure equations. So, in these equations if I assume theta value I can compute beta and psi, because they are 2 equations they are not linear because there is cosine beta sin beta cosine psi sin psi, when I know theta I can compute beta and psi that be the position analysis for this leakage problem here.

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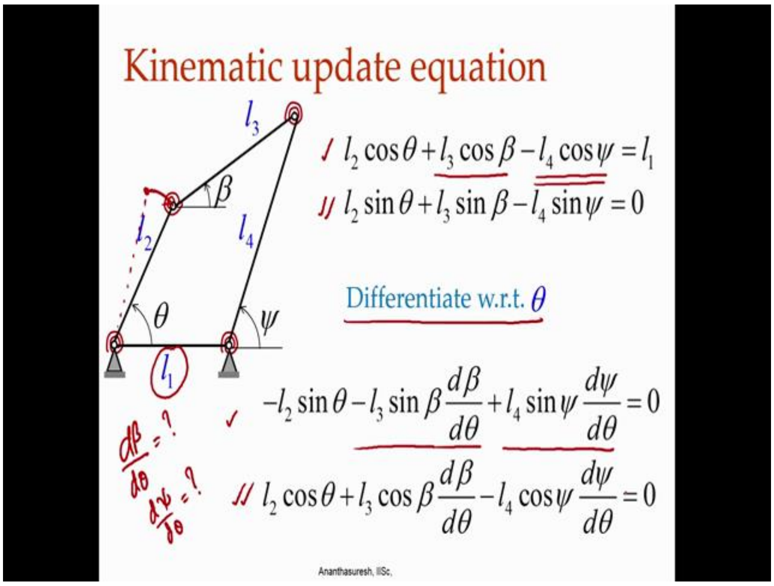

Now, our interest is to see what will happen from this configuration I move theta by little larger amount let us say from here I move this to a point it will place around a circle or a point. So, if this point goes over here that is theta is changing then how do, beta and psi change. So, what we want to compute is d beta by d theta what is that and d psi by d theta what is this in kinematics I want to find relationships easiest way to do it is using this infinite decimal changes that happen in theta and what will be the infinite decimal changes in beta and psi for that we need the derivatives.

So, I need to compute the derivatives of beta with effect to theta and derivative of psi with respect to theta, and then I know that how beta and psi change as theta changes. So, we will differentiate with respect to theta that is we take the look closure equations and differentiate with respect to theta then we get what is shown here the first one we are taking this equation and take its derivatives l 2 cosine theta becomes minus l 2 sin theta and then l 3 cosine beta is this term will give rise to minus l 3 sin beta, because cosine derivative is minus sign and then beta is a variable it depends on theta. So, we also need to take d theta by d beta that is what we want to compute that has come in equation likewise l 4 minus l 4 cosine psi becomes plus l four sin psi d psi by d theta, that is equal to 0 because l 1 is constant it just a length here similarly, if I take the second equation and differentiate it with respect to theta I get l 2 cosine theta plus l 3 cosine beta d beta by d theta then minus l 4 to cosine psi by d theta equal to 0.

So, here we can see that in a given configuration we can not only beta and psi that is a position analysis, we can also find d beta by d theta d psi by d theta because they are related by this equation that we get and we will be differentiate loop closure equations with respect to theta, where incremental we get that because in these 2 equations when theta beta and psi are known after position analysis is done we can also get this velocities if you well d beta by d theta d psi by d theta can be computed. So, that is what I have written.

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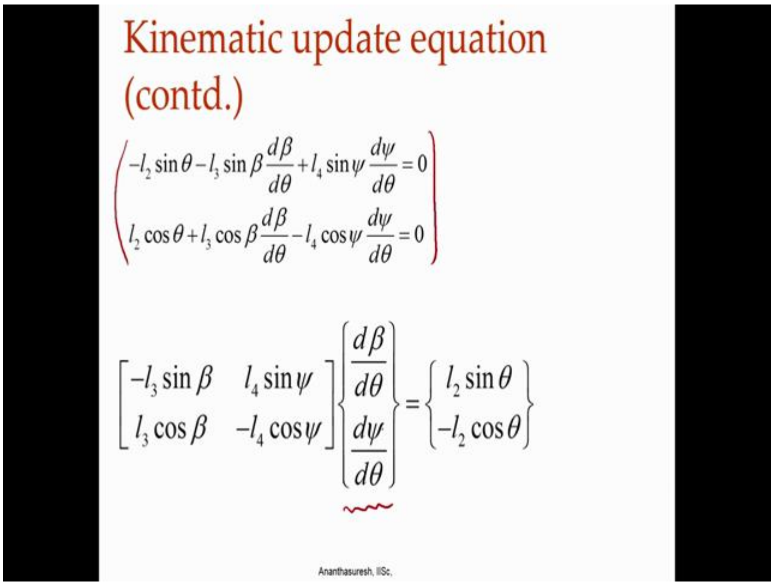

So, the equation that we got after differentiating that arranged a matrix form, because it us nice way to think about it here. So, where we put the unknowns what we want to find d beta by d theta d psi by d theta. So, that matrix will depend on the beta psi values here right hand side has is sin theta cosine theta meaning for a given value of theta we can find beta and psi by solving the position analysis equation that we had earlier in the last slide and then we want to compute this kinematic sensitivities as they are called d beta by d theta d psi by d theta we have this now to compute them.

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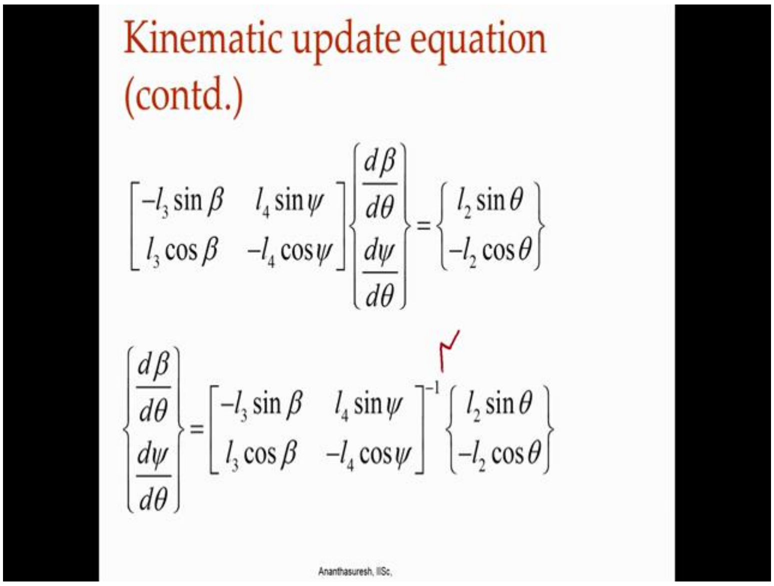

What we need to do is we have to take the inverse of this matrix you see we have to find the inverse of that matrix.

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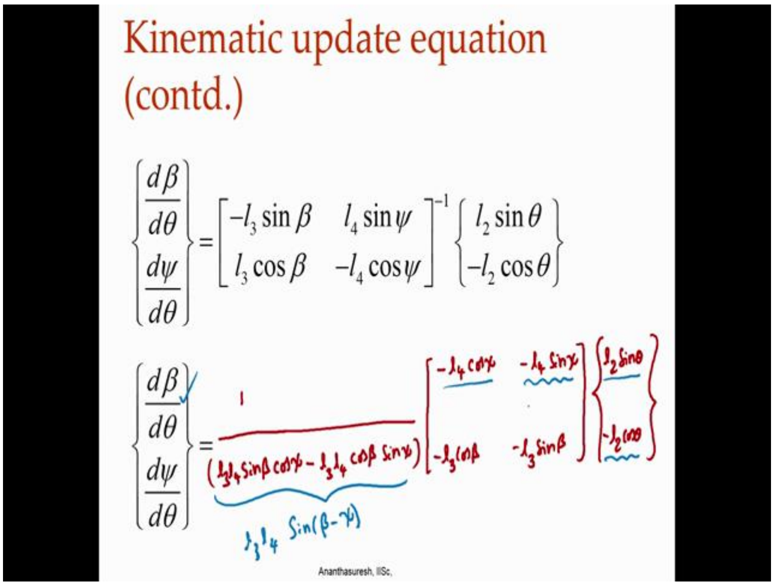

So, let us do that. So, if I want to write the inverse first thing is to write the determinant. So, determined of this matrix is l 3, l 4 sin beta cosine psi this multiplication in other multiplication minus l 3, l 4 cosine beta sin psi that is the determinant and then we have to find the cofactor matrix and then take the transpose of it, that will make this minus l 4 cosine psi and here will be minus l 3 sin beta this will become minus l 4 sin psi this will become minus l 3 co sin beta, this will be the inverse you can check it later this we have to multiply by l 2 sin theta minus l 2 cosine theta. When we do this multiplication we will get d beta by d theta as well as d psi by d theta we can already see that in the denominator here where we have the quantity here we can write this as l 3, l 4, l 3, l 4 is common to them we have sin beta co sin psi minus co sin beta sin psi there is nothing, but sin beta minus psi that is what we get.

Now, if I multiply these I have l 4 cosine psi multiplies l 2 cosine theta and then minus l 4 sin psi multiplies minus l 2 cosine theta in order to get this there again we have l 4, l 2, l 4, l 2 in the numerator what I will get is l 4 l 2 for common then I have cosine psi sin theta that is minus and then sin psi cosine theta. So, if you see that that also comes to sin difference of 2 angles if you work out this algebra what you get.

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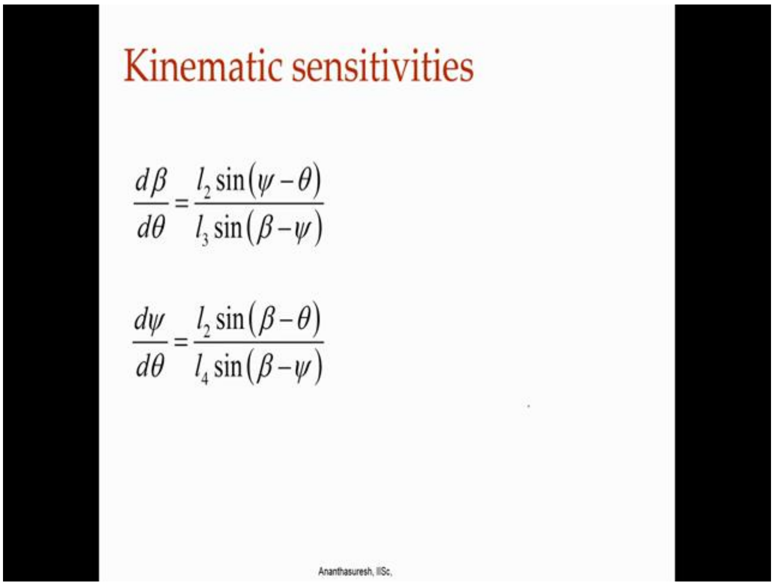

Will be this d beta by d theta is given by l 2 by l 3, l 4 got cancelled if you lets go back and look at it there is l 4 here there is also l 4 in both the terms that come here there is l 4 there is l 4. So, that gets cancelled. So, what remains for this first 1 d beta by d theta is l 2 sin psi minus theta, then l 3 sin beta minus psi and then we have similarly d psi by d theta equal to l 2 sin beta minus theta l 4 sin beta minus psi.

If we have this they are kinematic sensitivities meaning if I get if I know theta I can first compute beta and psi, I can also compute as theta changes what be the change in beta as well as psi for that we need this derivatives.

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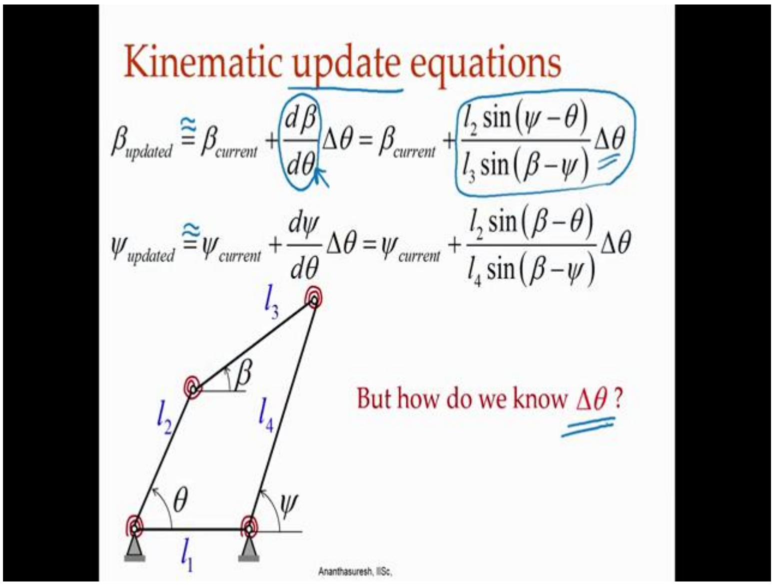

So, what we mean by kinematic update equations is that beta updated will be beta current plus d b by d theta times delta theta beta current plus d beta by d theta. So, that one is just what we derive as now times delta theta, the small change in delta theta we can get the small change in beta which is this whole thing, that is where the sensitivity co efficiency come in similarly psi updated will be this. So, if you take small enough delta theta and merge will be able to get this with this co efficiency if the delta theta. So, this relationship that I have put equal it is actually approximate it is not exactly equal; because we only take only first order term we did not take delta theta square delta theta cube and so forth.

So, if you keep your delta theta small enough then you can go step by step by step and get the curve that matches the exact curve very closely if your delta theta is small enough that is fine. So, we have figured out kinematics now I if somebody give me theta I can get beta and psi and somebody gives me update in delta theta I can get updates in delta beta and delta psi. So, I can get updated beta and updated psi here, but how do we know this delta theta. So, how do we compute it for a given problem that depends on the elastic behavior of the mechanism?

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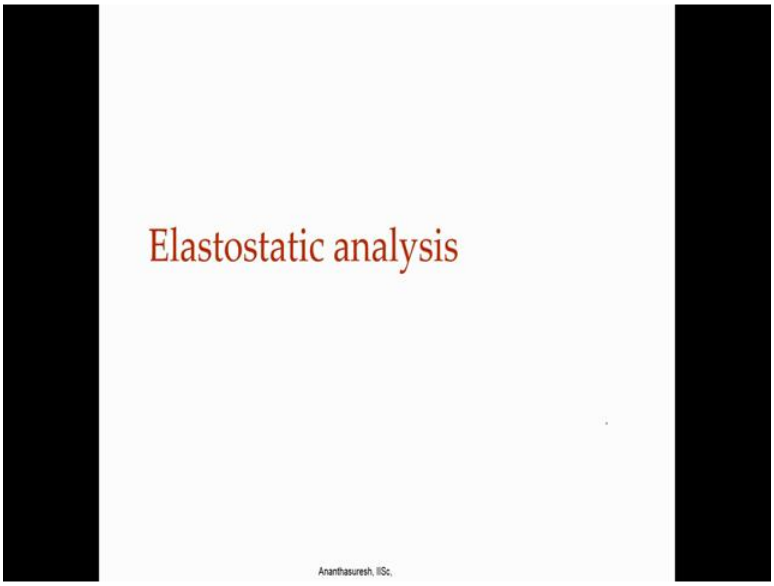

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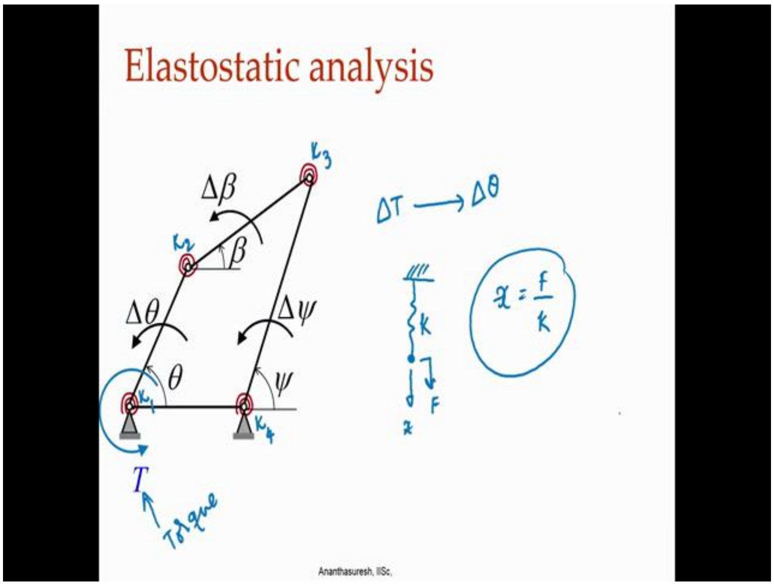

So, for this we have to do the electrostatic analysis for the electrostatic analysis now we need to go back and put the springs. So, let us say that there is a kappa 1 here and kappa 2 here and kappa 3 here and kappa 4 here in general let us say that they are all different now, there is a torque applied let us say at some point it could also be a force or torque whatever when we have that we want to find when apply let us say we are in this configuration now, when apply a torque I want to know what will be theta change that is if apply from this position a small torque delta theta, what will be delta theta? that is what we want to find out in the electrostatic analysis, for that if we just recall how we do if we have let us say spring and there is a force acting here, if I want to know how much is this x lets say spring is spring constant k immediately write x equal to f by k what we do for this will be exactly that and this we know you just write by force balance is very obvious there here, also force balance will be there, but may not be that obvious we have to write down some equations here instead what we will do instead of using force balance we will use the energy method.

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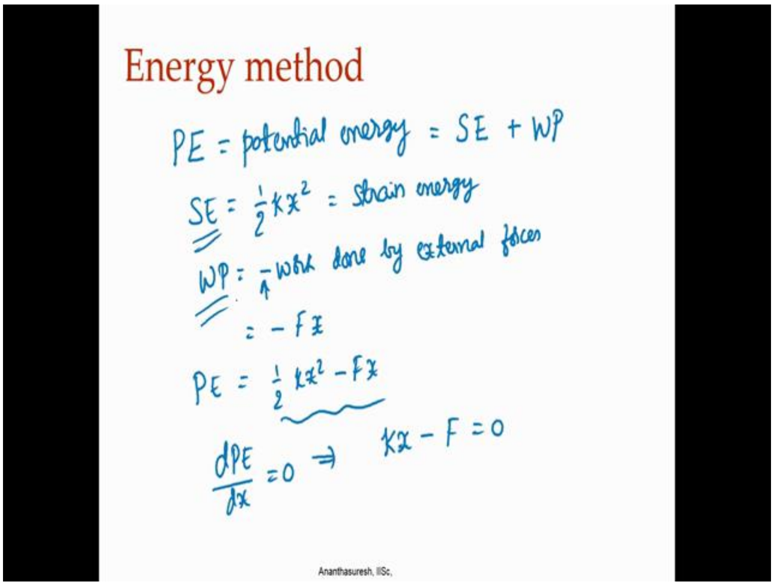

Energy method if I were apply to that spring I had the lets say potential energy, this is potential energy is actually some of two things strain energy plus work potential. So, strain energy is half k x square for this little spring and. So, this is lets write strain energy and this work potential is negative of the work done by external forces here, we do have an external force that is minus F times whatever work display that went through F x. So, potential energy here is strain energy half k x square and plus work potential is minus F x then there is definition itself has its negative sign now, we use this energy method that is we want to minimize this potential energy to get the stable equilibrium solution. So, for that we have to say that d p e by d x should be equal to 0, because that is a necessary condition for the minimum of the potential energy that gives us if I take derivative respect to x over here K x minus F that is equal to 0 that gives us x equals to F by k the similar method we will use by writing the strain energy and work potential for our linkage.

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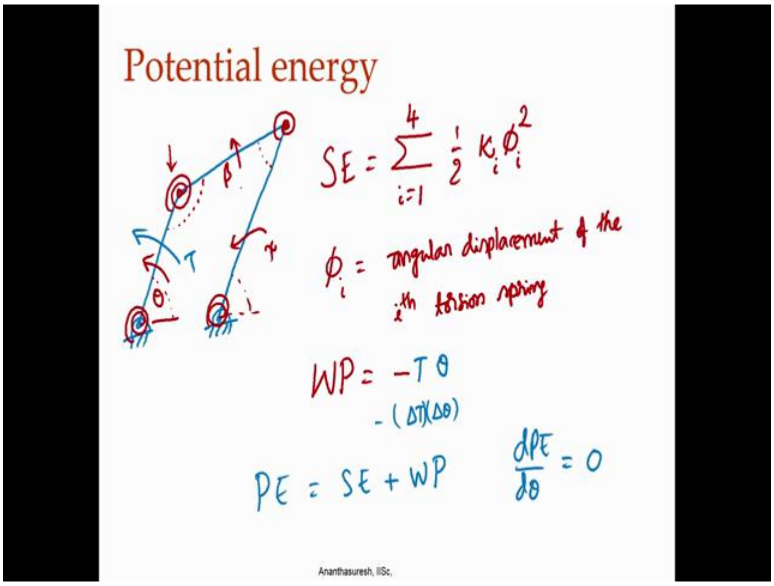

So, if I have the mechanism if I have the mechanism like this and I have the joints and the torsion springs there is a joint here, joint here there is a torsion spring there, torsion spring there, torsion spring there, torsion spring there, all 4 of them, I can write strain energy by saying I equal to 1 to 4 because there are 4 of them half kappa i times corresponding angle. So, here since I did not use the symbol phi i will just use phi i meaning the first one actually will be this theta we said this is beta that rotation that and this is psi. So, if I want to see this theta will be directly that phi over here I have to see how this angle changes beta and theta so both is changing. So, this angle is what will be the rotation experienced by this torsion spring likewise here beta and psi together will decide how this angle changes for the fourth one just change in psi itself is good enough just as for theta that was good enough. So, we can write that thing here and then square. So, phi i we can say is the angular displacement or rotation of i-th torsion spring which is a consequence of our pseudo rigid body modal, we when we have that we can get this half k x square or torsion spring half kappa i phi i square.

Potential energy if I have torque potential work potential right. So, before that work potential negative let us say I have some torque over here, if I have torque let us say over here then minus T times the theta that is absolute if it is moving only by some delta T and delta theta then you would write it as delta T delta theta will be the work potential occur the minus sign. Then we will get this potential energy as a sum of strain energy and work potential then our variable here is theta is a single degree of freedom, then I have to say d p e by d theta is equal to 0, when I do that I get an equation where unknown will be only theta why not about beta and psi, they are not unknowns because they depend on theta not only beta and psi absolute values in this configuration, but also when theta changes with delta theta delta beta and delta psi also change with this kinematic coefficient that we have derived earlier in today's lecture.

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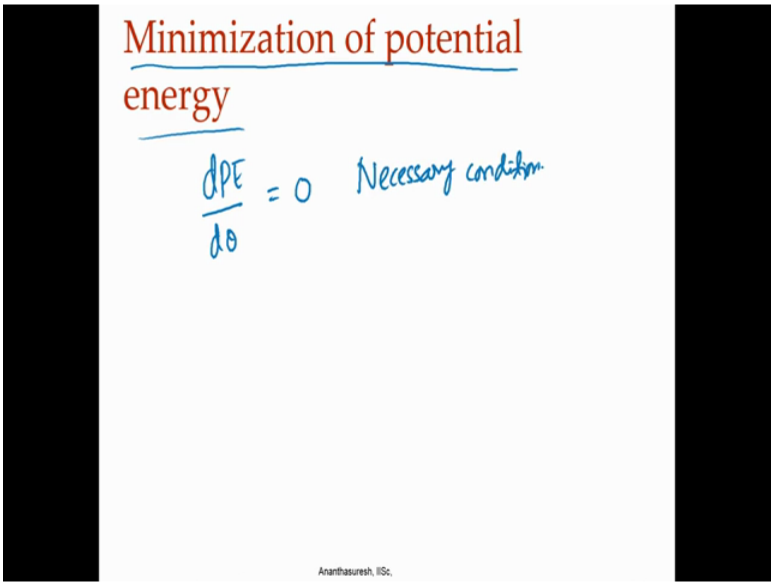

When we do that we this minimum potential principle gives us an equation which is d p e by d theta is equal to 0 for given torque or force whatever that will be going into the work potential that value is put in then we want to find what is the change in theta that will also give away the change in beta as well as psi, that is the minimization of potential energy which gives us the equation equilibrium of the static equilibrium for our purpose we are using necessary conditions, it could be minimization or maximization if it is minimization there is second order condition you can verify that will be the stable equilibrium if it is maximum it will be unstable equilibrium which we will see later on for now let us say we are using necessary condition which is what we are using now.

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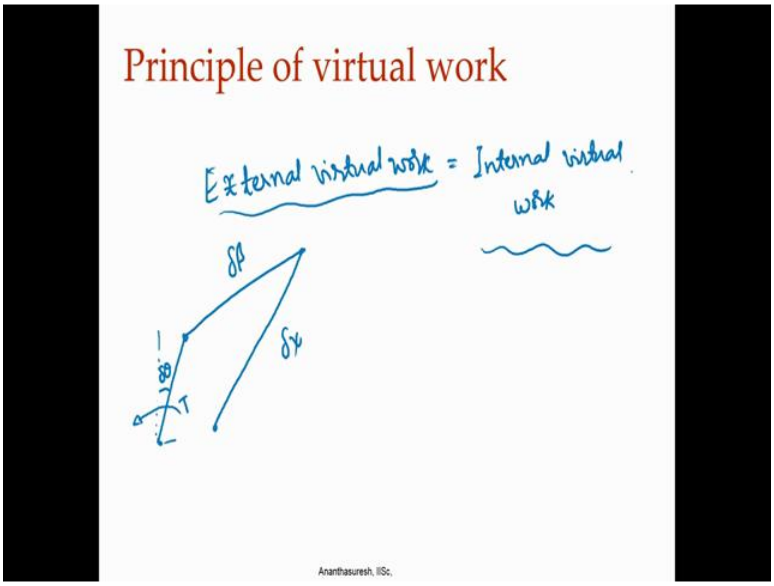

We can also think of this in terms of principles of virtual work which shares that external virtual work is equal to internal virtual work, for that you have to imagine virtual displacements just imagine that our linkage is there linkage is there now, we just say that this has moved we just imagine that it has rotated by some delta theta, then we have to also imagine delta beta here and delta psi here and based on that we have the internal torque here because the torsion spring that will go to this side external virtual work if there is a torque applied here that will go to external virtual work that also gives a same equation that would be obtained with the minimum potential energy principle the same equation that we get.

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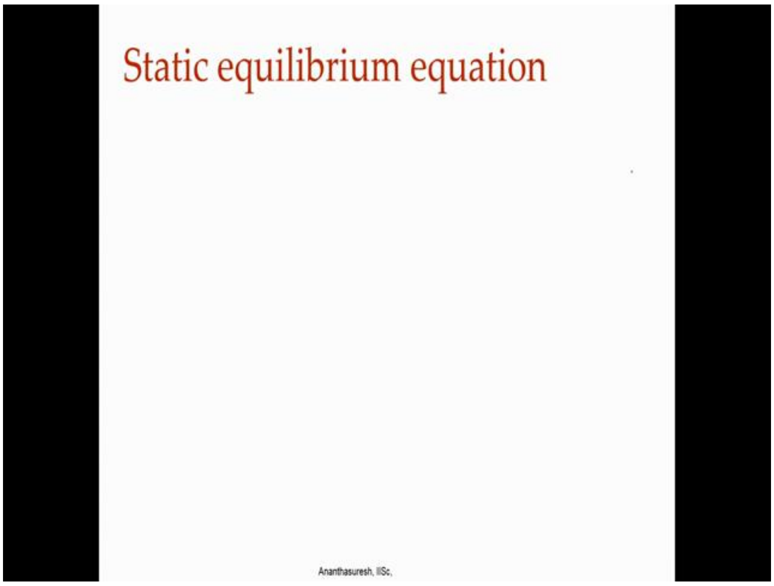

And with that either with minimizing energy or principle of virtual work we get static equilibrium equation, we will see through an example in the next lecture right now I have just given you how this method would work.

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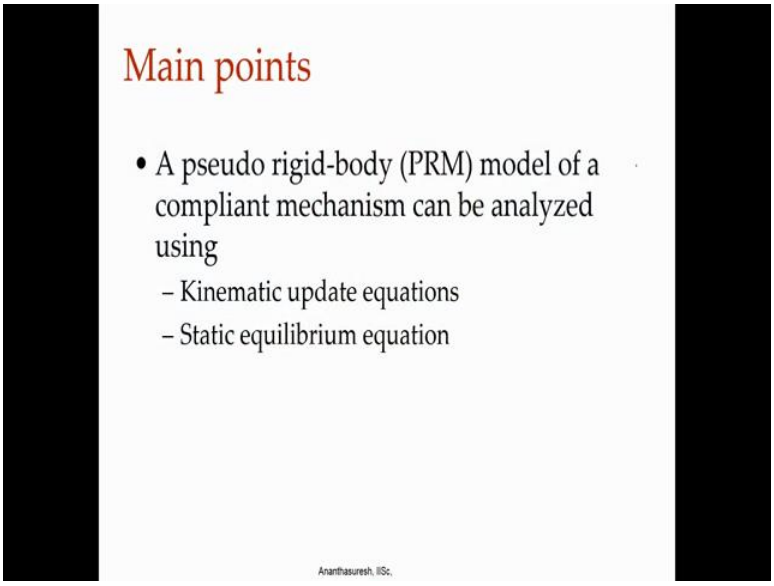

So, the main point of today's lecture is that when we have a pseudo rigid body model of a compliant mechanism, we need to have this kinematic update equations that purely kinematic no reference to forces or torsion springs or material properties after that we get the static equilibrium equation that can be arrived at using principle of mini operation energy or principle of virtual work.

If you want to read more about what we have discussed you can look at this book compliant mechanisms by glary Howell, and what will be doing in the next lectures would be to use this methods to solve few example problems and implement them in MATLAB so that get appreciation for doing analysis using pseudo rigid body model.

Thank you.