**Compliant Mechanisms: Principles and Design**
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

Lecture – 17
Howell-Midhas elastic model

Hello, in the last lecture we discussed the kinematic approximation for the locus of the loaded tip of a cantilever beam undergoing large displacements. Today we are going talk about elastostatic approximation and that leads to something useful later on. We are looking at elastostatic approximation of the locus of the loaded tip of a cantilever beam.

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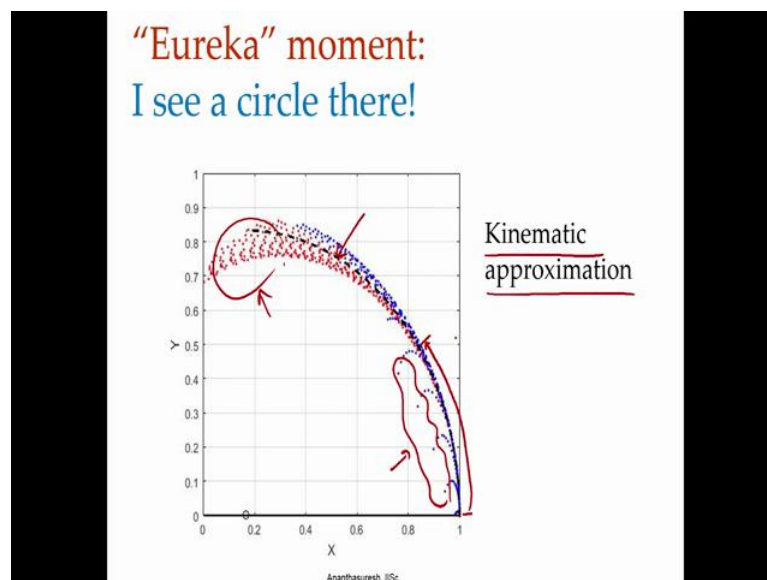
We had seen that there was a Eureka moment for Burns and Crossley then they notice that there is a circle here for the locus. All these collection of data points were obtained by applying on a cantilever beam, load that is inclined that need to has a transfer component as well as axial component and they all happen to be in this (Refer Time: 01:10) if we neglect some of these things, that happen when you apply very large axial load or very large. Axial load that is tensile here and compressive here, but more or less, if you were to notice there is a dash line there dashed curve, that is a circle approximation that is a kinematic approximation.

How is that useful because as you can see from here to about there; it is very close to this circle, baring a few compressive axial loads and if we go further it is deviating a little bite, but more or less you can approximate the motion of a cantilever beam tip with a crank that is a little sharper that is here we have about that.

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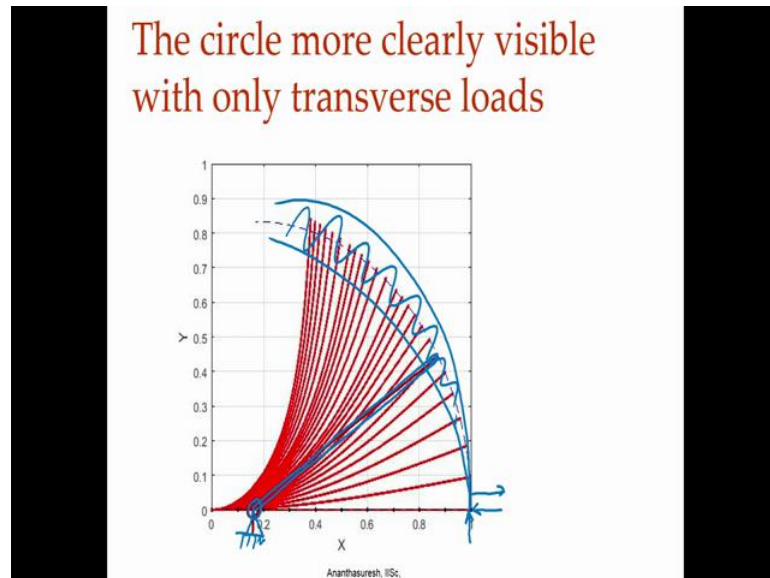
If I put a centre then I can put a crank that I think I should change the colour here. If I put colour, I want let us a blue colour, if I put a crank there which rotates about this point as if it is fixed there the motion of the rigid crank will be similar to the motion of the cantilever beam with load that can come from the transverse side or the axial side. Axial can be that way as well as this way if we do all that we get the sliver of thing as if we saw there it was something like this; which we can approximate all of the approximated with a single circular path that is the kinematic approximation.

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Kinematic approximation of the locus of the loaded tip

Burns, R. H. and Crossley, F. R. E., "Kinetostatic Synthesis of Flexible Link Mechanisms," Trans. ASME, 68-MECH-36, 1968.
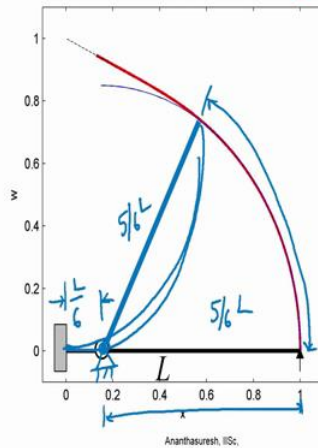
Burns and Crossley, 1968

The locus can be approximated with a circular arc for a very large range of bending of a cantilever.

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That means that we have the contribution of Burns and Crossley who said that we can approximate this width a pin joint there and we have a crank that traces this, if there were to be only transverse load; we can see how good the approximation is up to about this point all of this. That is a lot of angle may be close to 60 years so, and that much beams do not bend often in complaint mechanisms. That is with this load here the beam would be bending something like this. That much of definition in fact, it would not be 0 there it will be something like this, with the 0 slop there. That much deformation is actually not quite common in compliant mechanisms.

It is a good approximation if you were to replace this cantilever beam with a rigid crank whose length they said is 5 over 6 of the length; that is from here to here the crank length I would say 5 over 6 times L and you have to move that fixed connection by 1 over 6 L or L by 6 if we do that put a rigid crank then you get a very good approximation for the tip of the cantilever beam and that is the exactly what they said in 1968, 5 by 6.

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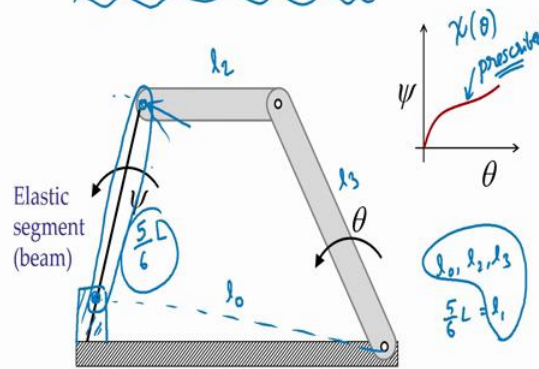
Now, we also discussed can we solve problems like this our answer in the last lecture was yes that is somebody asks us to generate a function, where psi is a function of theta; we can do that because all we have to do is over here we reduce this beam length by one sixth and add a rigid body here and say that this is where I extend this fixed one this all fixed now. Now I have this new length which is 5 by 6 of the original length I have rigid body linkage and for that I know how to do this function generation synthesis meaning if somebody asks us to generate a function psi of theta which is given or this is prescribed or it is desire, if we have that we can find this length L, this length let us call it l2, l3 and this length from here to here because that is a new pin joint over there. So, l 0 if I have these I can find l 0, l 2, l 3 and my 5 by 6l let us call that l1 is something.

All of these things we can find so, that this function we generated. After we do everything here, then we replace this particular rigid body with an elastic segment that is 1 by 6th more in length, if we do that then we get a compliant mechanism that has 2 rigid bodies an elastic segment and I would have designed it. That is the significance of this kinematic approximation because in kinematics we only care about how this is moving and we know that even it even if to a cantilever beam it would only move along a circular arc very close in approximation, but very close approximation.

When a linkage is like this the load at this point can come in any direction is a pin joint there will not be any movement as here, but the first can come in any which way right

and that is why we had considered on the cantilever beam loads both transfers and axial so, that the inclined load in lot of different directions. All of those we have approximate this with 5 by 6 L, Burns and Crossley said that you will be very close to what the cantilever beam would be doing you want to replace cantilever beam with this rigid body so, that is the kinematic approximation.

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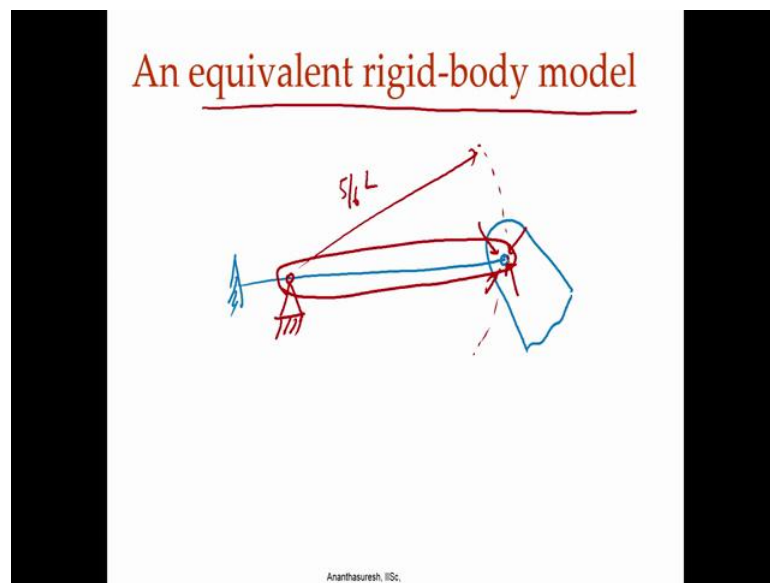
Now, we also ask this question when you make this equivalent rigid body model that is if I have a cantilever beam which is fixed at one end and other end is attached to some rigid body whatever it is. That rigid body at this joint is going to create loads in different directions. We do not know which way the load can come in any direction, it does not matter, whichever direction the load may come. We know that we can approximate this rigid body with that elastic body with rigid where we put that joint right that is the equivalent rigid body model.

As for as the motion of this is concerned right it will be symmetric when the force is down words it will be a circular arc around this point and that length is 5 by 6 L that is equivalent to body model, then we have go back to rigid body linkage synthesis or analysis and what there and then come back here and replay the rigid body with a longer cantilever beam. This works well for fixed and thinned elastic segment.
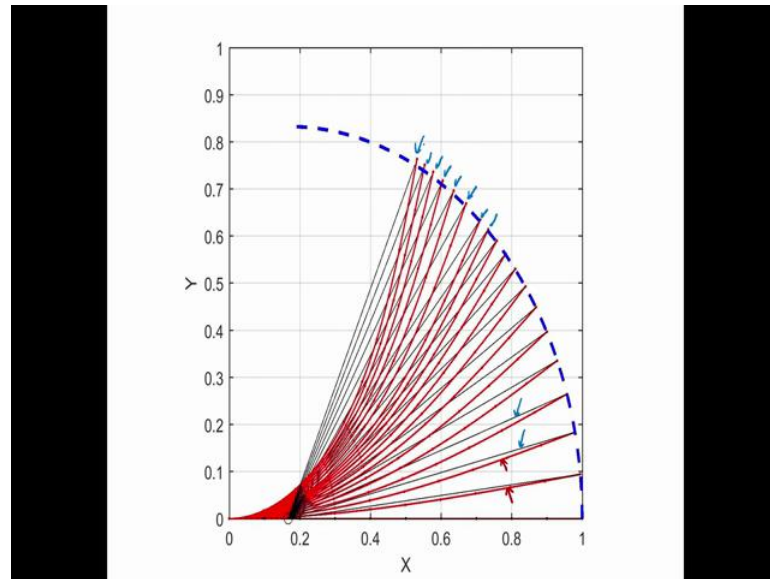
Now, if we look at this right. Here is where we have showing all that crank the black lines here or the crank directions and the red ones are the elastically deformed cantilever beam the red ones each of these here. These are all the beam deformations and then we have these black ones that are you see these black ones they are the crank rotations from here to here and both of them are laying on this curve and then of course, the deviate. Deviation starts to become apparent from this point onward they are slightly deviating and as the deflection becomes more they deviate more from the circle.
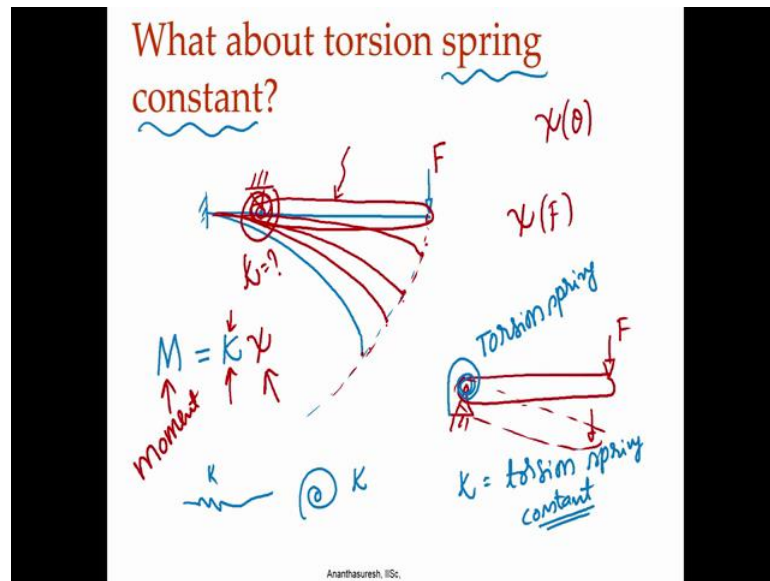
Now if we see that kinematically this works very well. What about the torsion spring constant because if I have a cantilever beam and I have a force here, I know that it deflects like this and we know that this thing if we have let us say this deflects like that this (Refer Time: 09:49) this circle there. If I replace that with a rigid body that is what we are been saying that if I replace that with a rigid body now. It is going to move exactly like that and that is what we saw. For a large rotation their coincident, but then a cantilever beam will offer some resistance to the application of the force; if I just put a rigid body like this kinematically for function generation problems, that is this function that we took psi as a function of theta it is perfectly all right because there these no information on the force.

But, if I what to also consider that relation between force and the angle psi, that is I want to know how psi changes with f not the psi is a function of theta which is purely kinematics; if I say psi as a function of f if that is what I want to know, then I have to get some notion of elastic model in my equivalent rigid body model. This is my equivalent rigid model. Now if I show just like this that is if I say just that I have a crank of length 5 by 6th right and apply a force, it just going to rotate where to go to dynamics right. It is not going to set let us a point like the cantilever beam would do because this lacks elastic element. What we say is that we can imagine that since it is crank if I imagine that there is a torsional spring. This is the torsion spring or rotational spring. Whenever we show a spring, we implicitly assume that it is a linear right. I mean it need not be in fact,

complaint mechanisms or non-linear springs, but whenever we see if I draw a spring like this immediately I want to know what the scale is right. Similarly if I show torsion spring like this I want to know what is kappa is.

Kappa is torsion spring constant. If it is linear torsion spring constant, what do we mean by linear; if I were to move this it is going to rotate there will be a reaction movement here due to torsion spring. That reaction movement if I call that M that is going to be equal to kappa times this angle. Let us call it is psi rather than theta. Let us call that psi times still there is a mod, let us call that psi. This is the moment and this kappa is spring constant, when it multiplied by this angle I get the moment here and that moment and this force f times that momentum will be equilibrated so, this will be in some kind of a static equilibrium. Where it turns like that there will be f here and then you have the momentum f l cosine psi that will be equal to this moment m which is given by kappa time's psi.

We assume that is it the case in this example is our question. That is the next question what about the torsion spring constant. If I replace a cantilever beam with an equivalent rigid body that is little shorter and if I want to put a torsion spring over here, what is this kappa here. First of all is it linear like this; meaning that a different points here is this spring constant the same value does it remain constant because for each rotation if I do that is f some value, increased another value as increase the force it is going to go from here to here to here meaning, the cantilever beam will bend like that and then bend like this and bend like this and so forth. As it moves with increasing force and each time between 2 increments, if I were to calculate this kappa is that going to be the same or is going to be different, that is what we would like to know and that is what I have done here.

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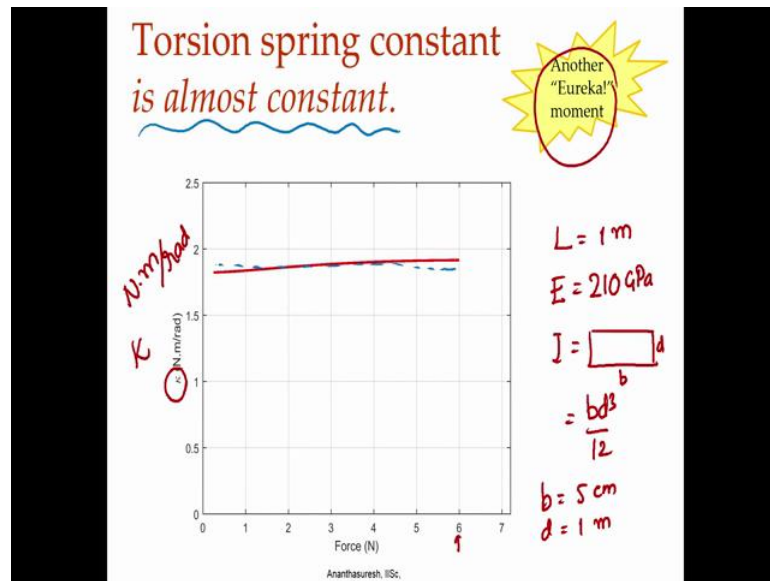
So, here for various values of forces starting from 0 to about 6 Newtons we calculated that value kappa here the unit is of kappa Newton meter per radian because if I multiply by angle I should get newton meter that is a moment like right. There is another Eureka moment that is almost constant it is not exactly constant to generate this I have taken length of the cantilever to be 1 meter and Youngs modulus I took that of steel that is 210 gigapascal and I have taken rectangular cross section that I will be b d cube by 12 and I took this b and this d, b I took 5 centimetres and d, I took 1 millimetre and varied the force up to 6 Newtons when we do that this torsion spring constant are to be almost constant well. It is not quite right; it is not exactly constant, but almost constant.

We can approximate the elastic nature of the cantilever beam now by introducing torsion spring constant this kappa is not going up and down; it is going up and down a little bit, but relative to the 0 base that I have taken it looks pretty much constant. If I take some average value in the middle, if I want to take something like that right whatever value that is then I would be approximate the elastic behaviour also here in a convenient way where this kappa is constant as the force increases. That is I just said that as a forces is increasing when I do incremental steps is the kappa the same it terms out to be in fact, very close to being constant.

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Matlab code

```
% Crank angle
psi(i) = atan2(wL(i), 1+uL(i)-1/6);
% Plot the PRB crank
plot([1/6 1/6+5/6*cos(psi(i))], [0 5/6*sin(psi(i))], '-k');
% Moment at PRB centre
M(i) = f(i)*5/6*cos(psi(i)) - f(i)*n*5/6*sin(psi(i));
% Torsion spring constant
kappa(i) = M(i)/psi(i);
plot(f,kappa,'-r','LineWidth',2);
hold on
axis([0 1.2*max(f) 0 2.5]);
xlabel('Force (N)','FontSize',20);
ylabel('\kappa (N.m/rad)','FontSize',20);
grid
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The torsion spring constant is almost constant. That Eureka moment well and this the MATLAB code, that you can also try along this lecture there will be supplementary files where you can see. We find the crank angle psi that is if I have a beam and when it deflects like this, I take the rigid body pivot over there, cantilever beam was fix there it will go like this. Now I am joining this from here to here that angle I am denoting by psi. That psi atan2 function gives you the right quadrant wL and then 1 plus uL the way uL was defined in that particular code so, basically what we are saying is that I want to take this vertical thing, in this horizontal thing and that is what we have, this psi once you have psi we can calculate the moment assuming that here the load has both a transverse component as well as axial component. We can get the moment here and then that kappa is obtained by dividing moment by the angle there and then plotting kappa and how close it is to being constant and we saw that it is close to being constant.

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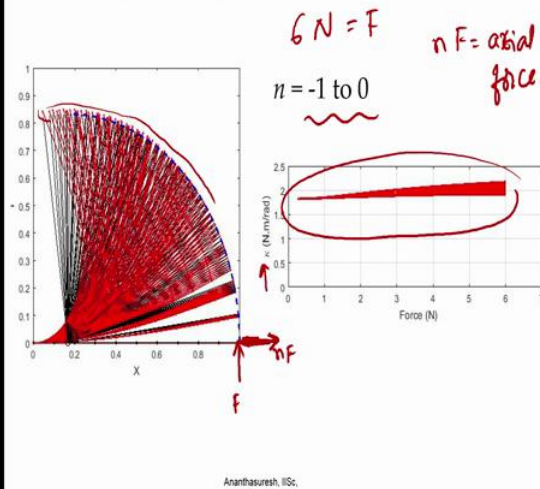
These inclined loads; if I were to take some value 6 Newtons that I would show there right. Each time I also take an axial force which is that if I call this F. I would take n times F as axial force; n times F as axial force, that is if there were to be transfers force F there is n times F actually n it be positive is that way when n is negative it becomes compressive axial force. So, n direction is this way n times F. With various things applied here and each time to equivalent crank, if I were to do that were just for 1 force 0 to 6 Newtons no inclined loads now we have taken inclined load also and then see that the spread here that you see again is not very much just like the sliver of the fixed tip if you see how close it is to the circular arc right similarly, the kappa here is also pretty much constant.

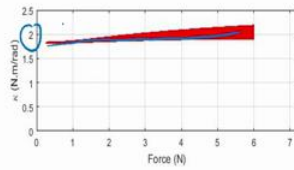(Refer Side Time: 19:18)

More inclined loads

$n = -1$ to 1
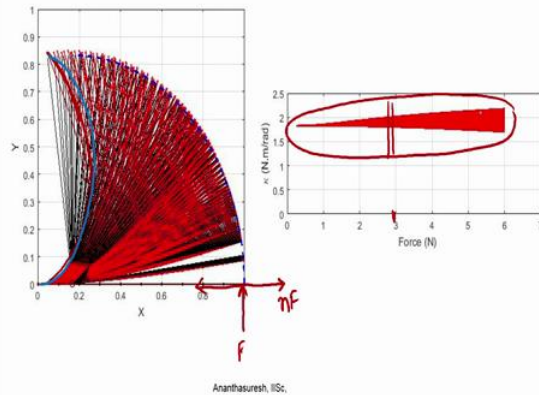
Now here is another one where you can take more inclined loads varying n from minus 1 to 1 meaning that as I am varying this force 0 to 6 Newtons, I am changing force all the way from positive 1 to negative 1, that n F that we are taking n varies from minus 1 to when compressive to tensile if we sweep all that, it increases a little bit more, but again not deviating too much and nice thing is that as force increases this spread increases; that means, that for lower forces let say I am applying only 3 Newtons meaning I am not deflecting the beam a whole lot then I am closer to being constant than it is for larger forces. You can see the force here the beam is deflecting all the way like this right. It is deflecting let me change the colour. It is deflecting like this; that means, that means there is axial force and compressive axial force and tensile transfers force which reflecting so, much and yet we see closeness to being constant.
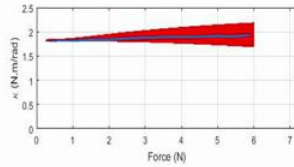
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If we just see the spread again when it is only compressive load minus 1 to 0 verses minus 1 to 1, the spread is increasing, but it is narrow, narrow enough to be approximated to be a constant kappa.

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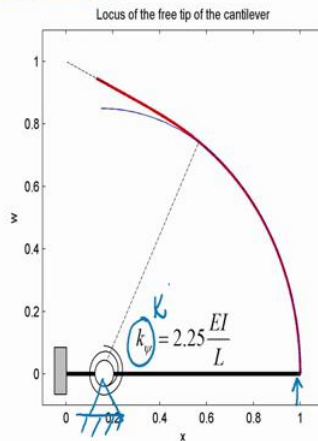
Kappa being constant is the main usage here, that is the Eureka moment and that was a contribution of Howell and Midha. Professor Ashok Midha was a professor at the time board university and (Refer Time: 21:08) was his student they observed this very nice

thing, that this kappa is constant and that is what I call elastostatic approximation, when Burns and Crossley did that in 1968 for kinematic approximation and they said it 5 by 6 L they said were not only that this kappa that we have they used k.

I am let us call it kappa, has been constant that I can just fix it here and if you close your eyes whether I put a rigid body with a torsion spring or an elastic beam this point here in response to the force that is applied here will not be able to make any difference. That is if I present you a model of a rigid body which is slightly shorter than the original elastic beam and for torsion spring constant and give it to you, you will not be able to tell the difference between this and the cantilever beam to begin with at all. That is we have both kinematic and elastic approximation or elastostatic emphasising were dynamics may change static conditions we do not find the difference and that is the nice thing about this and this is the paper where they published in 1995, several years after Burns and Crossley, but they had this wonderful observation that this kappa is constant.

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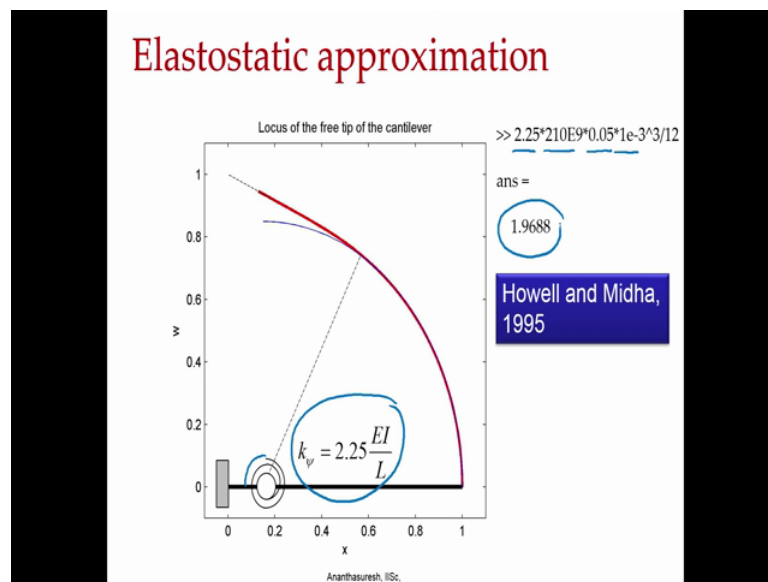
That is their thing in fact, in this particular case I put that they had given this expression also 2.25 times E I by L, will come to how they arrive at this value, this particular formula for that torsion spring constant. When I put that 2.25 E and then I here, I took 5 centimetres and then 1 millimetre b d cube by 12 and that 1.9688 and that was very close

if you see. Let us see yeah not here, but let us go back if you see it was very close this is 2 Newton meter per radiant and it was there if even if I have change n or all the way from minus 1 to 1, that is compressive axial completely to tensile axial while the force was acting as much as the 6 Newtons, where it was deflecting a lot it was very close 1.98 will be somewhere just like that over there.

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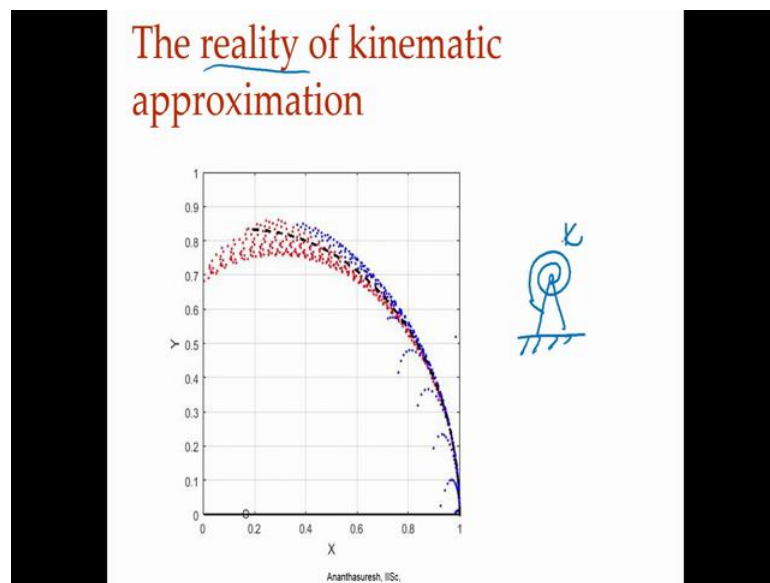
That is what anybody who take if I were to a constant and their formula agrees with the example data that we have taken here. Again let us see the approximation if you consider the locus of the cantilever kinematically, the reality is that it is not a circular arc, but you can take it and get a way for most situations. In a similar manner the elastic approximation of putting that torsion spring over there right, that kappa being constant will be similar to this. If the spread were we saw and now we say that; that is constant if you take then we are ok for most of the situations.
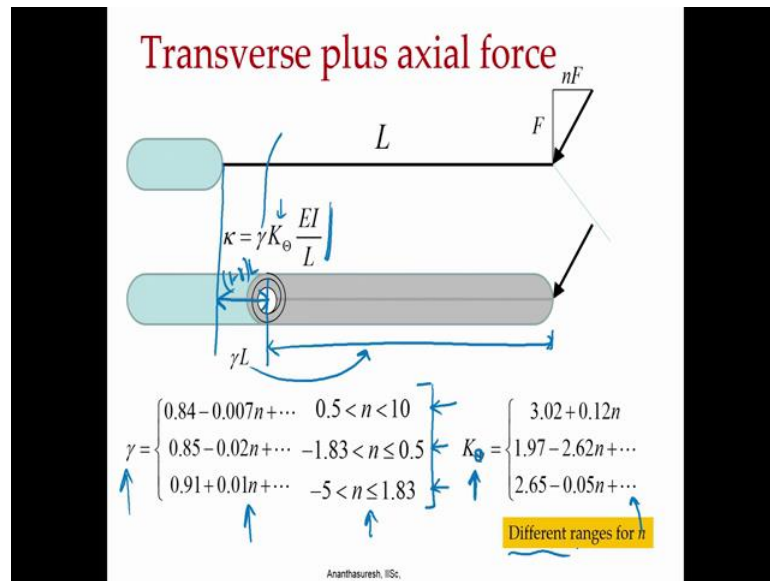
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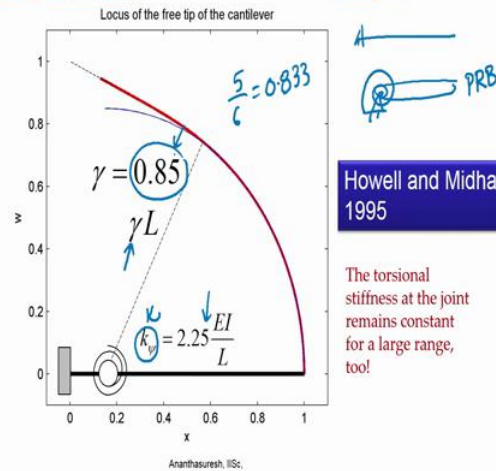
Howell and Midha call that pseudo rigid body model that is you have a cantilever beam that is a real cantilever beam, but your approximating is behaviour both in terms of kinematics as well as elastostatic by a rigid body and a torsion spring. This is again kappa. That is what calls a pseudo rigid body; pseudo is not real, it is a rigid body model for sure they called it pseudo rigid body model because you are replacing a cantilever beam with a rigid body. Which is fixed over there and there is a torsion spring. This is the pseudo rigid body model, these days people use that PRB model; pseudo rigid body model. That is built up based on two things kinematic approximation in an elastostatic approximation. They did one more thing they define this gamma, which they called characteristic length factor they choose a value of 0.85 rather than 5 over 6, that Burns and Crossley had used 5 over 6 is 0.833 right so 0.833 easy to correct 0.8333 right. Instead they went for 0.85, why did they change it and they actually introduce later that thing also here. We will see how they did it how they arrived at this formula and why they went for 0.85.

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Transverse plus axial force

$$\kappa = \gamma K_\Theta \frac{EI}{L}$$

$$\gamma = \begin{cases} 0.84 - 0.007n + \cdots & 0.5 < n < 10 \\ 0.85 - 0.02n + \cdots & -1.83 < n \le 0.5 \\ 0.91 + 0.01n + \cdots & -5 < n \le 1.83 \end{cases} \qquad K_\Theta = \begin{cases} 3.02 + 0.12n \\ 1.97 - 2.62n + \cdots \\ 2.65 - 0.05n + \cdots \end{cases}$$

Different ranges for n

Ananthasuresh, IISc.

What they did was they did very extensive calculation using elliptic integral solutions as well as finite element analysis, where they took a beam there is a cantilever beam of length L with some Youngs modulus cross section area and so forth. Applied the force so, that it has transverse component and axial component both in terms of tensile as well as compressive for the n F here just like what I said. They did that and then they fit circular arcs to that data, if you remember I also showed this spread of these points is the locals.

When they took that and they approximated with circular arc because that was already known and they try to see what the best circular arc radius is and where should that be, that is they try to obtain this value gamma. What we call characteristic length factor that is from here to here will be this gamma L; where gamma they took it as 0.85 in fact, what they found was different ranges of n as shown here. When n varies from 0.5 to 10 that is when n is tensile, the axial load is tensile they gave something 0.84 minus 0.007. In fact, these are all approximations in another slide I have the 6th decimal place accuracy they give for this fitted curves to find this characteristic length factor. Where would you fix it from here to here will be basically 1 minus gamma times L. You move the fix support to here make it a thin joint support; put that torsion spring of spring constant given by that formula and they said that depends on n value, when n can be varying when n varies for each range of n the first range, second range and third range they gave a formula for gamma, similarly for this K theta also which goes over there that

is given by this one there they given some values of course, range of n for this and range of n for this are different.

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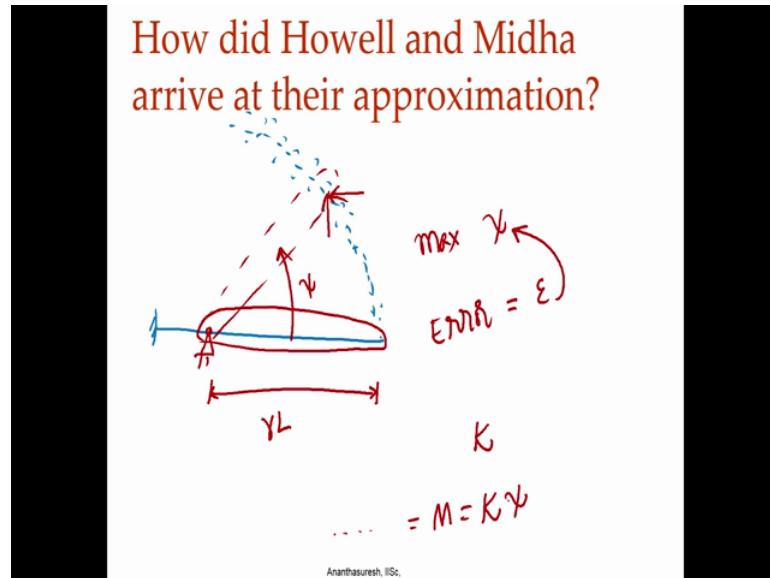
It is all by fitting the curve and in each case what they did was they approximated this with a circular radius arc radius of gamma times L, where gamma is a current shift factor and this kappa, what they did was when they get this spread of data; that is if I take a cantilever beam it is going to give some points like this right and we have seen that within the earlier slides they will be like this.

Now first in they did was if I were to approximate this with a rigid body crank somewhere there, these a rigid body crank of length in their notation gamma times L where this is gamma is current shift factor. They try to find this gamma through fitting in such a way that the angle, if this were to go like this. This angle from here to here let us say this they try to maximise this angle. To what extend I can go in terms of rotating this crank in other words bending this elastic beam so, that the L r between the circular arc and this spread of data is minimised. They try to maximise this psi for given error. You may say error should be less than some epsilon, for that they try to maximise this rotation and based on that they came up with this values of gamma, and at the same time they also came up with the values of kappa because, kappa again they would try to see if it is

go there, then the moment here has to be equal to kappa times the angle psi, the moment of course, comes from if I have the force that component as well as this component let say, you can take moment about this and you will put that there that should equal to K times psi. That is how they did this curve fitting so, that they can get an approximation for these things.

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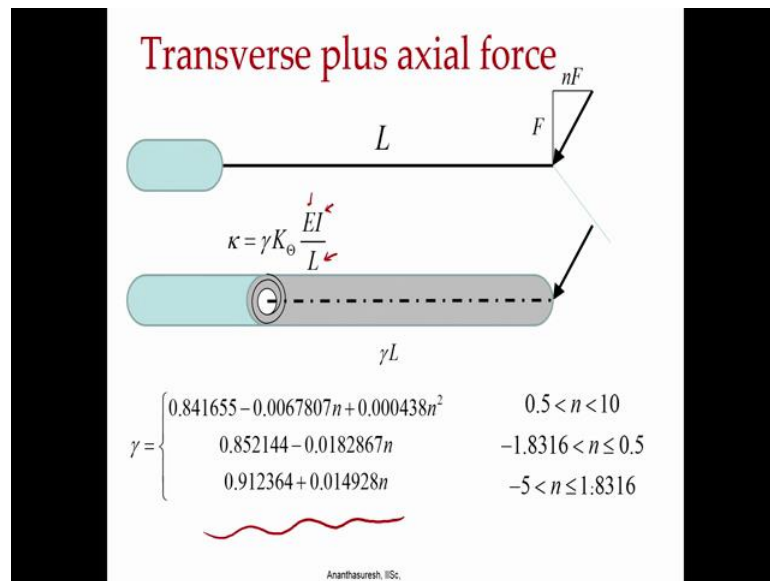
Again this is a gamma K theta E I by L, we are putting the material property, cross section property, length L and gamma is given by this. Now given a more detailed expression they have put 841655 and then 0.0067807 so, many decimal places based on their epsilon that they had chosen error they fit that for different ranges and varies from 0.5 to 10 and then minus 1.8316 to 0.5 and so forth.

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Detailed approximations

$$\kappa = \gamma K_\Theta \frac{EI}{L}$$

Howell, L. L., *Compliant Mechanisms*, John Wiley, 2001.

$$\gamma = \begin{cases} 0.841655 - 0.0067807n + 0.000438n^2 & (0.5 < n < 10.0) \\ 0.852144 - 0.0182867n & (-1.8316 < n < 0.5) \\ 0.912364 + 0.0145928n & (-5 < n < -1.8316) \end{cases}$$

$$K_\Theta = \begin{cases} 3.024112 + 0.121290n + 0.003169n^2 & (-5 < n \le -2.5) \\ 1.967647 - 2.616021n - 3.738166n^2 - 2.649437n^3 - 0.891906n^4 \\ \quad -0.113063n^5 & (-2.5 < n \le -1) \\ 2.654855 - 0.509896 \times 10^{-1}n + 0.126749 \times 10^{-1}n^2 \\ \quad -0.142039 \times 10^{-2}n^3 + 0.584525 \times 10^{-4}n^4 & (-1 < n \le 10) \end{cases}$$

Ananthasuresh, IISc.

They have given different ranges for these values you can get n and then you can also get this K theta for which from the book by Professor Larry Howell have taken this image where this K theta, you can see this n, n square, n cube, n to the 4 and you know all that, all the fitting that they have done to arrive at this.

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PRB for a fixed-fixed beam

L

$$k_\psi = 4.50 \frac{EI}{L} \qquad k_\psi = 4.50 \frac{EI}{L}$$

$$\frac{1-\gamma}{2}L \qquad \frac{1-\gamma}{2}L$$
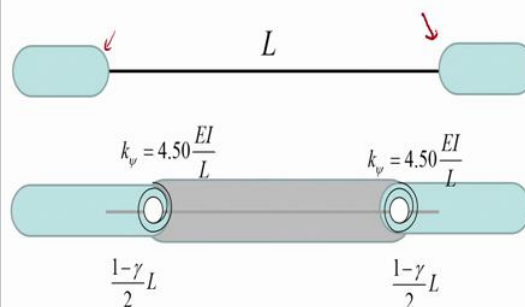
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So, that you can get something that approximates for certain angle, in fact, for each of the approximations they give to what extent, you can assume that this will be accurate

within the epsilon that you take. Error you pre define within that error this kinematic approximation as well as elastostatic approximation will be accurate to given error and that is how they have taken. They have extended this pseudo rigid body modal not only for the cantilever beam, but also for the fixed-fixed beam when I have fixed and fixed and this is fixed here as well as here, we can approximate like this and for many other situations which will considered in the next lecture is to summarize now.

(Refer Side Time: 32:57)

Burns and Crossley found a very good approximation for the kinematics and Howell and Midha found a very good approximation for an elastostatic.

(Refer Side Time: 33:19)

Further reading

- Howell, L. L. and Midha A., "Parametric Deflection Approximations for End-loaded, Large-deflection Beams in Compliant Mechanisms," Journal of Mechanical Design, Trans. ASME, 117(1), 1995, pp. 156-165.
- Howell, L. L., *Compliant Mechanisms*, John Wiley, 2001.

Both of them put together leads to this pseudo rigid body modal, which will discuss in the next lecture in little bit more detail. If you want to refer to something further I have given reference to the book by Larry Howell as well as this fundamental paper, that has published in 1995, which is worth reading where they explained the details of how they fitted the curve lot of paints taking word, after their initial observation that this torsion spring constant indeed constant pretty much with very little error over a large range of rotation of the cantilever beam under inclined loads of all kinds. Compressive tensile, axial loads and transverse loads together will lead to this nice elastostatic approximation.

Thank you.