Compliant Mechanisms: Principles and Design
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

Lecture – 16
Burns-Crossleys kinematic model

Hello, this is the 4th lecture this week, last 3 lectures we discussed Elliptic Integrals the theory and the 3rd lecture we also some implementation in mat lab. Today we go a little further and then see what we can do with Large Displacement Analysis of a cantilever beam. So, we look at a particular idea today which is called Kinematic Approximation of the locus of the tip of the cantilever and what we can make out of that. So, let us look at this particular concept of kinematic approximating the locus of the loaded tip of a cantilever beam.

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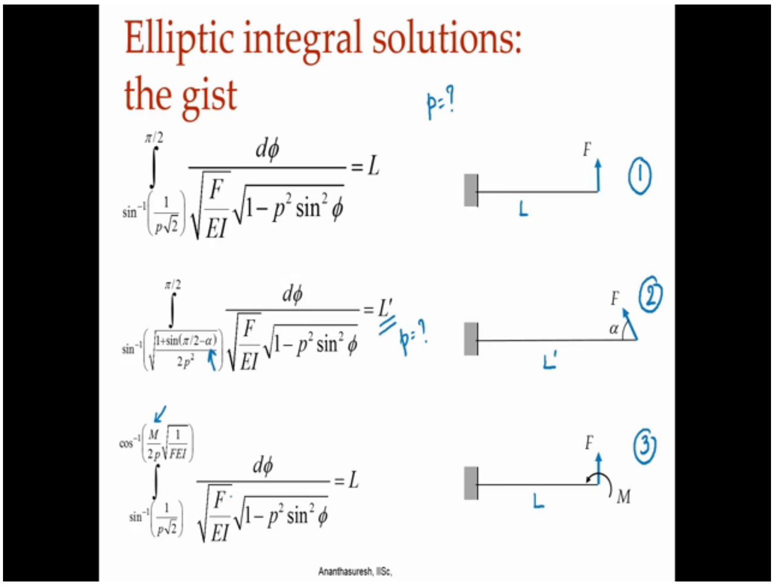

So, let us recall our Elliptic Integral Solutions the essence or the gist here we considered 3 cases; where there is Purely Transverse Force like in this over a 1st case and there is a Inclined 1 2nd case there is a force and a moment we can also consider a case where there is inclined force and moment and so forth. When each of these 3 cases that we discussed there were corresponding equations that we needed to solve, that is in this particular case the 1st 1 we need to find the value of p in all these cases we need to find p, we have to solve this Elliptic Integral Equation to find p once we have p we know everything else, similarly in the case of inclined load the alpha is there for given alpha and force and E and I again we have to find what p is for given length L prime here if this is L this is now L prime and this is again L, the moment and the force we can see where the moment is and of course, force is where it is.

So, in each of these cases we have to solve the respective equation to find p once you have p everything is known about the elastica that we have talked about in the last 3 lectures.

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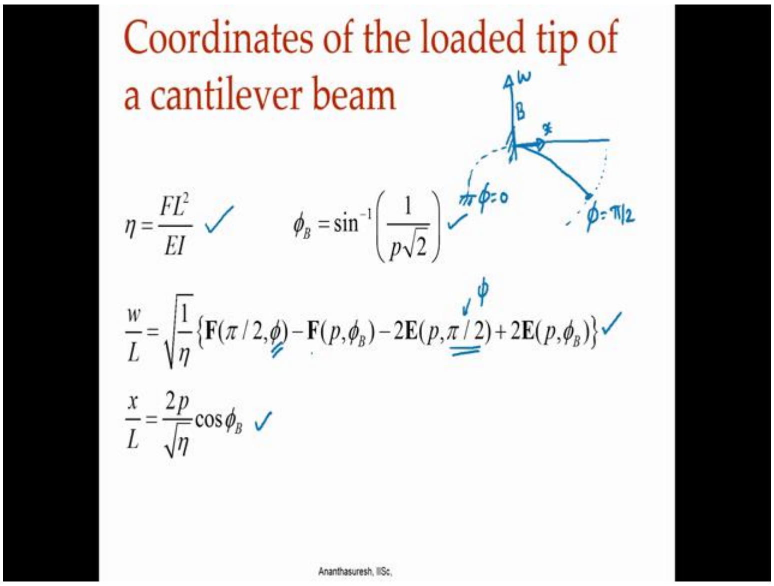

Now let us see what we can do with this for Compliant Mechanisms, let us look at this loaded tip and its coordinates. So, that we can trace that we call the locus, locus are loaded tip as the force is increased from 0 to some value and how does the cantilever deform is what we have seen, now let us pay attention to the locus of the loaded tip for that we start with this non dimensional number eta and then this phi B which as p in it that p we have to be find as we just said once we find that p and this phi B is known to us that is the change of variable corresponding to theta equal 0 where the cantilever beam is fixed.

So, when the beam is like this point is what we called B, that is why I am calling phi B rather call me phi 0, phi 0 is when this bends like this when we extend it over there this is where phi is equal to 0 here phi is equal to phi by 2, if we recall that is what we have then we get w and x by fixing the coordinate system over here, this is x and that is w the tip here what kind of locus it would have is what is given by this 2 equations once we solve for p, as you vary the angle phi that goes from phi equal to 0 all the way in fact, this angle that we have this is should not be phi by 2 it should be also phi B we have and then should be phi for any angle phi by 2 at the end. So, this should be changed to phi that is what we have we can get any point.

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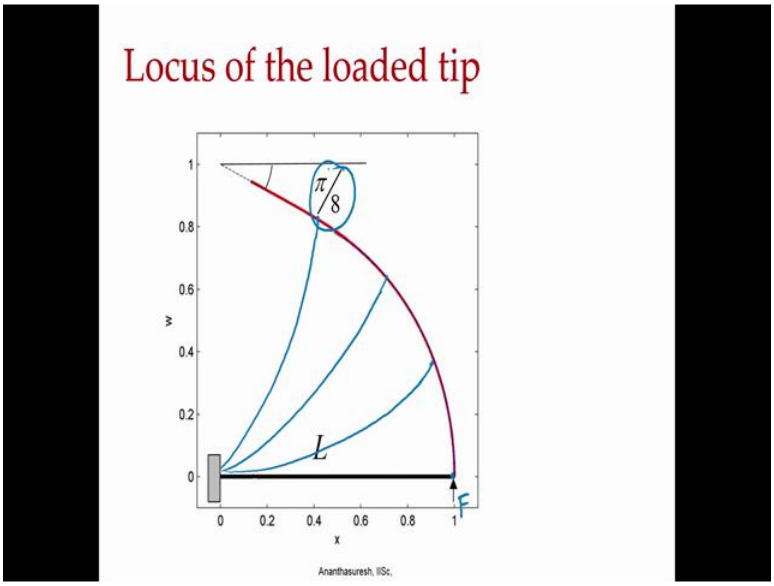

Now, let us plot it that is what I have done here using elliptical solution this is plotted the red curve it has an interesting property as a change this force from F equal to 0 larger larger number the cantilever beam which goes which deforms like this that what we have seen. So, the tip is what we are plotting here coordinates of the tip and it as an interesting property that it subtends an angle of pi by 8, no matter what this length is what material it is that is E value whatever it is and whatever I is that is cross sections property 2nd moment of area irrespective of what those are if you were to apply large enough force the locus of this loaded tip will trace a curve and that curve eventually will make with the horizontal that is parallel to the beam and angle pi by 8, this is something that we can derive from those x Land x by L and w by L equations. It is not straight forward, but you can get there. So, there is something very fundamental about the way the kinematic works here I say kinematics because kinematics is about geometry in motion whereas, cantilever beam is in motion because of the force that is applied it has some inherent invariant pi by 8.

So, there are many such insights that 1 can gain in this large displace analysis of beams, but what we do that with that is our question now.

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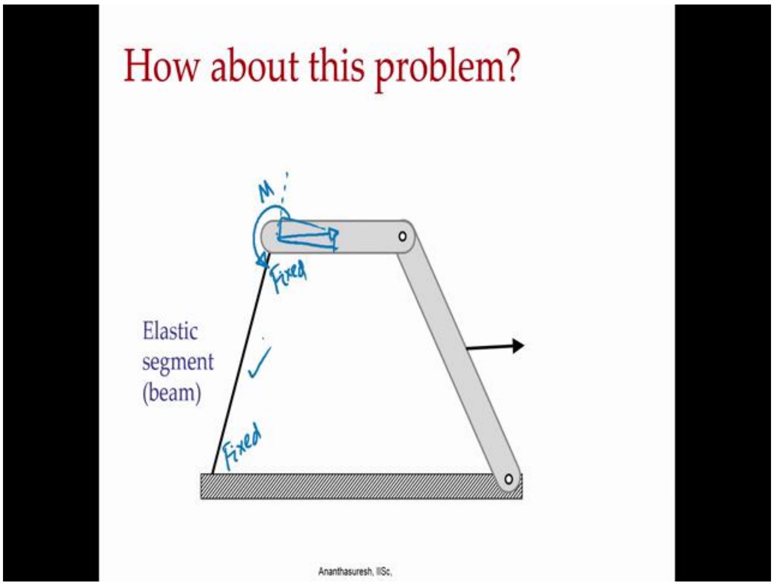

Let us say we take this problem we have a beam here which is fixed at this point and it is pinned over here and I have 2 rigid bodies. So, we can call this a partially Compliant Mechanism, if I apply some force over here its bounds to move how does it move how do we solve this problem, that is if we are given this beam we know its modulus 2nd momentum of area and of course, the length we know this let say when apply a force here how does it move its like a R R rigid body change with some spring added if I look at the elastic segment or beam as a spring then it is spring loaded and it will then apply a force is going to move by certain angle of rotation here, as well as angle rotation here and then here how does it move? If I want to ask that question can we do then the answer is yes we can solve this problem because we know how to deal with the large displacements of beams now it is fixed and pinned when this moves, let us say apply force here or apply a torque over here this is going to move this beam will have because a pin joint will have force in some direction as this pulls, when it pulls it is going to have a component along that and a component along this there is a transverse force, axial force we know how to deal with that based on the Elliptical Integral Solution. So, we can actually solve this problem.

How would this problem what I have changed now is that it was fixed here in the earlier slide it is also fixed now, in this case because of that force if there is a force, let us say in that direction here because we are not allowing it to rotate there will also be a moment that comes over here. So, we have force at this point which can be again resolved perpendicular and parallel we have 2 components this and that transverse force and then axial force, but it will also have a moment. Since we do know how to handle transverse forces, axial forces and moments we can actually deal with this beam hence we can actually solve this problem, in all of these cases the p value is the 1 that we have to find when we do that how to draw the default profile of the elastic segment.

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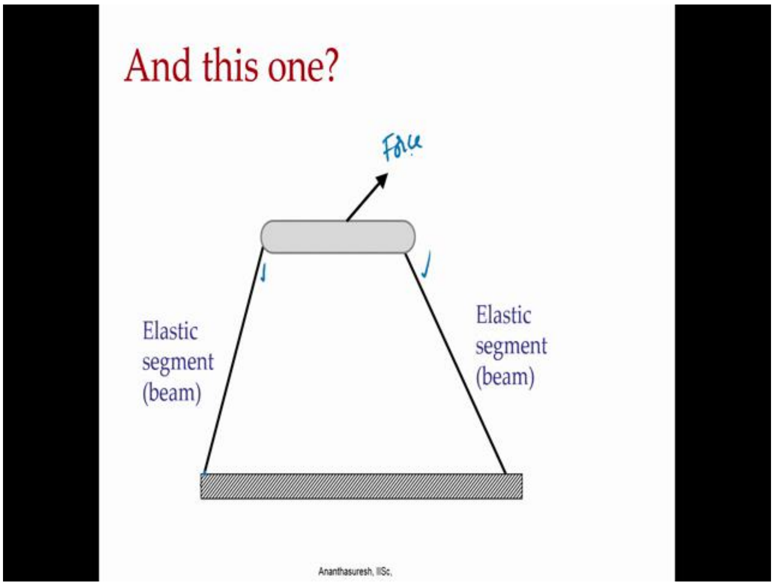

Another variation here we have 2 elastic segments and then we have force here, can we do this? Yes we can because this is a force. So, this point there will be a force in some direction, force in another direction which will balance this force let us say this may be other way whichever they balance then there will be again transverse axial transverse 0 I mean transverse axial have to do perpendicular to that we have that and this. So, we can actually deal with this problem as well since we know how to deal with elastic beam segments that can experience transverse axial moments.

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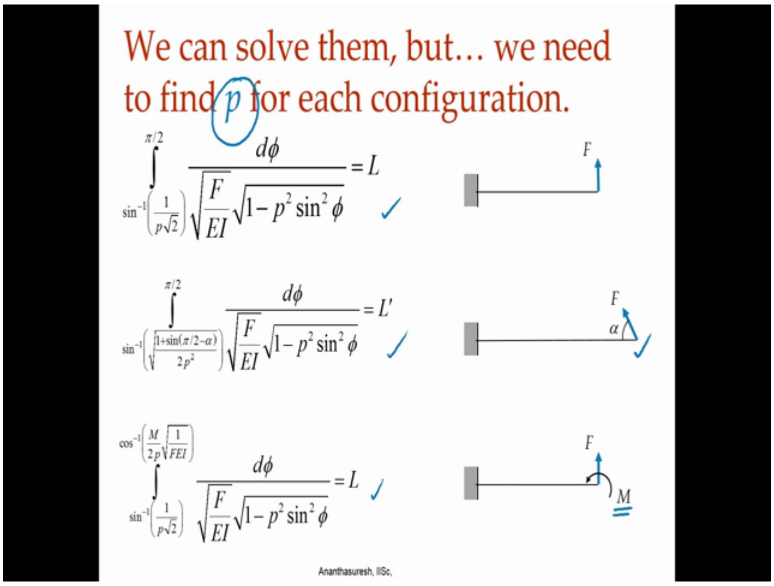

Another variation so here we have fixed and fixed on all sides this also we can actually do, once again in order to do that what we need to is take this different cases and find the value of p in each of this cases and this p will vary from configuration to configuration meaning that if we have a configuration given here in this instance when apply a force I need to know what p is for that force, when it moves the beam condition would have changed because beam has moved and the force might be different we have to again find the p. So, from configuration to configuration we have to find p. So, we can get the deformed shape that is changing continuously as applied force that can be definitely done, but it will not be trivial to do, but once we understand elliptical solution 1 can implement such a procedure, but it is going to be complicated. So, what we say is that we have to find p for each configuration depending on how the loading is if it is a pin joint will only have an inclined load if there is a fixed connection then we will have the moment also coming to the picture along with transverse force and possibly axial force meaning it can be inclined load with a moment we can do that, but it will involve lot of complication it would not be computationally inefficient to do that, but we have to do a lot of work to set up the equations and solve this equation.

Again solving this equation is not that difficult because the range of p is very small it will go by go from 1 over square root of 2 to 1. So, that is what is going to happen in any problem. So, it is not that difficult to find the value of p using any simple numerical method because we saw that p is well behaved with respect to calculating the length of the elastica part, but still we have to do lot of work.

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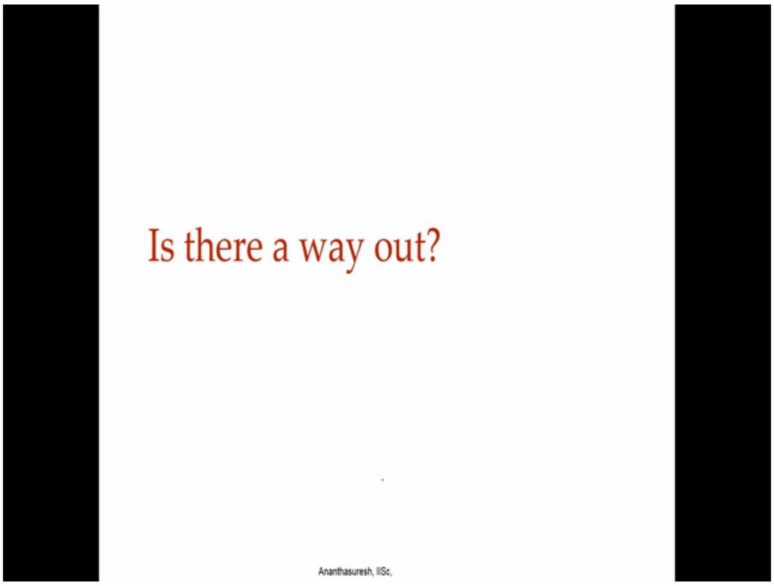

So, we ask is there a way out, is there something where we can avoid all that complication that we have to deal with and yet we able to deal with Compliant Mechanism that have elastic beam segments where we do not want to go to finite element analysis and do something about it is there a way out, it happens there is before we discuss that.

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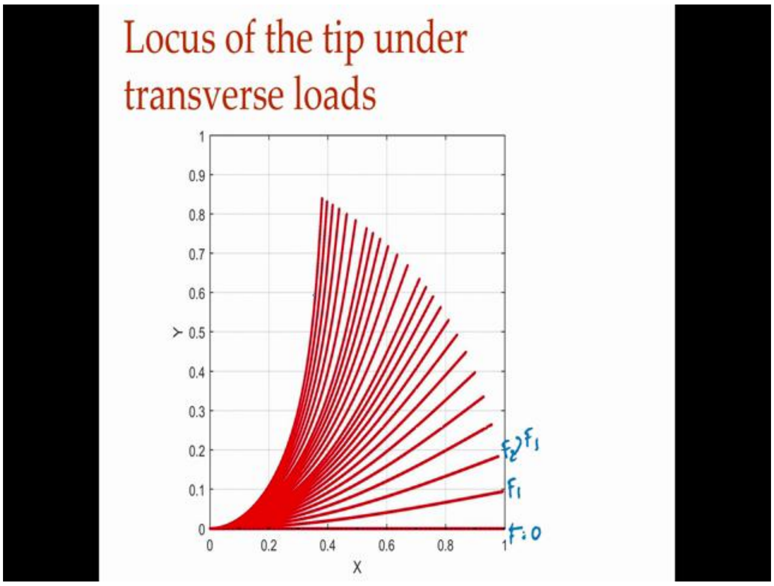

Let us look at this Locus of the tip under various transverse loads. So, transverse load is 0 and then its slowly increasing load is force is 0 here force is some F 1and F 2, F 2 is greater than F 1and so forth. So we have force increasing its going there when we see this do we seen any pattern is the question that we should ask do we seen anything interesting here looking at the locus is than any interesting.

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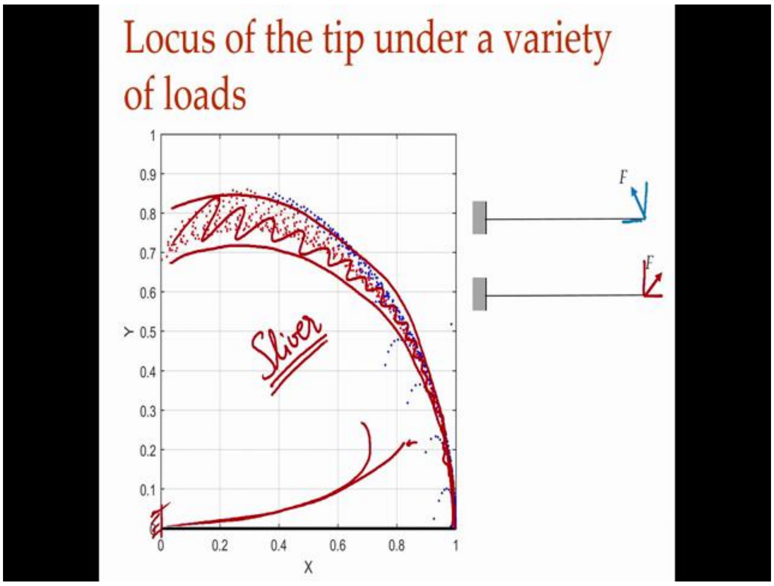

It may help to look at this picture where I have drawn it using the mat lab course that we saw in the last lecture, where I have taken the case where F is the this way meaning there is transverse force and there is a compressive axial force and the other case we have transverse force and there is a tensile axial force. So, that is the color coding here we have the red 1 and the blue 1 and you can see in some cases it going down like this some case it has gone.

But all the points if you see the locus is highly constrained and I varied this forces quite a bit over a wide range of forces before Finite Element Analysis Elliptical Solutions say that no something bad is going to happen, that is beyond which there will be something bad happened to this cantilever meaning that we have really pushed it to the limit and if we see it start out like a tiny sliver that is what it is. The tip does not go outside this range no matter what you do we are fixing it here apply all kinds of forces only transverse loads right now I have not applied moment here, moment will make it go anywhere actually I can take this and put it wherever when I do that I am not only applying transverse forces, but also moment load, but in this case I am applying only transverse forces that is force can be that way or this way this is what you are getting. So, is there any pattern that you can see we see that kinematically the loaded tip is constrained to move in a very narrow, if I neglect this which is like a kind of buckling when I am compressing load and applying if I have a beam like this let us say a transverse force now we apply an axial load there is actually tends to turn back a little bit. So, it will this beam would turn something like that and those things coming little inverse that where you see these little blue arcs and these things have tensile force if I have axial force in that direction it will stretch out a little bit. So, I can go little farther internal force, so red ones have gone little beyond for the same force, but a little component acting axially.

But what you need to notice is majority of the thing is that that is in enclosed in this small sliver that is where the tip is able to move. So, kinematically the tip is constrained to move in a tiny sliver light part of a moon soon after the new moon the tiny sliver is the area in which the tip is moving.

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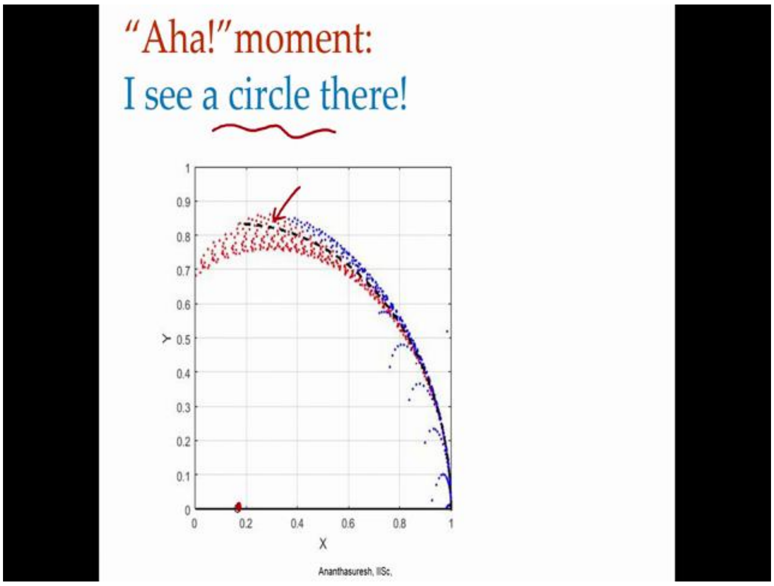

And that should give us a clue as to what the kinematic constrain here is with an elastic segment, if you have not gone in that now then I would say that here where it is it is like a eureka moment or "Aha!" as we say I see a circle there where is the circle the black one that I have the dash line the black 1 with some center somewhere we see that I can actually draw a circle right and that means, that sliver can be approximated with a circular arc.

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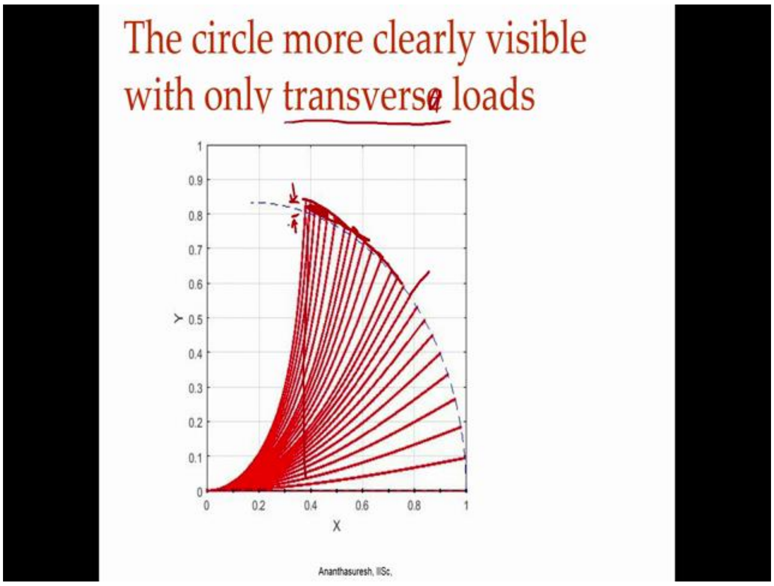

And the circular arc can be more clearly seen if I only apply the transverse load not the axial load just the transverse load, when I apply then I can see that a circular arc approximates the locus of the tip quite well I would say up to this point it right on that circular arc and after that deviates a little bit you probably do not have compare that actually undergo this much of displacement even if it is the error is very small as you can see this part is very very tiny if you look at the coordinates of this 1. So, in a height like that from there to here it is only that much is the error absolutely little bit.

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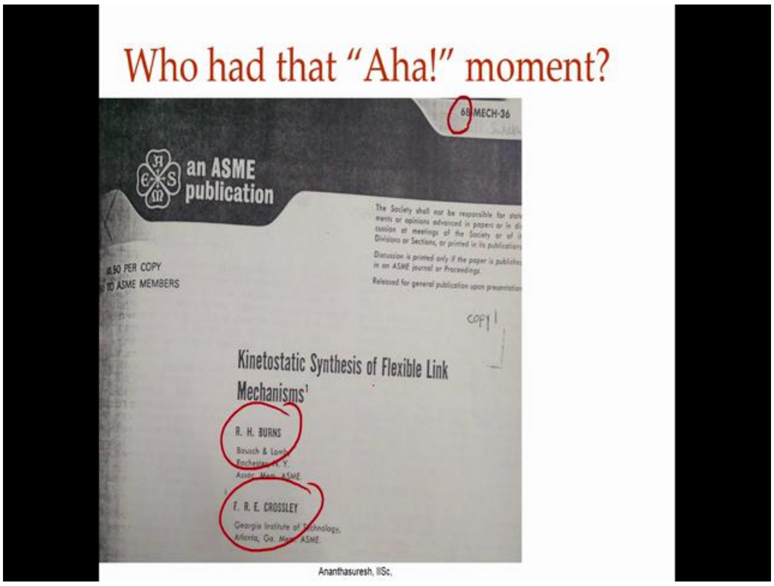

So, circular arc approximation is a pretty good 1 and who had that "Aha!" moment it were these 2 people who did this wonderful work back in 1968, it is ASME transactions paper 1968 where they are talking about Kinetostatic Synthesis of Flexible Link Mechanisms, at the time the word compliant was not coined yet (Refer Time: 17:47) they did it much later.

So, in 1968 they had worked on this and they saw this Kinematic Constraint that exists in the Locus of loaded tip of a cantilever beam under transverse loads, axial loads, but no moment.

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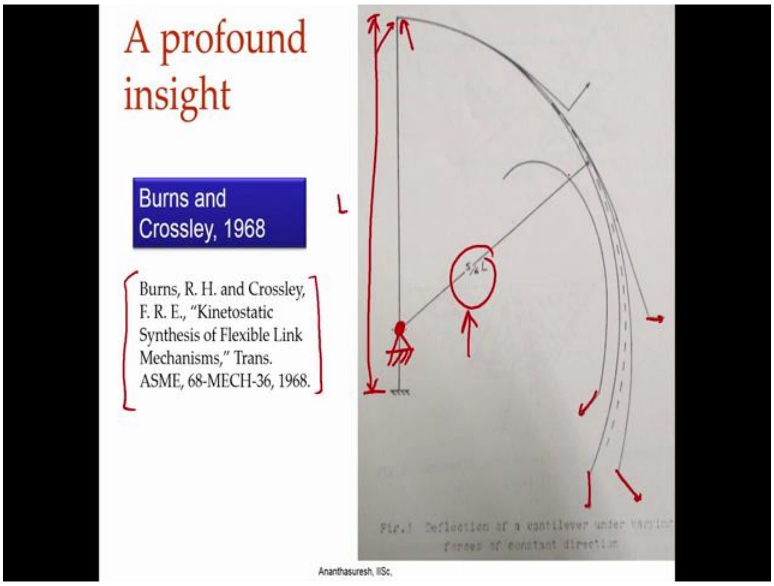

What they do. In fact, they had this profound insight into this problem this is the citation of this wonderful paper in 1968, they actually said there is a figure from their paper this if you have a beam of length L they said that if I take only 5-6th of that length put a center there, then the locus of this now here it is fixed 1 and there is we are applying load in all kinds of directions because we said that when an elastic segment is part of a mechanism with rigid bodies or other things, at the end there will be loads in all directions if there is a pin joint if there is a fixed 1 is also a moment, in this particular paper they consider a pin joint there. So, loads can come in any direction meaning that there is a transverse component, there will be a axial component in that case they saw whatever sliver I showed they actually have drawn this with a force in the direction beam may bend like that force in this direction bends like this force here it bends different way.

So, for each case they have actually drawn these at that time just coming then. So, they had done this calculation and they saw that and they came up with a suggestion that if you take 5-6th of the length and move over the fixed pivot and put it over here right this kinematically approximates that circular arc right circular arc approximates the locus of the free tip that is a profound insight that is where going back to their work more than 50 years later.

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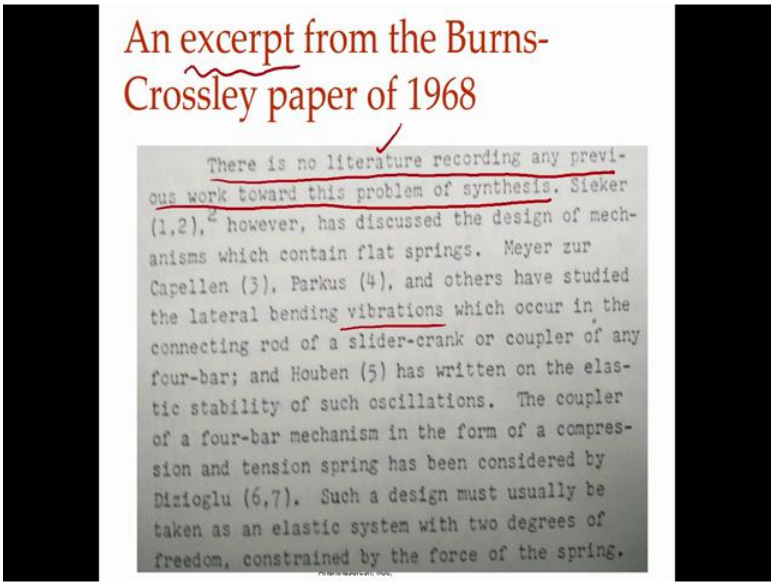

And here is a small except from their paper, where they say there is no literature recording any previous work toward this problem of synthesis that they have done, they have not stopped at this profound insight they actually do synthesis with this meaning that they design Compliant Mechanism and they give something Sieker how we ever discussed the design of mechanism that flat springs and they are giving credit to others, but somebody did vibrations some large oscillations and so forth. But this particular way of synthesizing with this kinematic abstraction they have was really new since I have not found any work before this and they are actually right there is no previous literature for this a pioneering paper in the area of Compliant Mechanisms.

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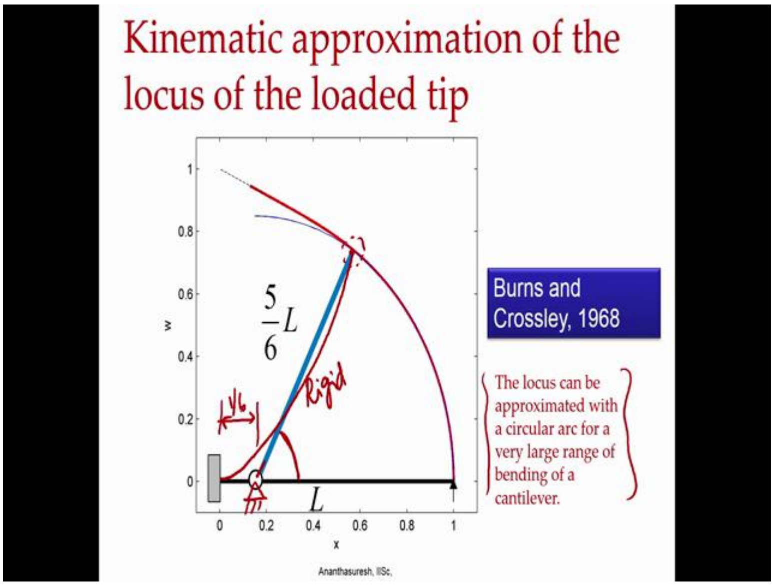

And this is what we said no here is the tip and you say that it is fixed here and we have the rigid now we can assume that this is rigid body and whatever the beam is going to do this rigid body is going to do as per as this point is concerned that is what is important kinematics.

So, the point that attaches to something else if we know how it is going to move we can deal with it, that is what is Burns and Crossley contributed to this. The locus can be approximated the circular arc for a very large range of motion you see this angle that we have, this is deforming you can see this if it were to go there it will be something like this, there is a large deformation of a cantilever beam and they are able to get away with a rigid body replacing the beam and that is the wonderful thing about this concept 5-6th of the length and you have to move the fixed pivot by 1-6th to come here that is 5-6th this is going to be 1-6th.

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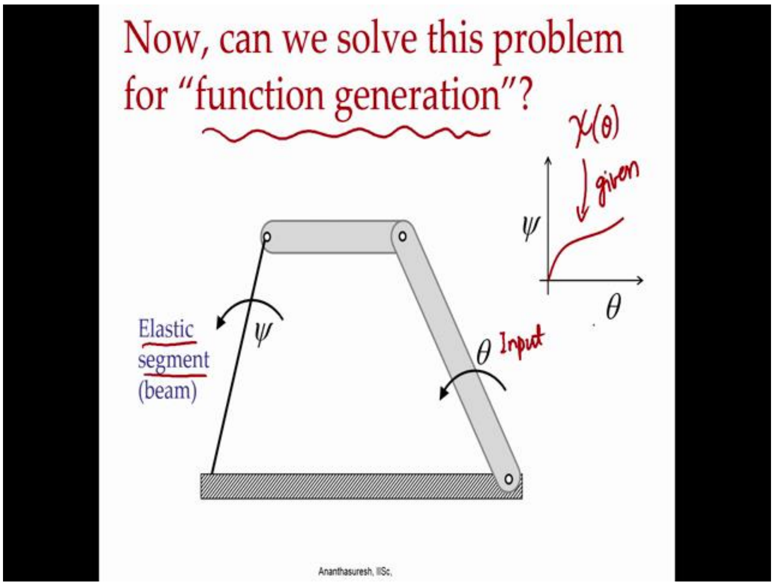

So, now with this insight can I solve this problem; this problem is something called Function Generation in Kinematic Synthesis that is what they have done in this paper. So if I were to give this theta input if I have some partially Compliant Mechanism like that 1 elastic segment elastic beam, input is theta rotation of this is psi because we do know that cantilever beam here when it deflects end 1 is going to have an angle that is important for the middle 1 right if you call that psi if this function is prescribed let us say I have function like this function psi is a function of theta, if this is given to us can I synthesize the mechanism like this that is the synthesis problem what is called Function Generation. Before computer there is to be a mechanical computers where most of the mathematical things could be done mechanically that is where this Function Generation terms come, that is if you say I want a sign arc here or logarithm or something then I can design the link lengths of these or rigid body lengths and in this case they wanted to remove a rigid body put an elastic 1 which they called Flexible Mechanism what we today call Compliant Mechanism they were able to do that.

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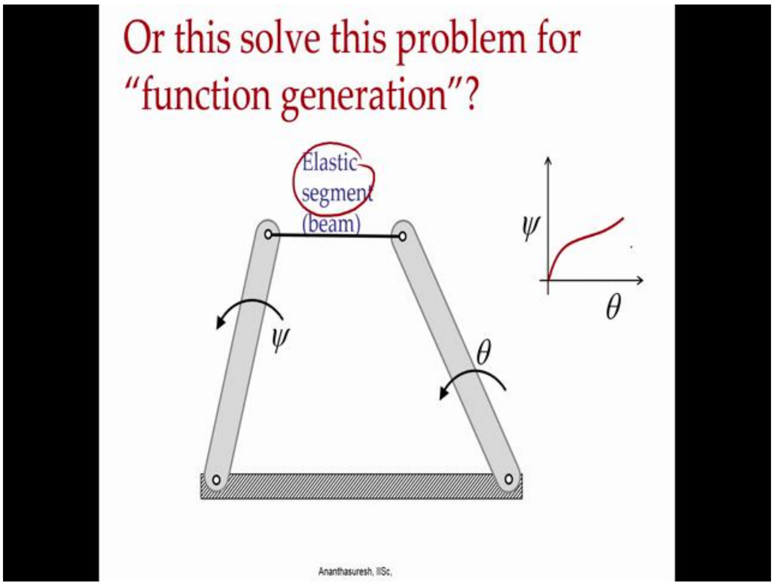

So, given a function if you know what you want coordination between theta and psi. So, you could actually do this, they also dealt with 2 rigid bodies and here is where the Elastic Segment is in the middle then again we can generate different functions the previous 1 we have sudden capability is a another capability in terms of the kind of functions it can generate.

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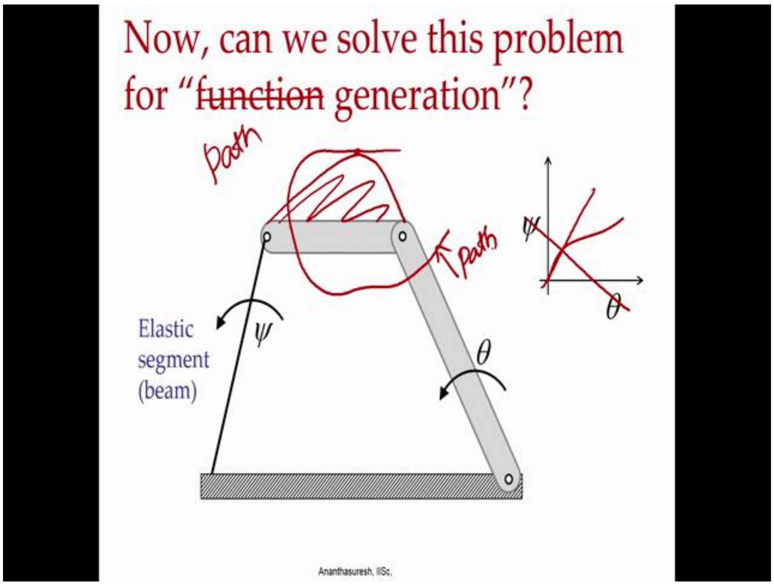

Or we can also have instead of just the function that what we have we can also have a path instead of function here, we can also do Path Generation meaning if I were to extend this body a little bit that this body I can chose a point that point is going to trace a path what we call a coupler curve that can also be done.

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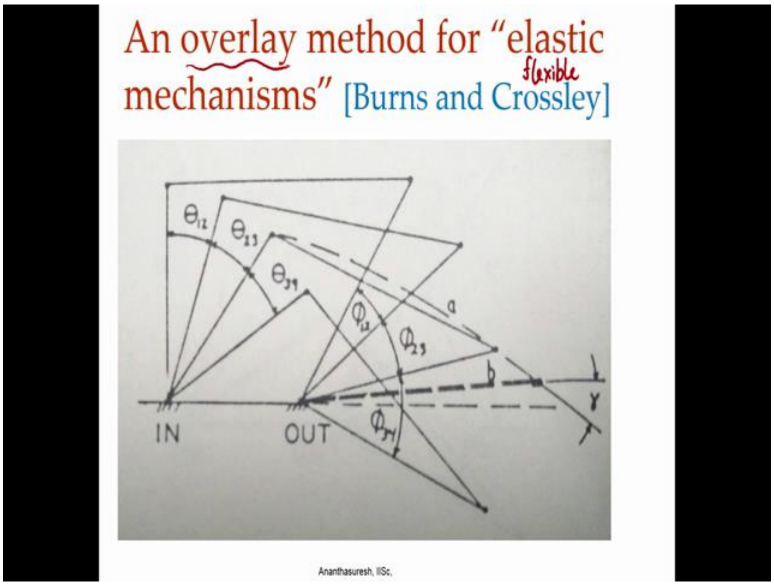

Now, it is not between psi and theta we can actually generate a path that is also is possible. So, there are number of things that we can do what in this paper they were able to do what is this over lay method using what call a Flexible Mechanism or Elastic Mechanisms they use the word Flexible Mechanisms, what it did was if we know theta and phi what I call psi they are using the symbol phi.

So, if this moves by this much they should be move by this much. So, that is a geometric way of synthesizing mechanisms of that time, they were pioneering in sense that they actually did that when 1 rigid body is replaced with a an elastic beam and there were able to develop this over lay method to design mechanisms, that why they said there was no prior literature where they had anybody had done that kind of a synthesis methodology where 1 element was replaced with an 1 body was replaced in a elastic segment.

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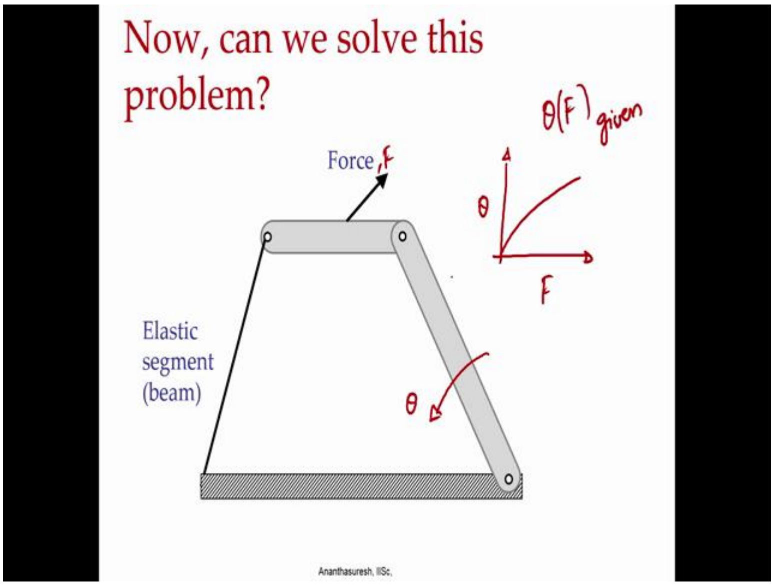

Now, with that thing can we solve this problem? That is when we are given a force here an elastic segment can I do this, instead of movement that is Function Generation where theta is given and phi is there now we are giving a force and we want some output, in order to do that we have to somehow capture the elastic behavior of the cantilever beam beyond this kinematic.

So, we have discussed in this lecture the kinematic abstraction of the large displacement of a cantilever beam, but we also need to look at its elastic behavior meaning what resistance this cantilever beam offers to the deformation if we considered that, then if I force and I say this is this was our theta if somebody gives us this function that is force F and say that it should like this can I get it, that is what is given is this now theta as a function of F applied here if that is given can we design this mechanism that is exactly what will be equivalent to designing a spring here, I am showing angle versus force I could have shown force verses angle that becomes a spring a non-linear spring that is what Compliant Mechanisms are in some sense can we do that in order to do that, we are to also capture the elastic behavior of a cantilever beam as it undergoes large displacement that is what will consider in the next lecture.

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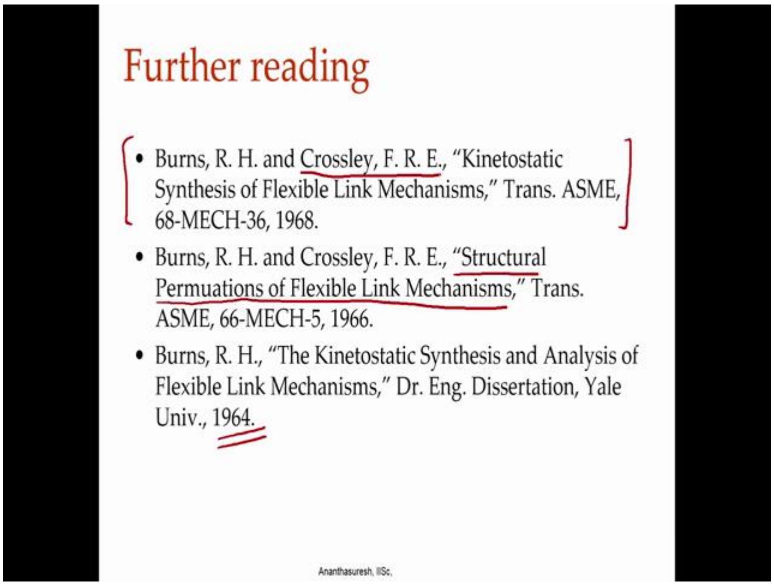

For this lecture further reading needless to say that this paper of Burns and Crossley is very important and they also had written a paper 2 years before that, which is Structural Permutations of Flexible Link Mechanisms, that is which if we take a 4 bar linkage you can replace a the crank either this or the 2nd crank the elastic segment or the coupler or if I take a larger linkage such as Stephen Sense what's want to change. Then you can replace rigid bodies with the flexible thing they had done this permutations 2 years before that and 2 years prior to that there is this wonderful page with thesis at Yale university of Burns and Crossley as adviser, by the way Professor Crossley was a pioneer in Kinematics in many ways for Compliant Mechanisms, he is really a pioneer because he had this wonderful insight and these are some of the papers that we can learn from to see how this field started with this very important insight of abstracting the kinematic behavior of large displacement complaint beams or flexible beams and that makes it Compliant Mechanisms a mean able to Rigid Body Motion Analysis and we will discuss the elastic aspects in the next lecture.

Thank you.