**Compliant Mechanisms: Principles and Design**
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**Lecture – 14**
**Elliptic Integrals and their use in elastic analysis**

Hello, in the last lecture we looked at large displacement analysis of a cantilever beam with a load at the free end of the beam; one end is fixed other end there is concentrated load f and we used Elliptic Integrals based solution which is completely analytical where we were able to convert the differential equation in the transverse displacement w in terms of only one variable called p which was related to the slop at the free end or the loaded tip and was able to solve using elliptic integral solution.

Today we will extend that concept and in fact make it even more profound by using what is known as elastic similarity principle. So, let us look at this elastic similarity principle which enables us to solve many other problems than a cantilever with a load at the tip. So, this elastic similarity in the large displacement analysis of beams is quite interesting and this is a slightly difficult topic. So, we should pay attention to whatever we are doing.

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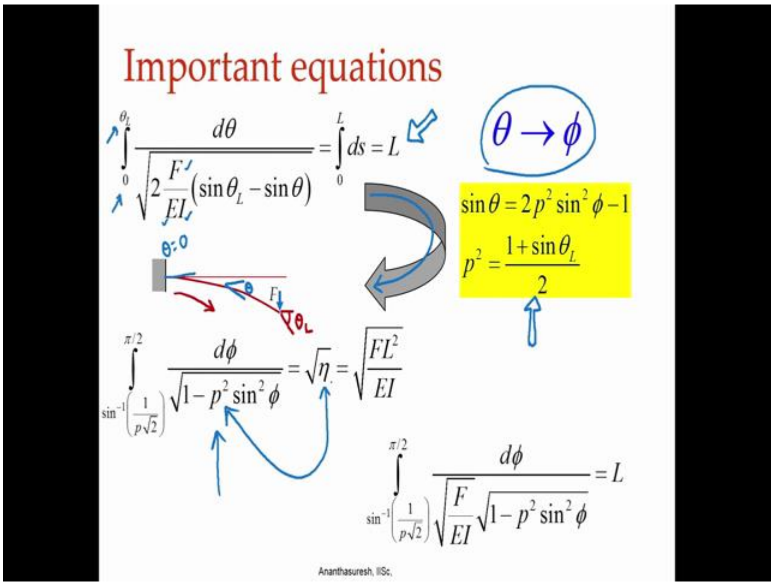


So, let us recall some of the important equations that we had began with in the last lecture and these are indicated here. If you look at the first one it is going to indicate this theta L and theta L is the slope here that is theta L where we have arc length. So, we go in terms of this parameter s which is measured along the deformed beam and that s is equal to L that theta L any other point if I take let us say intermediate point let me use a different color if use intermediate point over there now the slope at that point if I were to write that will be just theta.

So, what this equation here shows is that when I integrate from 0 to theta L; then the whole thing is equal to L where theta is the variable here which when you go from here; here theta is equal to 0 because the slope is 0 there and then increases up to theta L when I do this limits from 0 to theta L; this particular integral which involves the force F and young's modulus is E movement of area second point of area I, then I will get back my length L, on the right hand side we have 0 to L d s all this came from d s by d theta which is curvature and we came from that equation. We also did a change of variables from theta to phi. So, here we have the same equations, so these are same equation that now is in terms of phi rather than theta. Our limits also have changed as per the change of variables equations that we had used. This was done in the last lecture.

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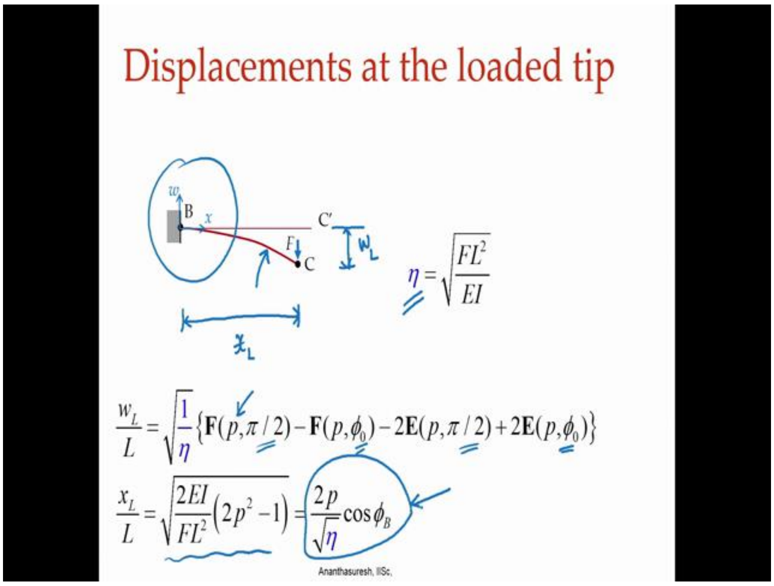


We also did in the last lecture; the following that when we have this if I indicate this as the w axis and x axis as shown over there; that w L by Lx L by L that is w L by L will be from the original beam to the deformed one, that is w sub L and x L is from there to here. Note that we are indicating the coordinate system in the deformed profile of the beam and not in the un-deformed one. So, this is the deformed profile of the beam that is the characteristics of geometric non-linear problem are large displacement problem.

So, here we indicated this in non-dimensional form w L by L equal to something which depends on this non-dimensional factor eta which is square root of F L square by E I where we know all the quantities; F is the force, L is the length of the beam before and after we assume that length is the same, young's modulus E, second point of area I and then there is this p that needs to be solved; if we go back to the previous slide that p is over there that depends on this value eta, if I know eta calculate p and by vice versa.

So, we have that p that depends on eta the whole thing is a function of eta which is non-linear function because, these are capital F is the elliptic integral the first kind complete because it is pi by 2 incomplete phi naught or that value at p and then we have second elliptic integral E complete and then incomplete. So, non-linear function of eta and x L by L also can be derived like this we had written this, but we had not done this part, but if we follow the derivation there where we had the horizontal span of the distance between the point that you have taken the fixed one to where is deformed that will come out to be this; that was not in the last lecture that in the further reading if you see you will get that thing it is not at all difficult if you follow the last lecture. So, we have done this.

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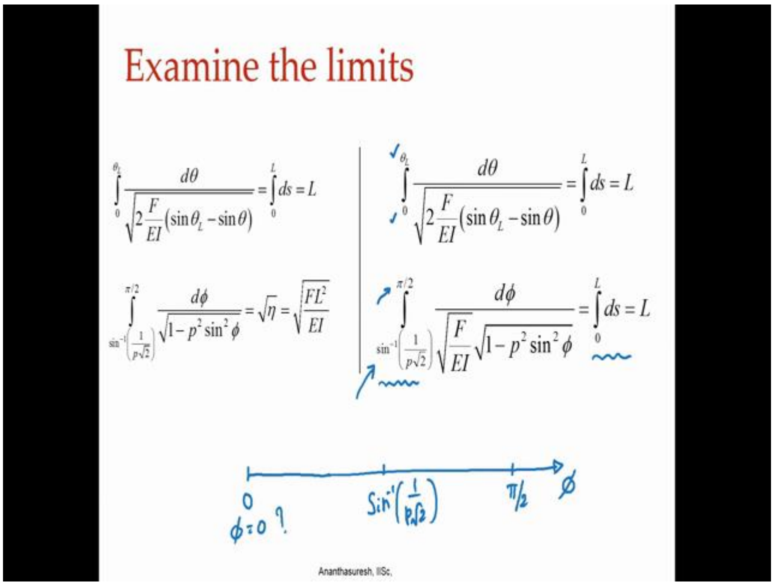

Now, let us look at this limit of this integral. So, it was from 0 to theta L over there that gave us this L on the right hand side and the on the left hand side we rearrange little bit that also comes in terms of this 0 to L d s with this limit is sin inverse 1 over p square root of 2 to pi by 2 for phi theta anyway is a slope that varies from 0 to theta L at the loaded tip whereas, the phi has this strange thing that somehow starts at some value which is sin inverse of 1 over p square root goes up to pi by 2; pi by 2 is some notion of completeness is there it goes to 90 degrees, but what is this? Why is it starting from here and what happens if we go less than this?

So, we have let us say if I do the phi axis here let say that pi by 2 is some kind of an end for this because, theta L these an end of the beam. So, let us say that pi by 2 for the phi is the end. We are starting somewhere here which is sin inverse 1 over p square root of 2, what if I go to the other side go all the way to 0 that is phi equal to 0 at this point phi is equal to 0 what does that mean? Let us look at that.

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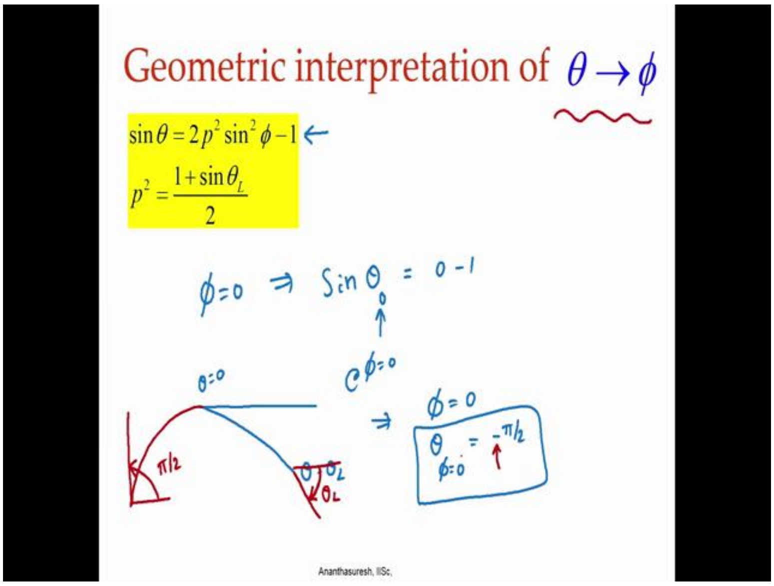


So, I substitute phi equal to 0 then the first equation that is this one let us go back that is this one here gives as sin I would say theta 0 let me say that is now it corresponds to where phi is equal to 0 I should be clear when I say that I mean at phi equal to 0 that will give me 2 p square sin square p but sin phi is 0 that will become 0 minus 1. So, what I get at that point is when phi is equal to 0 theta corresponding to phi equal to 0 will be because sin of that is minus 1 that is minus pi by 2.

So, what is happening is that if I have the beam like this and this is what we had said this angle at this point theta equal to 0 at this point theta equal to theta L, but now we need to go to a point where this theta correspond to phi equal to 0 where it is minus pi by 2; that means, that this one with the respect to horizontal that is what we are measuring right; so here if I want to use a different color; this theta L that we are talking about we are measuring with respect to horizontal. So, with that it should become minus pi by 2. So, that we get if I were to extend this curve until this becomes something like this; where this is pi by 2 since we are measuring from here to here from where it is to vertical horizontal line. So, this actually minus pi 2 we are going the see here we are measuring from horizontal in the clockwise direction as theta. Here when it goes counter clockwise it gives the minus sign. So, we have to go like this. So, in order to have the geometric interpretation from theta to phi; we take the clue that when phi is equal to 0 theta is minus pi by 2. So, with that we will be able to guess what this geometric interpretation of phi is.

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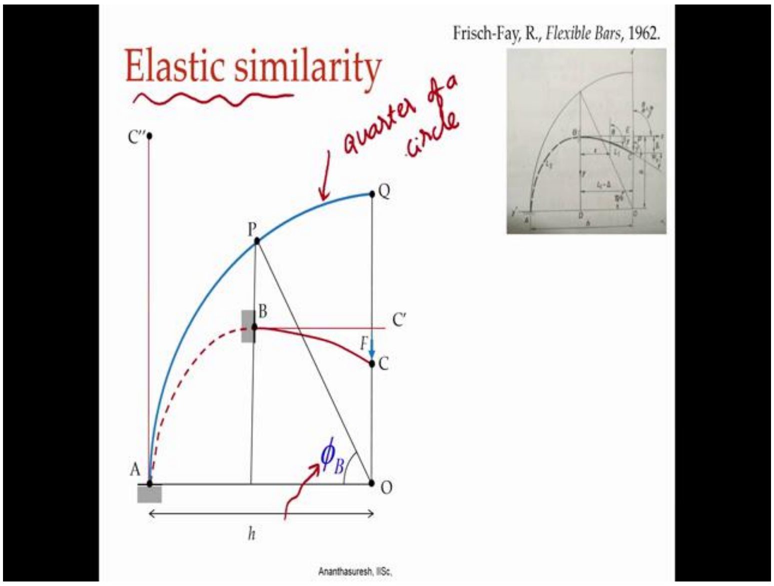


So, let us look at this principle of elastic similarity now; keeping in mind what this phi geometrically means. It is all taken from the excellent book by Frisch-Fay called Flexible Bars written in 1962 and there is a picture of that taken from that; the notation used in that different from one that I am using. So, we should not get confused we should see what symbols stand in his book and in this presentation.

So, we have this one now; what if I extended it beyond B that is what we said to take that phi back to the 0 value; if I do that then I can interpret this as a point A which has a vertical column A C double prime. So, originally B C prime deform to B C. Now we say that the vertical column A C double prime when we apply a load it would buckle to a shape that is A B C; if I fix at A; then I can actually afford to remove the support at B; I will still be able to hold the shape of the complete bend profile of the vertical A C double prime column (Refer Time: 11:37). A C double prime when you apply a load F over there it would bend like this; I should not show F here because the F is not acting in that configuration. F is act actually in the deformed configuration. So, again if I were to remove the support B it will stay the way it is.

So, what is phi now? We have just shown going back all the way from B to A with this interpretation; now it turns that if I extend it and imagine that there is a point A and there is A C double prime vertical (Refer Time: 12:14) then I can join this line horizontal line from A and then let this point go all the way to Q then I get this circular arc or quadrant of a circle A P Q. So, here is where our phi B is.

So, we have to take this B extended vertically up to P and then join to O remember this is a quarter of a circle and that is how this phi is defined and that phi is measured in this manner. So, from horizontal line A O and then O P that inclination is phi B. Again notice how we got phi will bent from B up until meets a quarter of a circle and then come down. So, the phi that we measured is actually this angle that we have shown there and this is the elastic similarity; the way the cantilever beam from B to C prime would deformed to B C would be same as the vertical column A C double prime would buckle to A B C and one we fix at A we do not need the support at B and this is the Elastic Similarity between a cantilever beam and the buckling of a vertical column are abstract as Frisch-Fay calls.

Now, we have this h here that is the horizontal span or radius of the quarter of a circle; we need to find that.

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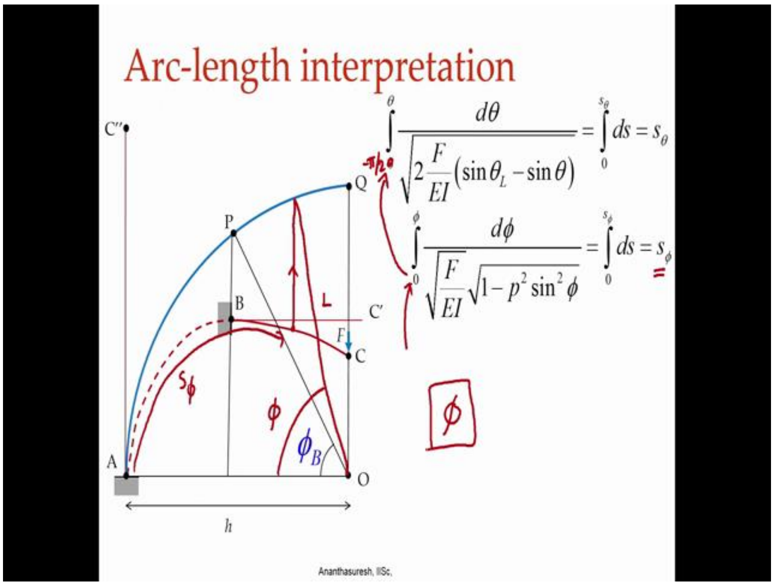


Before we do that let us understand this arc length interpretation in this Elastic Similarity. Now I am going from theta 0 to theta or phi from let us say the corresponding thing here would have been not 0, but it should have been our arc length whatever that sine inverse 1 over p square root of 2, but what we have done now is I am varying this phi from 0 to phi if I do; it will go to 0 to some s phi; likewise theta I can go from let us say a point B or some other things corresponding to this; it should not be 0 0 they do not correspond some other limit over there.

When I go to theta; I would get s theta that is from 0 to s theta and the 0 again in fact, correspond to 0 this as we already said it is minus pi by 2, when I go to minus pi by 2 theta that will be 0 to s theta; d s if we do s theta. What does it mean? When I take a point let us say somewhere over here; then from here to here there is an arc length s; s phi if I take this point over there again how do I complete the phi? I have to extend vertical line there and join that to this; this would be my general phi; that phi that I show here this should be calculated. I have to draw vertical line and join this with respect to horizontal I get phi when I do this integral from 0 to phi what I get will be that s phi that is this arc length from here to here; that is interpretation here when I go from here to B I will get this arc length that in the dash line. When I come from B to C then have to subtract whatever arc length I have from here to C that is A B C minus A B; if I do then I will get the original length of the cantilever beam that I had as L. So, this interpretation

Elastic Similarity you can imagine a number of situations using this diagram or this especially the interpretation of this phi that is the key concept here.

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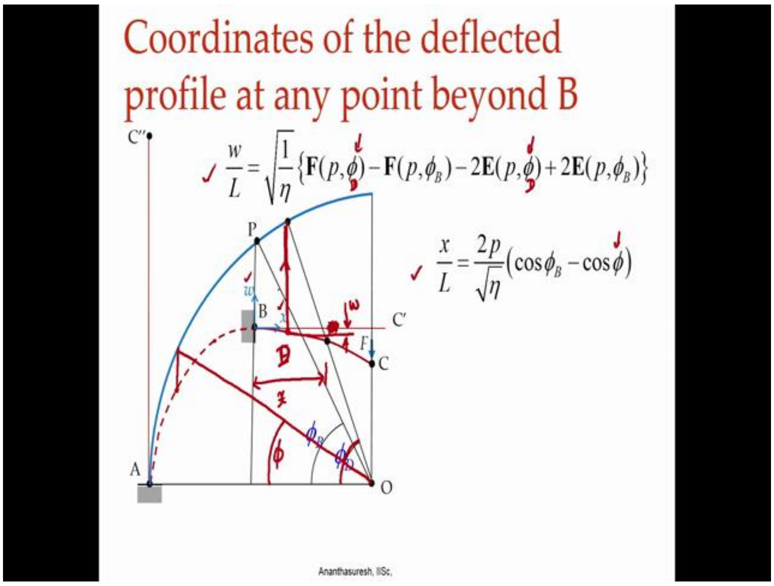


Now, let us also look at by knowing that what are the coordinates of the deflected profile at any point B beyond B any point D beyond B. So, if I take a point D again I have to take this interpretation in fact, this is not the point I should take. So, we should actually imagine this will be let us called this point E or something. This is of no significance if I take a point here; I have to draw a vertical line and join this that becomes my phi D. So, actually that is not D so, we can call instead of D; E that should be the phi D. So, again we have to go up here and join that that becomes my phi D. So, if I have that the coordinates here if I take the w x thing w by L that is this little distance that is very little how much ever that has deformed that will be w and the horizontal coordinate in this x that will be x here; that w by L is given by that and x by L is given by this in terms of whatever angle that we have it general phi D have called so that can be you know phi D and then from phi B we are measuring and this also will be phi D any point. You also know the coordinates; this arc and interpretation whatever point you can take a point here, here let us say I take point here; I have draw by vertical line and join that to this that becomes my general phi. So, I can take that over there, over there and over there; then I would get the w at that point from here to there and x from here to there in this coordinate system of w and x; that is how the this interpretation is graphically and equations are as what there shown on this slide.

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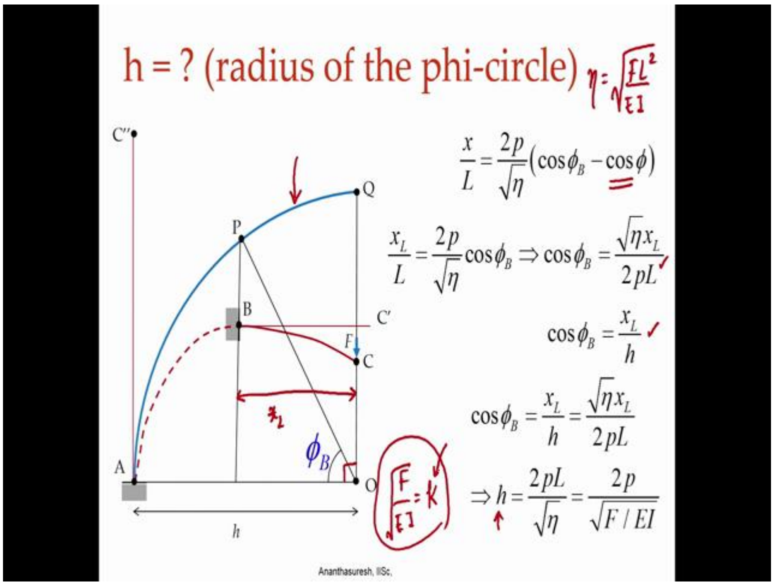


Now, what about this h; we are not talked about that yet. So, what is the radius of this quarter of a circle that we have what is that h; that h can be interpreted by knowing that x by L that from the previous slide where we have discussed it is 2 p by eta again eta is non-dimensional number that is square root of F L square by E I times cosine phi B minus cosine phi that is a general one.

So, now what happens when we go all the way to end that is point O where phi actually becomes 90 degrees. So, that is 90 degrees; then if I go all the way then that cosine phi that will be 0. So, I get x L that is x is general for any phi; if I take that all the way to the end which is phi equal to pi by 2 that is from B all the way to C when I do this becomes 0. So, I get x L by L is 2 p by eta cosine phi B implies that cosine phi B is equal to square root eta times x sub L by 2 p L. What is x sub L, x sub L is from here to here that is my x sub L.

Now, you can also see that cosine phi B is equal to x L by h; this is x L and h is this radius here; cosine phi B is also equal to x L by h by comparing these two; we can get that h. So, cosine phi B is x L by h from here which is also equal to square root of eta x L by 2 p L and from here we can get h to be equal to 2 p L by eta are 2 p because eta is F L square by E I; if you substitute it will become L gets cancelled; square root of F by E I. So, this square root of F by E I; Frisch-Fay calls it k if we are referring to the book this will come handy. So, h is 2 p by k in his notation. We are sticking to what we have symbols that we have use; I am not introducing additional symbol k in this presentation. That is why we can get h also.

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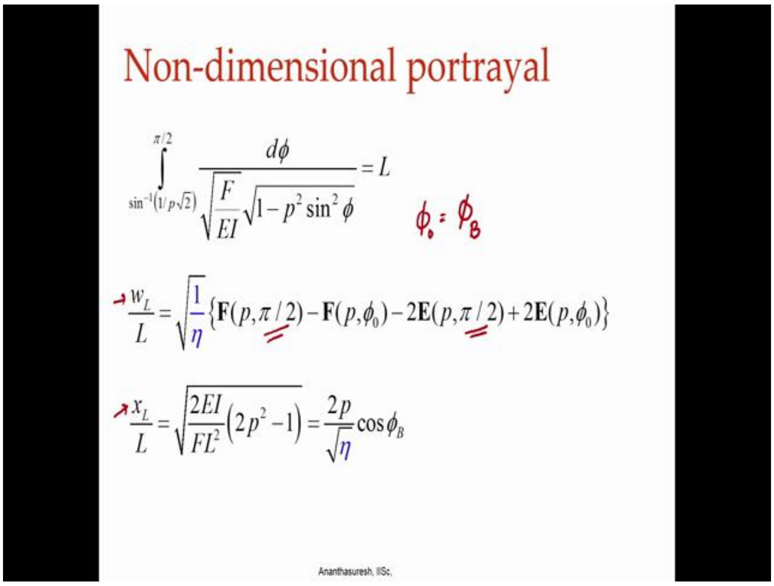


Now, if we look at non-dimensional portrayal again. So, w L by L, x L by L we had for complete one here. So, we have that phi going all the way to pi by 2 starting from phi naught which is what we had in the previous lecture; in this lecture that phi naught we are denoting as phi B because the 0 corresponds to in this presentation; where phi becomes 0 which is actually in theta equal to minus pi by 2 are this phi becomes 0. So, this is for going complete L both horizontal; horizontal here and then transverse w L.

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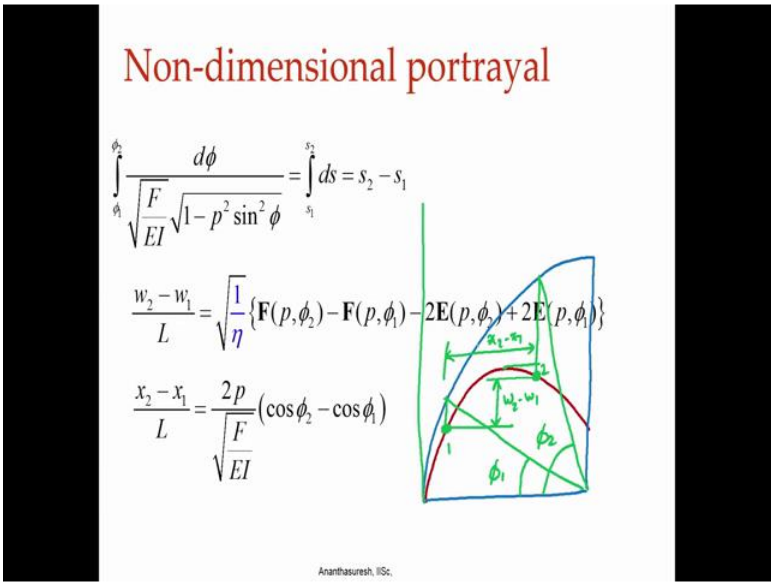


Now, with arc-length Interpretation any other point I take; I will get if I go from phi 1 to phi 2 then w 2 minus w 1 by l is what I get. I can go from phi 1 to phi 2 both elliptic of the first kind capital F second kind E; likewise horizontal x 2 minus x 1 by L; 2 p by the square root of F by E I cosine phi 2 minus cosine phi 1.

So, we able to get any length that is if I have this one going all the way something like this if; I take any two points let us say that let me draw that circle with same blue color we had used; let us I have something like this that is a quarter of the circle; if I take any two points then I have to let me change color again for clarity; I have to do vertical line and join this vertical line, join this then I get the angles let us call this phi 1 and this is phi 2. Then what I get if I say w 2 minus w 1 what I get is from the horizontal that we had; so whatever distance that is there between this and this that will be w 2 minus w 1 and likewise the horizontal distance from here to here that would be x 2 minus x 1 by taking 2 points 1 over here another over there. So, this is the point 1, point 2 we can get that in a non-dimensional form.

So, we can actually solve a number of problems starting from (Refer Time: 23:57) that was vertical like this and original horizontal that we had and anything in between. So, this means that if there were to be an Inclined Load;

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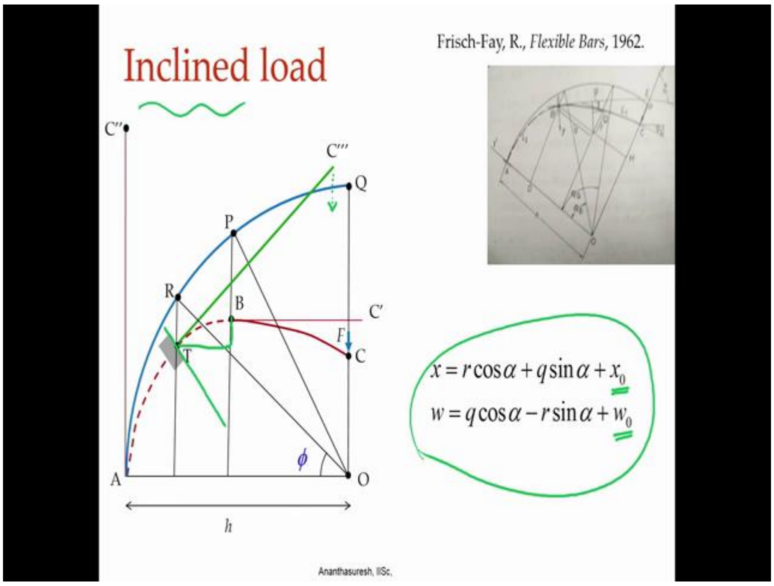


So, how does it come up about Inclined Load? So, for we have taken a load that was transverse to the beam, but now if I take a point such as what is indicated over there and draw a line tangent to this profile that we have general profile that we have; this green one still has force; it goes from let us say T to C triple prime that would deflect to T B C profile under the action of this load F. Again I do not show force F here, because that has not deformed when that force acts as C triple prime that will change it form T C double prime to T B C and again we do not have to put the support at D when we do that just like we had argued about this vertical straight: A C triple prime if we fix this then I do not need that. So, in the inclined load I can again take this general phi angle; draw a vertical line to T R and then join R to O I get this, then I can remove this support at B. This is what again Frisch-Fay gives in his book as if we have turned coordinate system you see this coordinate system what you had like that has been turned over there; based on the angle. That angle is whatever angle that this makes with this; we can call it alpha that is what he also shows an alpha angle here; again other symbols of different in the book compared to the symbol that we have used in this presentation, but what it means is that you just have to rotate the coordinate system and do the translation because our measurement is from here because this is where our x is and our w is.

Now it has moved over here and has been rotated. So, you to do a coordinate transformation by doing this some translation from here to here and also a rotation because now it has changed the coordinate system is changed with respect to that if there is inclined load because the load were to be here it is inclined compared to the original one which were to be transverse like this an inclined. So, the inclined load if you do; you have to do this coordinate transformation based on this all that information is right here in this diagram.

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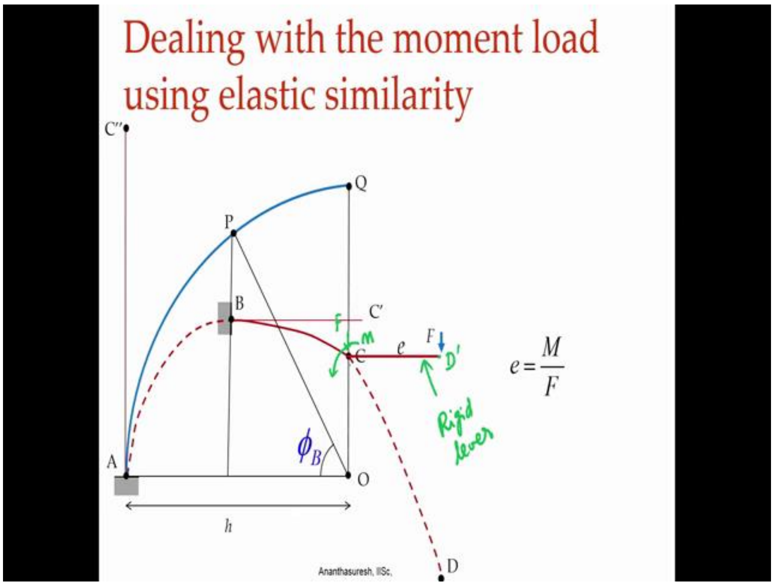


Now, how do we use this Elastic Similarity Principle when there is a Moment Load? Now in this original in this force F it would have deformed like this. Now there is additional moment M; then how do we do this? It will not be the way it is shown here because this is for when the force F is acting; now there is moment acting everything here that is all this portion will change. How does it change? So, what Frisch-Fay recommends is that imagine now there is a rigid lever. So, this is a rigid lever attached at point C horizontally and now at a distance length of the rigid lever e which is taken as M divided by F; we want to bring in the effect of movement at the point C.

So, what he recommends is; let us move this force F from point C to another point over there which is distance e from here. Now if we see the way we have defined e if I multiply this F with this e then I get moment here which will be M because e is depends M over F when I multiply this e with F; I get M and I get back the force F here. So, I am getting moment and force over here effectively. So, I get the movement M; I get the force F; if a mass in that there is an extension the beam which let us say deforms to something like this. So, this C to whatever point this is let us call this D prime that would deformed to C D and the whole thing also changes.

Now the previous one that we had that h p all that the quarter of a circle will change now because there is F and M are rather we have F at a farther point beyond C. So, we have to change that circle exactly same construction only thing is it will become larger as we will see here.

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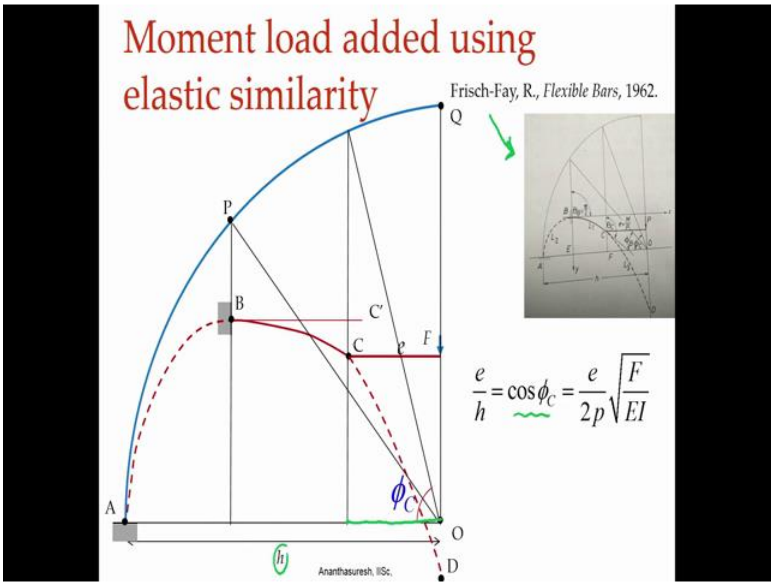


Now, I am putting that point this O has changed correspond to D at to get right above that D that D; C D is the extension of this C F which will be longer where the force is being applied to get this thing.

So, now we have a new thing and I need to get that new h here; in order to get this new h we already know the e the e again if you go back to this e equal to M by F. Now we can get that e in terms of geometry here that e by h that e is from here to here which is same as that h is the new h that we need to find radius of the circle that is equal to cosine phi C which we also know to be equal to this e by 2 p square root of F by E I that comes from the way the cosine phi C is defined here. So, from here we can relate this h to e when h is there h is again related 2 p here.

So, we know everything in this diagram again it takes little bit interpretation; I have put the original figure from Frisch-Fay's book; if we look at the geometry we get this relationship then we know the e is a function of h; h is the function of p only unknown is p now. This p is not the same as that what we had when there were to be only vertical force at point C; now the fore is acting at this end of the rigid lever or extension of this to create the effect of moment here. So, you have to recalculate that p; how do we get that p?

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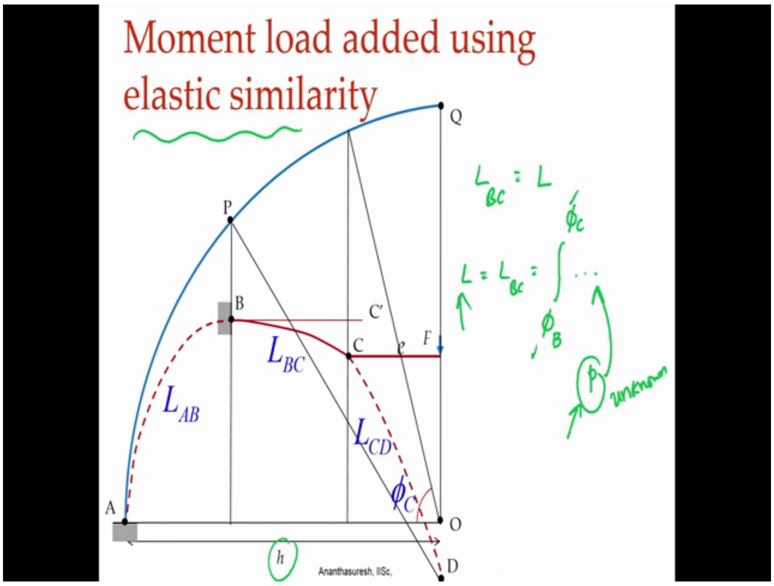


In order to calculate that p what we need to know is; this length A to B is L A B, B to C is L B C, C to D is L C D; what we are given is only this L A B, L B C; L B C equal to L is all we know, but we can write that with our arc length interpretation this L B C or L equal to L B C is an integral that goes from phi B to phi C.

We already know phi C from the previous thing; cosine phi C equal to this; we already have h was p by square root of F by E I that we have derived. So, phi C is known to us and phi B is anyway known to as because that was sine inverse 1 over 2 square root of p. So, these two limits are known. So, I can get this L B C by doing that elliptic integral of the first kind that we had; from the arc length interpretation that has the unknown which is p. So, p will be the unknown in this equation. So, since I know L; I can calculate p; once I know p I can get everything here that is when we change that force over here our radius h has changed h depends on p that p is obtain from this equations.

So, you can solve using this principle elastic similarity a case with moment load as well; you can get even more complicated what if there are Two Transverse Loads?

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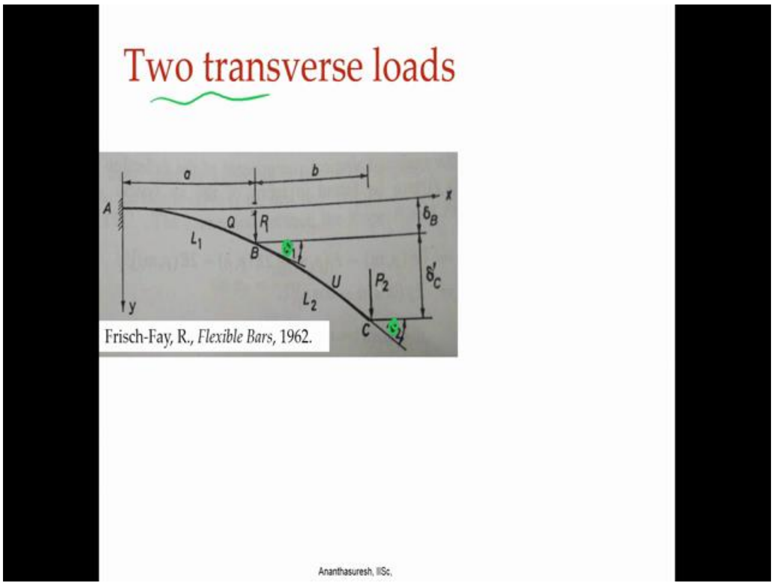


So, there is a P 1, P 2 here I taken the figure form Frisch-Fay book directly and again his symbols are different he uses psi where we have used theta and let us not get confuse and of course, use P for force where using F for force and so forth, but the concept is clear when there are two transverse loads like this we know in the linear case we use principle of superposition whatever we get with P 1; displacement we add with the displacement we get when there is for P 2. So, that cannot be done with non-linear, but this elastic similarity principle lets you do almost that it is not linear superposition it is some kind of non-linear super position using this graphical construction and the corresponding equations. So, if we do that here; this is a diagram from Frisch-Fay's book where we have this picture where the original one is shown with h equal to 2 p by k.

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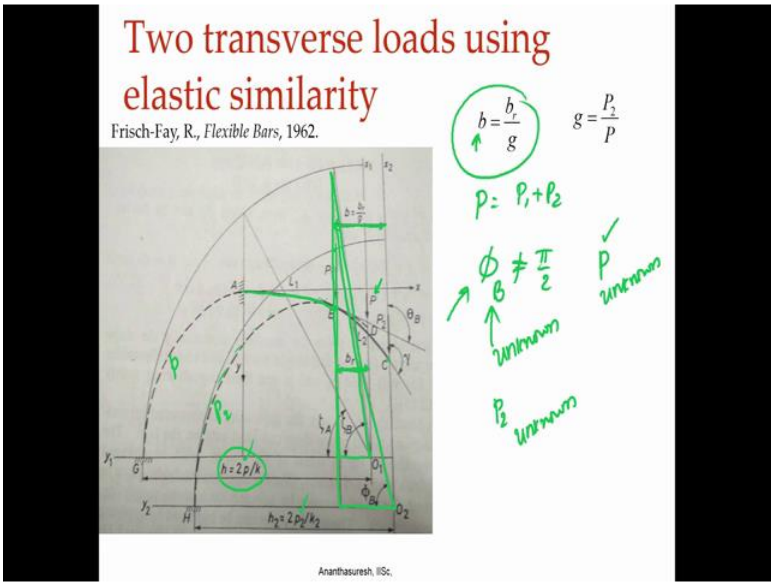


Again I said uses a k which is square root of F over E I that is a new symbol we have not introduced, but he uses and he says that form A to B there is a P 1 and then after there is a P 2 over here at point C. So, if you want to know what is A B deformed; he says introduce in a new force P that is over here you see that P over there he say the introduce P such that this P is equal to P 1 plus P 2 because at point B you would feel the effect of P 2 as well as P 1; when we do the shear force diagram and also you would feel a moment which is this P 2 times whatever this distance b is that distance b So, it defined this b as b r by g where is g is P 2 by P 1 and b r is this distance; it is a lot of geometric construction correspond that each of that there is an equation that we can see, but what he is saying is that if you want to know the profile from A to B; you should have the total force P 1 plus P 2 put at it distance which is b r such that the movement here that is felt is P 2 times that distance from here to here which is b that is how you puts that.

So, we gets that and corresponding value of P which is over here you have one equation the end point unlike the previous one; it is not equal to pi by 2 that phi B here is not equal to pi by 2 which was the case earlier. So, that phi B is an unknown this is an unknown here and of course, the P is also is an unknown. We have 2 unknowns here and then we have this other one B to C which he says we simply another one of this where it is at B imagine like a cantilever beam from going to that C which is straight forward which we have discussed earlier that gives you B C.

So, A B is controlled by one P and then B C is controlled by another one which is P 2, this P 2 is an unknown or unknowns now are three. P 2 this P the original P is there and then phi B what equation do we have again uses geometry that this length b r can be obtained in two different ways because we have we know various distances here that is if I look at this distance b that comes from cosine one and b r comes from the cosine for the first case if I take this triangle from here to there, here to this b r goes from here to there to here, I will get b r other one goes from here to there to here, I get that b, b and b r are related the way he defined. So, he gets that additional equation. So, we get previously from the length equation here we get for P 1 equation and phi B comes from the cosine equation.

Now, the third equation is this relating this b and b r. He gets 3 equations, three unknowns and solves it. It is a lot of geometric construction where the corresponding equations it works studying carefully to see that it actually works out in a way that when there are two transverse loads you have three equations and three unknowns. Unknowns are phi B, p and P 2; that p corresponding to this, there is a P 2 corresponding to this; we are superposing in them; these are non-linear thing yet you are able to superpose in order to get the combined profile.

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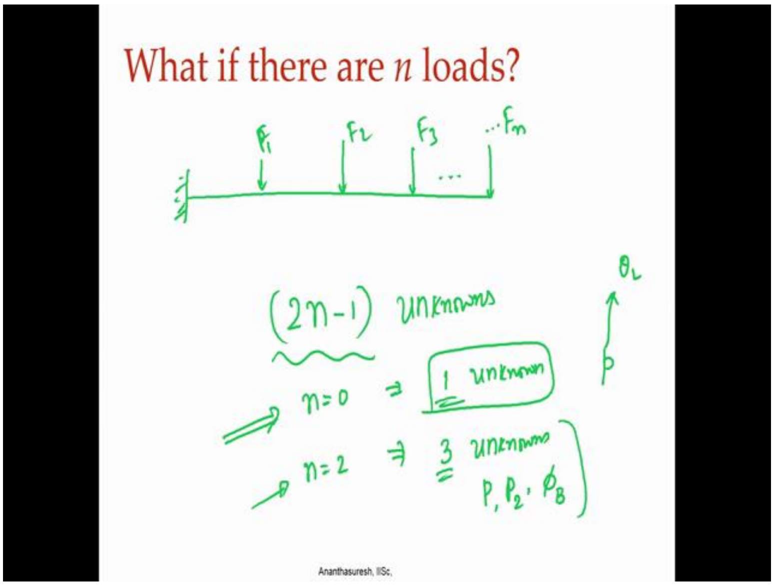


In fact, he argues that if there are n loads let us say I have a beam where there are n loads any number there is let us say F 1, F 2, F 3 and F n if you have that then you would do these diagrams each one of them and unknowns will be 2 n minus 1 here and so many equations you have to get. When n equal to 0 which are the first case we have done in the last lecture we had only 1 unknown which was that p that we had or that p was related to the slope at the end theta L; when n equal to 2 which we briefly discussed now unknowns are 2 times 2 4 minus 1; 3 unknowns which were 2 piece which were p and another P 2 for the second arc that we considered and then there was this phi B at that point where it is.

So, when you have more loads; you will have more and more unknowns, but what is interesting is the differential equation originally was convert equation 1 variable; when there are 2 loads then it becomes 3 unknowns; eventually it becomes 2 n minus 1 unknowns. It is a very efficient where compared to Finite Element Analysis; it may be difficult to believe, but we have few words; instead of solving differential equation here we solved only elliptic integral equation with one unknown p. In the case of two loads we will have let us say two differential equation load 1, load 2. We are not supposed to add them up linearly; we had to have non-linear one. So, it becomes a difficult one to do, but now we are able to do it in few word variables and able to get the solution.

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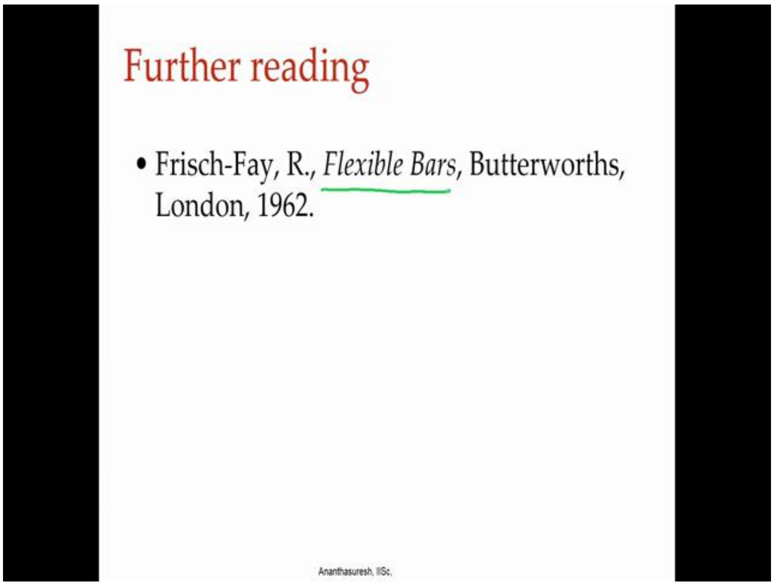


All that I have discussed today is from this book called flexible bars. Those of you are interested to study this book in the chapter 2 in that which explains all of these in lot more; what we need to understand now is that using this principle of elastic similarity;

we can solve a cantilever beam with that tip load in any direction, Inclined load is possible and moment load is possible, multiple loads are also possible. So, anything we can be done using this principle of elastic similarity. It was a work done in 1960s; somehow it seems to be forgotten, but there is a big need to get it back if you want to do large display analysis of flexible bars.

What we will see in the next lecture is a set of examples to understand whatever we have discussed in the first lecture and this lecture that is this week first lecture and second lecture; third lecture of this week we will do some numerical examples to understand this; later on will see how this insight can be taken further to be useful for compliant mechanism analysis as well as design.

Thank you.