Compliant Mechanisms: Principles and Design

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Lecture – 13
Deformation of a cantilever under a tip-load, using elliptic integrals

Hello, this is the third week of lectures on the course on Compliant Mechanisms Principles and Design, last week we looked at flexures as well as finite element analysis use to analyze complaint mechanisms, which primarily consist of a elastic beam segments or what we called elastic elements and we emphasized non-linear analysis in particular geometric non-linear analysis and we saw that finite element analysis can simulate just about any complaint mechanism whatever the non-linearity might be, but then it is not. So, useful for design purposes always and usually getting some useful insight is also difficult with finite element analysis.

If we have an analytical solution, it will be much better which is what we will do this week try to get the insight in to large displacement analysis of beams, in particular with cantilever beams. So, let us begin by looking at a cantilever beam and how it behaves where it under goes large displacements, under very large transverse loads.

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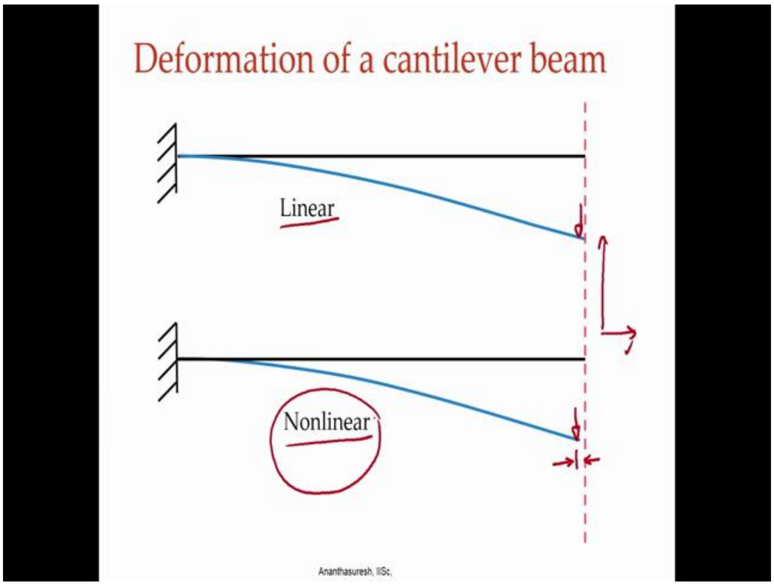

So, looking at the cantilever beam, we had seen this picture in this last lecture where if someone shows cantilever beam deformation, when there is a load, transverse load at the free end, how does it deform? We have shown both linear and non-linear how do we tell them apart we had said that if we actually observe by looking at whether there is displacement in the axial direction. As the loaded tip starts moving its going to have displacement not only in the transfers direction that is this direction, but there is also displacement in the axial direction that is in this direction, which happens only in non-linear case, but on the linear beam theory case.

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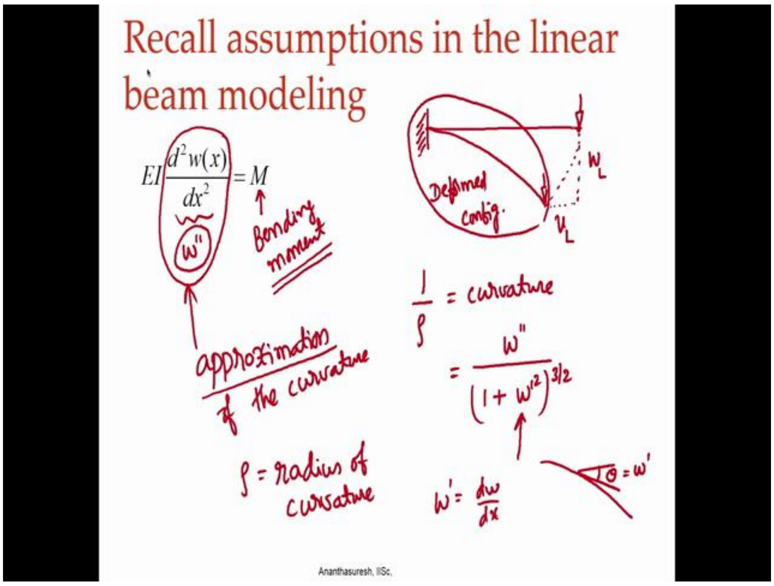

And now today we are going to discuss this non-linear displacement in detail analytically. So, let us recall the assumptions made in the linear beam theory, where we are all familiar that will have young's modulus E moment of area I, second moment of area I, and then there is this w double prime that is second derivative of the transverse displacement w x with respect to x and this is our Bending moment, this what we call Euler Bernoulli theory, where we say that this particular quantity that we have w w prime is actually an approximation of the curvature, this is an approximation of the curvature.

So, curvature which is inverse of a reciprocal of the radius of curvature, where this is curvature, where row is the radius of curvature, which is given by w double prime plus divided by 1 plus w prime square raise to 3 over 2. Now when we see it is in approximation, we are neglecting this w prime. So, for small displacement this will entail only small rotations.

Hence w prime which is d w by d x, when a beam deform like this if I draw a tangent, this will be the slope theta that is w prime d w by d x ,when that is small that is bending is limited then you can neglect this then this approximation by w w prime is valid. That is 1 assumption we had made in linear beam theory, other is more settle is this bending moment, we write bending moment in the original configuration. If I have a beam like this, since we are discussing cantilever beam let us considered that when apply a force anyway, but in our case we are going to apply the load at the free end transverse load, we write bending moment diagram in this configuration rather than in the deform configuration.

So, we saw that in the case of non-linear analysis there will be a transverse displacement we can all it w L as well as u L which is in the axial direction, it is not axial deformation in the sense of stretching of the beam it is just that the locus of this will have both components. Transverse component as well as the Axial component and you have to write the bending moment in this deformed configuration rather than the original configuration, that is the difference and also the assumption that we make linear analysis is that bending moment or equilibrium written in the original configuration, is good enough per as not good enough in the case of non-linear.

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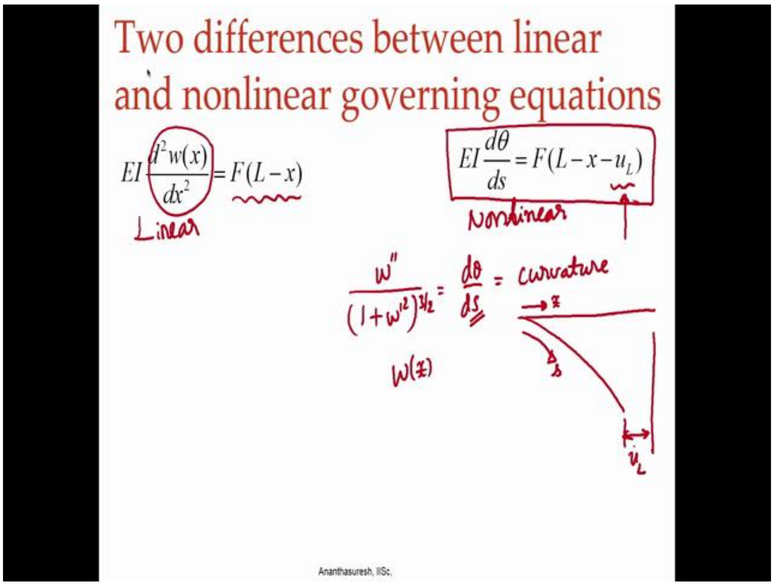

So, those 2 differences which we just discussed are apparent here, you can see this is the linear case that we just discussed and this is the non-linear case that we are going to discuss now and you can see that d theta by d s which is nothing, but the curvature, how the slow of the beam is changing with respect to arc length parameter that is to be noticed s.

So, the beam is like this, we measure as along the bent profile of the beam rather than original 1 may be like this, that will be x where are this is s and d theta by d s is a curvature. So, we not we had in approximation here, now here it is not there. The other is if we look at the bending moment this is in the original configuration, so that simply F times L minus x, but here we have additional which we said is u L the beam what to be like this when its bent, what is this? Displacement u l at the x equal to L or s equal L, that is what we have these are the 2 differences between the linear and non-linear governing equation.

So, this is the equation that we need to solve now and you can see that this equation is in was d theta by d s, which as we saw is non-linear because there is w double prime divided by 1 plus w prime square raise to 3 over 2. So it is non-linear and we have this extra variable u L. So, we have a differential equation to solve for w of x, but also in another equation to solve for this scalar u L. So, we need another scalar which we will see as we go along.

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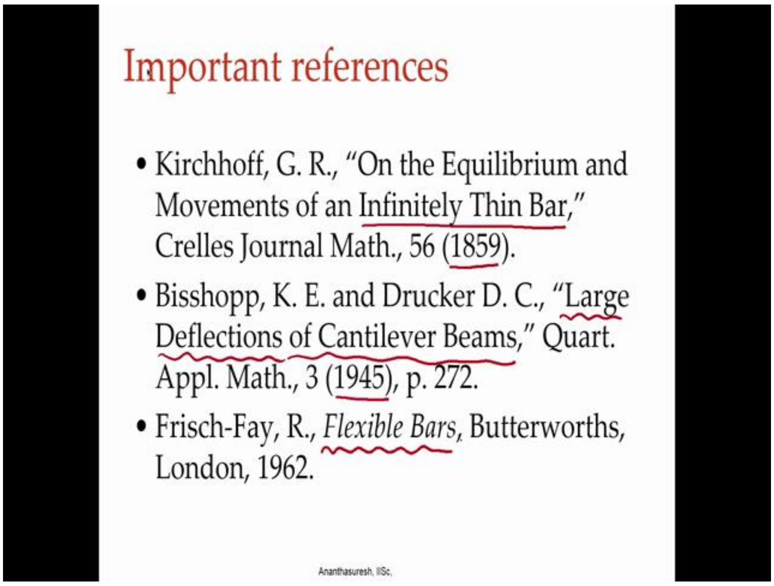

These are the 3 important references for this a content of this lecture, you can look at the dates 1859 Kirchhoff, the same Kirchhoff of circuit theory and who had also work done plates, he had worked on this infinity thin bar people is to call it a Bar now we call it a beam, on the equilibrium and movement.

So, we have done some fundamental work we will refer to what he had done and Bisshopp and Drucker 1945, they had looked at is large deflections or displacements of cantilever beams; much a what we discuss today is from this paper and they had done this analysis in the context of analyzing the fishing rods, which are very flexible and they needed to analyze I guess they did for in the context and there is this beautiful book called "Flexible Bars" written by Frisch-Fay and this book is a comprehensive set of chapters that deal with various beams and under various loading conditions.

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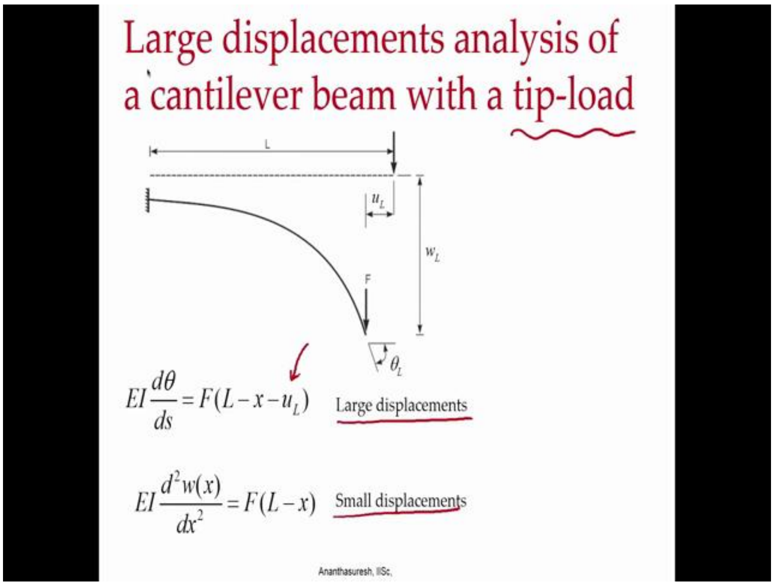

So, this reference will be a good to use to understand more than what will discuss in today's lecture. So, once again these a small displacements that we have and large displacement governing equation with a tip load that is all we are taking and we have this u L appearing in the large displacement case in addition to w of x.

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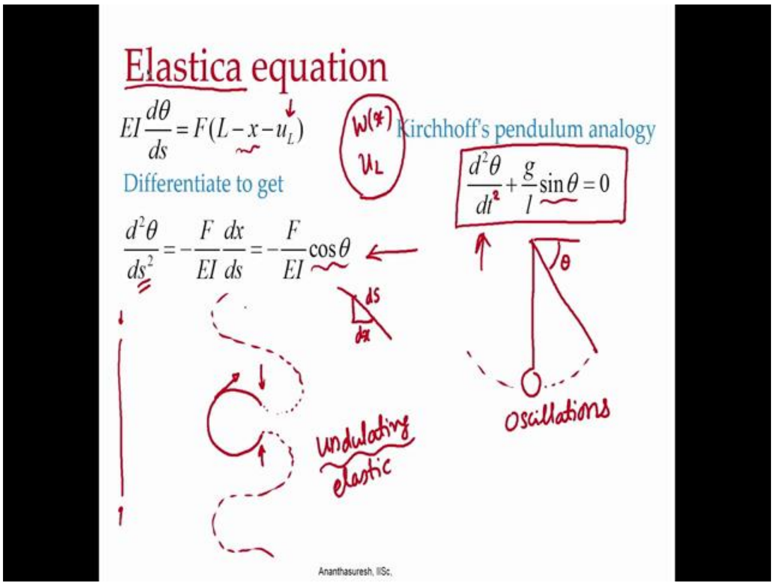

This equation is called "Elastica equation" not elastic Elastica equation; the solution of this will be called an Elastica. So we have EI d theta by d s equal to F times L minus x minus u L, we think about how we might solve? There is a clever technique that Bisshopp and Drucker and others have found. So, I will explain it in simple steps we can appreciate what they had done.

So, we have a differential equation and also we have a scalar unknown. So, our unknowns are w of x as well as u L true unknowns, 1 function 1 scalar. So, we have a differential equation. So, function is taken care of, but in other scalar equation to solve for u L, where differentiating with respects to s the arc length parameter. So, that becomes d theta by d s, I have taken EI the other side.

So, F over EI with minus sign because this is a minus x, this become d x by d s and we also note that this d x by d s is equal to cosine theta. So, when a beam deforms like that if I take this element and small element infinite element that call it d s, then there will be vertical and horizontal component we are here (Refer Time: 11:07) this horizontal component d x that will be d s and then cosine theta.

So, we are replacing in that d x by d s with cosine theta and then now see the analogy between the large displacement oscillations of a pendulum, this is Kirchhoff's pendulum analogy and the equation that we got. So, we have second there is d square theta by d s square here d square theta by d t square the square forgot. So, d t square and then we have cosine theta here sin theta that does not matter because if at draw the pendulum, normally we denote this angle as theta instead I can denote this angle as theta that becomes cosine theta what we have here.

So, you can see that the large oscillations of a pendulum, and the deflection of a beam are governed by the same mathematical equation; does it mean then that there are some oscillations also in this beam, when you take a cantilever apply a load is a static equilibrium equation, where are the oscillation turns out that there are indeed oscillations, if you extended it little further let us say I take a long beam and bent it in to a form like this, imagine that there was a vertical beam will come back to that later we apply 2 loads.

That is I have taken a beam we buckled it basically applying 2 loads. If it buckles both sides you going to become something like this and this 1 if I in beam what to be longer, it will actually go like this. So, this undulating Elastica that what it is called undulating elastic. Elastica is the de font profile of the beam is undulating meaning oscillations, just like the oscillations of the pendulum that under goes a large traverse. So, this normal we study small displacements in which case sin theta is approximate as theta, then we can solve it easily when there is sin theta as it is which cannot be replaced with theta then it is a non-linear equation that will have a feature that we have seen here right now.

But 1 thing to remember even though we are calling undulating in talking and about oscillations, what is oscillating here? It is a tangent, if I take tangent every point along this path of the beam that is very long and a buckled it in to a shape by applying forces, this tangent will be oscillating that is the explanation that we can give, but let us understand that when 2 different systems are governed by identical mathematical equation, it does not mean that always the physics of the 2 is the same.

Here the physics entirely different. This there is inertia for this term here for pendulum, here there is no inertia because a static model, yet the equations are the same, but physics is different. But there are oscillations in this undulating Elastica if extend it will little further.

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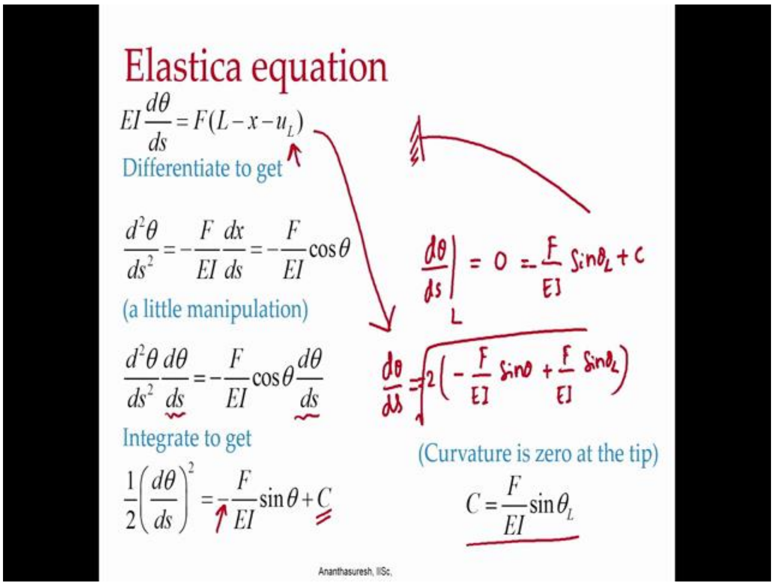

Now, coming came to the equation we just to get the truly talk about Kirchhoff's analogy for the beam that we got now. Now let us proceed by doing a little manipulation, d square theta by d s square that we have, we multiplied by d theta by d s on both sides of the equation and then, we integrate now in order to get half d theta by d s square, if take derivative I get back this 1 and the right hand side also we have integrated.

Now, we have a constant C, that C how does be valued? Will use the fact that at the free end of the beam, when a cantilever beam deforms like this and the free end moment is 0 and hence curvature is 0. Curvature is 0 means that d theta by d s at s equal to L these equal to 0.

So that means that, F by EI sin theta L plus c is equal to 0 that gives us that C to be equal to there was a minus sin here, it is that there is a minus sign that came from the differentiation. So C equal to F by EI sin theta L, so we got an equation now which is are form I can write d theta by d s equal to, minus F by EI sin theta and then we also have F by EI sin theta L and then 2 goes here this 1 and then 2 and you are take a square root .

Because this d theta by d s square, you got an equation. So, we got an equation that transform from here to here now, in the process who has managed to eliminate that u L that was there. But we need the equation to solve for it as we will see a later.

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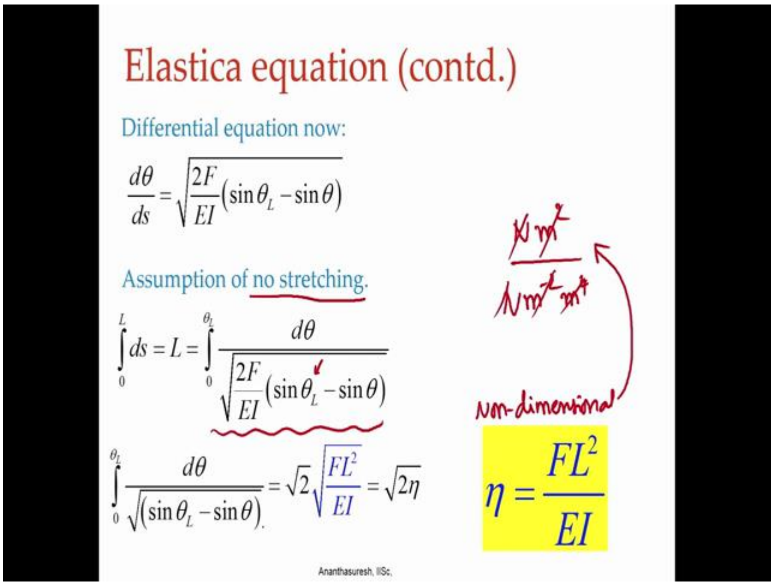

So, let us proceed with this equation that we have, d theta by d s is to F by EI. I just took it common sin theta L minus sin theta. Now we bring in that, we eliminated that u L what the still an equation to solve for that extra scalar that we have u L. So, we use this assumption of no stretching of the neutral axis of the beam, which is to say that if I integrate from 0 to L along the arc length parameter d s, I will get the full length L, there is no stretching there is no contraction. Since we have this d theta by d s, we can take d s other side bring this down stairs. So, what we have is this equation that is equal to L. So, if we notice what is done now, this is a scalar equation because this just a 1 equation what we have here is a scalar equation. So, the unknown in this should be also a scalar which is this theta L because once you integrate from 0 to theta L this quantity that is going have only theta l it s a function of theta L. So, we have a function of theta L is equal to L this equation that we got.

So, even though we had differential equation by using this couple of tricks, we manage to get any equation in 1 variable that is a highlight of this method. How do we solve this? Well before that I will just put in to slightly different form, let me erase a few things that I have written. So, you can see the equation better.

So, what we have here is, the same thing written slightly differently by defining a non dimensional quantity denoted by eta. It is a non dimensional quantity, which is a very important parameter in this analysis, is a non dimensional because force has Newton’s and then this is meter square and then e is Newton per meter square second meter area is to the 4.

So, it is non dimensional we can see it. So, Newton this is m 2, this m 2 no dimensions. So, what you have here on the left hand side, is actually a non dimensional because it only involves the angle theta.

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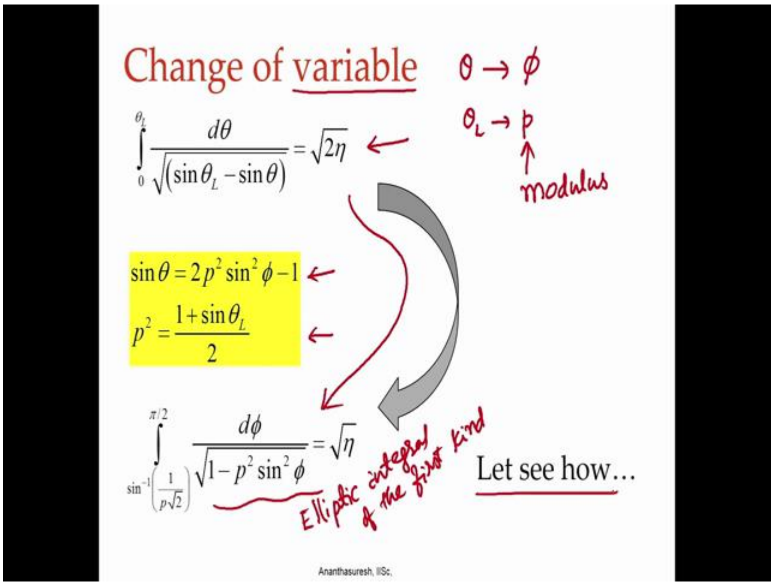

Now, how do we solve this? This equation that we have at this time, how do we solve it? You have to solve we use change of variable, we are changing theta to another variable called phi and these are the equation that we are using. So, sin theta is 2 p square sin square phi minus and then p square 1 plus sin theta l by 2.

So, there are 2 change of variable I should say, theta 2 phi and then theta L to something called phi, which is called the modulus of an elliptic integral that we see, actually we have it right here. So, this is an elliptic integral of the First kind. So, we are using 2 changer with theta 2 phi and theta in 2 p, which is what are indicated over here and let see how this thing comes from here, how we get here we will try to figure that out.

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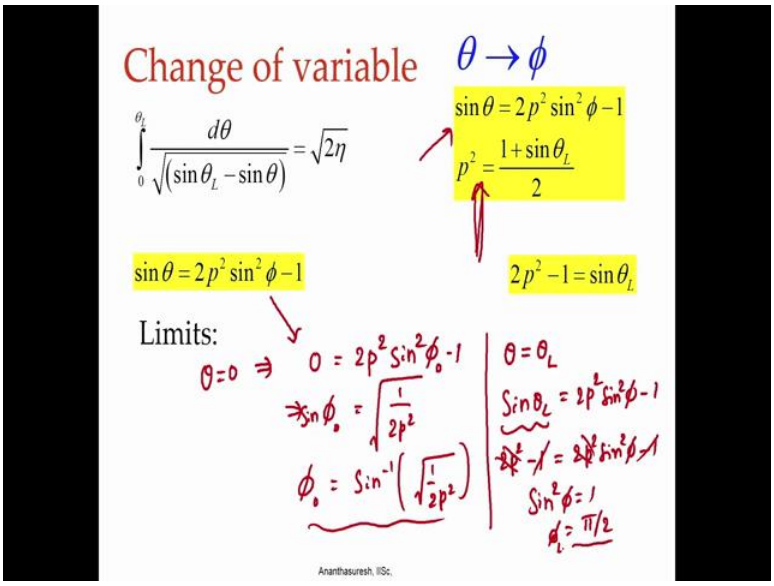

So, First let us look at the limits, when we change theta 2 phi when I take the case of theta equal to 0, is implies that that is sin theta which is 0 from here equal 2 p square, sin square, phi naught let us call it minus 1, that means that phi naught the lower limit is going to be 1 over 2 p square under square root and then it is actually sin inverse of that. So, let us we put sin phi naught here and then say phi naught is sin inverse of square root of 1 over 2 and then p square, I can take the p outside does not matter that is phi naught.

And what happens when there is theta equal to theta L which is the slope at the loaded tip, when you have that then we can go back here and then see what it is. So, we can see that we have I think it will be yes 2 1 of this. So, we can see that when theta equal to theta L, will be go back to this First equation to itself, sin theta L that is this equation, equal to 2 p square sin square phi minus 1 and then we know sin theta L from the second equation is 2 p square minus 1 and this side we have 2 p square sin square phi minus 1. So, now, minus 1 minus 1 gets cancelled, 2 p square gets cancelled after that there we get sin square phi equal to one or phi equal to pi by 2, it can be 3 pi by 2 and so forth where you do the First 1 is undulating Elastica we taking the First instance of that. So, we have our limits from phi naught 2 pi is our phi l.

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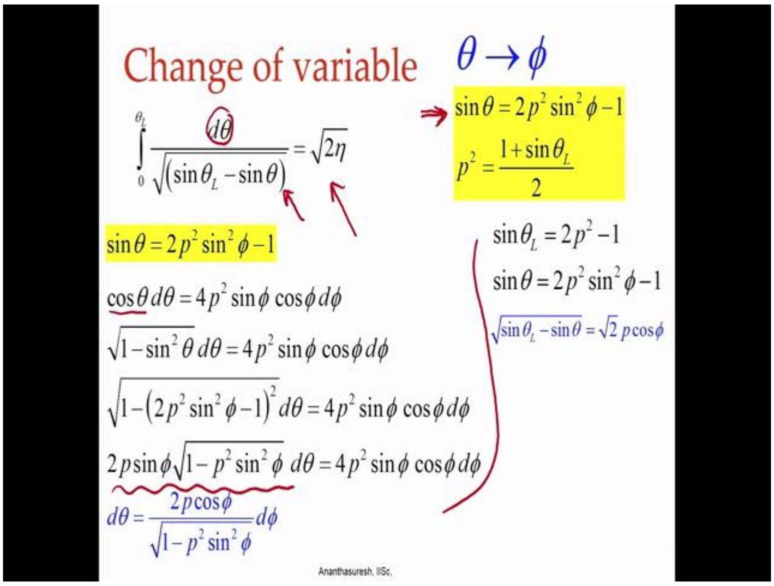

So, how in done that? Now let us look at further now we of converting this to the new form there are 2 steps here, 1 is we have the First equation which I am showing here sin theta is 2 p square, sin square phi minus 1, let us differentiated in order to replace d theta to d phi. So, we have first taken this equation differentiated with respect to theta, cosine theta times d theta 4 p square sin phi, cosine phi, d phi and cosine theta is written as square root of 1 minus sin square theta and then right hand side remains a same.

And then sin square theta we substitute this equation for sin theta. So, 2 p square sin square phi minus 1 square root and if you simplify you will get this to be that 1, 2 p sin phi time square root of 1 minus p square, sine square phi, d theta right hand side remains the same you can see that sin phi sin phi is there and 2 p that gets cancelled, then we get d theta when you replace we will have to put d phi and then this quantity 2 p cosine phi divided by square root of 1 minus p square sin square phi.

And let us also look at let us separate it out, sin theta L because we also known to replace this in terms of phi. So, sin theta L is to p square minus 1, sin theta is to p square sin square phi minus 1, if I take the difference of these 2 then I get the sin theta L minus sin theta becomes 1 1 gets cancel then will get this 2 phi square, times 1 minus sin square phi and then1 minus sin square phi square root was there. So, it is becomes square root of 2 p cosine phi.

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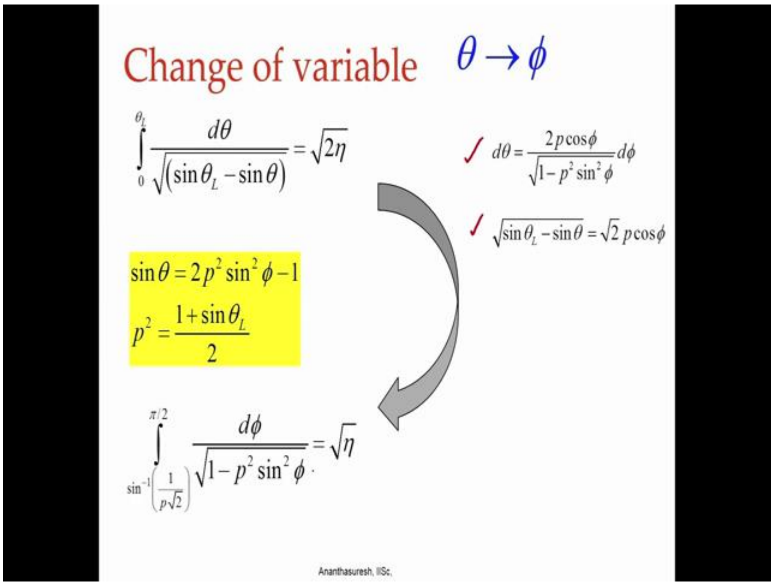

Now, both of these results we substitute over here and the limit also we calculated let us see what we get. So, when you do all this with these 2 things that we did in the last slide when substitute over here, we get on the limits that we did sin inverse 1 over p square root of 2 to pi by 2, and then d phi divided by 1 minus p square sin square phi, is equal to eta, we have taken this 2 also or the other side, when this square root of that square root of 2 will cancel cosine if that is there here this goes to denominate that goes to numerator they all cancelled finally, we get this one.

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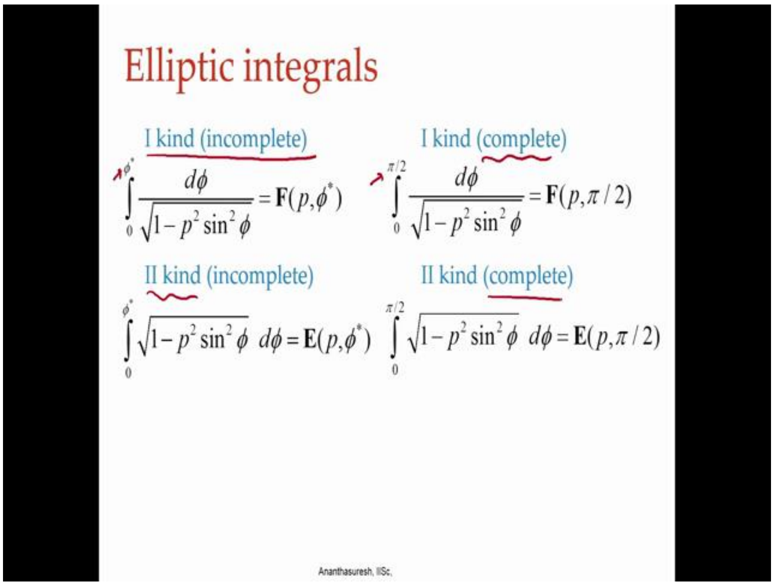

Now, have we solved the problem; In fact, we have and that is because what we got here is what is known as the Elliptic integral of the First kind incomplete, when it is something less then pi by 2, the upper limit when it is pi by 2 that is call the complete. These elliptic integrals are very important in mathematics for many reasons in a First are rules in terms of computing the arc length of an ellipse, circle we know how to do its simply arc times theta, but if are in ellipse is involve this equations if normal trigonometric functions are called circle of functions, the corresponding functions for an ellipse or elliptic functions related to that are these elliptic integrals.

So, we have these equations are the First kind, when what is the denominator goes to the numerator that becomes an elliptic end of the second kind, again incomplete and complete. We have a third kind which you do not need in this particular a context.

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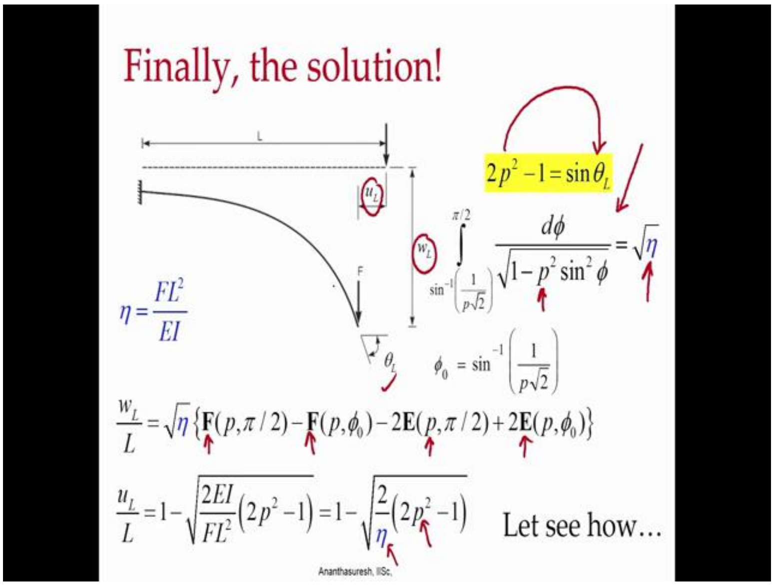

So, we actually have a solution now because we converted the differential equation that we had and that scalar equation u L with a no stretching assumption, we convert in to in equation that looks like this, we had this eta which involves force, length, young's modulus and second movement of area all of these things are captured in this eta here, if I know eta they can compute here the value of p, that is call the modulus of the elliptic integral and the p as we know is related to theta L.

So, I can get this theta L for given value of force, how is it useful? In fact, we have found theta L1 we have theta L, we can get the other ones as well, what are other ones? We want to find out what this w L is? And what this u L is? So, for both of these we have now the F denotes of elliptic it has a First kind complete, this is incomplete, this is second elliptic integral complete and then incomplete because is phi not is there and u L involves only this p and this eta.

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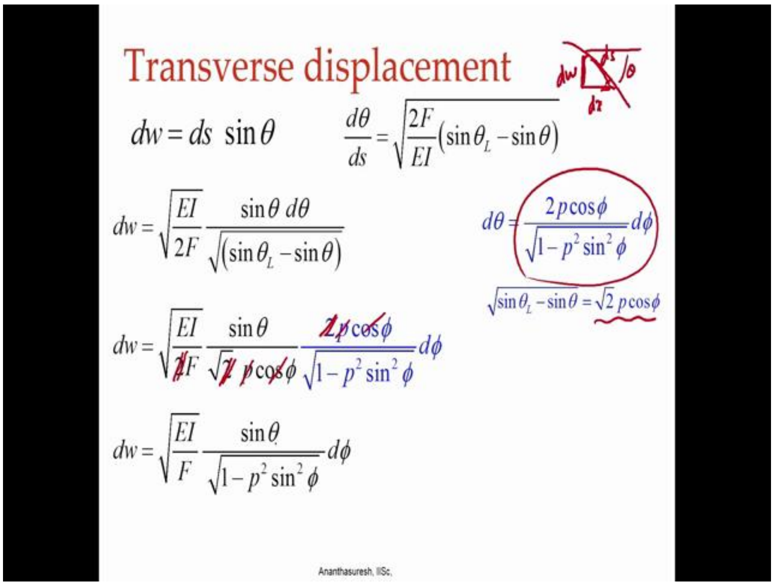

So, variable to get them under free tip, we can also get everywhere in between as we will see now. Let us see how this works? For transverse displacement as we already saw that if we have an arc, if this is d s then this vertical part is the transverse displacement d w, this is d w this is d x for that segment.

So, d w is since this is theta this is sin theta, d s sin theta we take the d w and we substitute for this d s from this equation. So, d s that we have, this will be d s goes that side d theta divided by this and then your sin theta which has appeared here and we had just done how to change variables from d theta to d phi. So, d theta will substitute this whole thing in terms of d phi change of variables and likewise we have this other equation square root of sin theta L minus sin theta, which is square root of 2 times p cosine phi substitute we get something like this.

And we cancel a few thing p cosine phi p cosine phi, square root of 2 here square root of 2 here with that 2. So, we cancel those we left with this thing and we also note now that the this sin theta that we have over here, that can be written also in terms of what we already know these what we got in the last slide.

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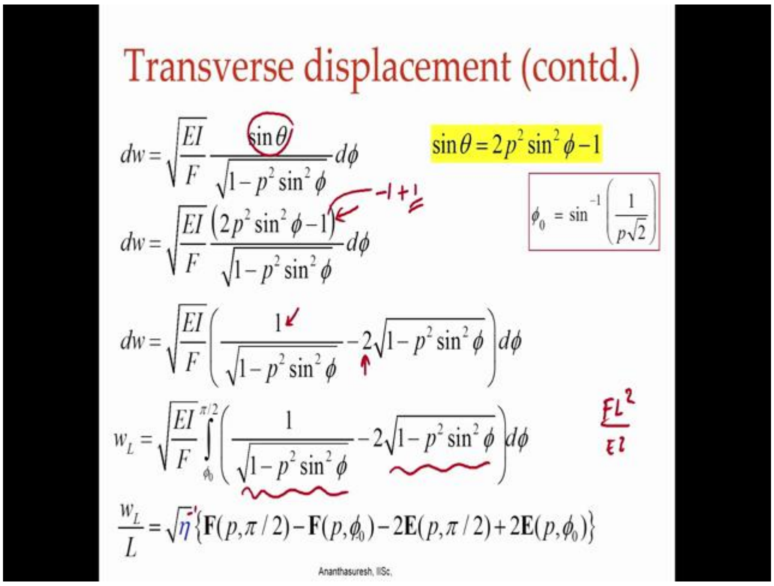

The sin theta is 2 p squares sins square phi minus 1. So, I substitute for sin theta this quantity over there then it wills manipulation.

We add plus or minus 1 to this we add a minus 1and a plus 1. Why do we need to that? If I add minus 1 this become 2 p square sin square phi minus 2, I can take 2 outside that becomes square root of 1 minus p squares sin square phi because in the denominator there is a same quantity, 1 minus p square sin square phi and the plus 1 gives raise to this one just to 1 by this one.

So, this is minus 2 times 2 p square sin square phi minus 2 or 2 times p square sin square phi minus 1 and 2 is here with a minus sin, since there is square root of the same quantity downs stairs that becomes square root of 1 minus p square sin square phi. Now if you notice what we have really these elliptic under the First kind here and then second kind here. So, that is why with the limits put in from phi 0 to pi by 2, we get w L by L, I made it non dimensional here by dividing by L throughout. So, that is why here I get eta because eta is F L square by EI. In fact, it should be 1 over eta so minus 1. EI by F that we have here, so that is what we get and then we have this p and then phi naught which all depend on the theta L the n point displacement.

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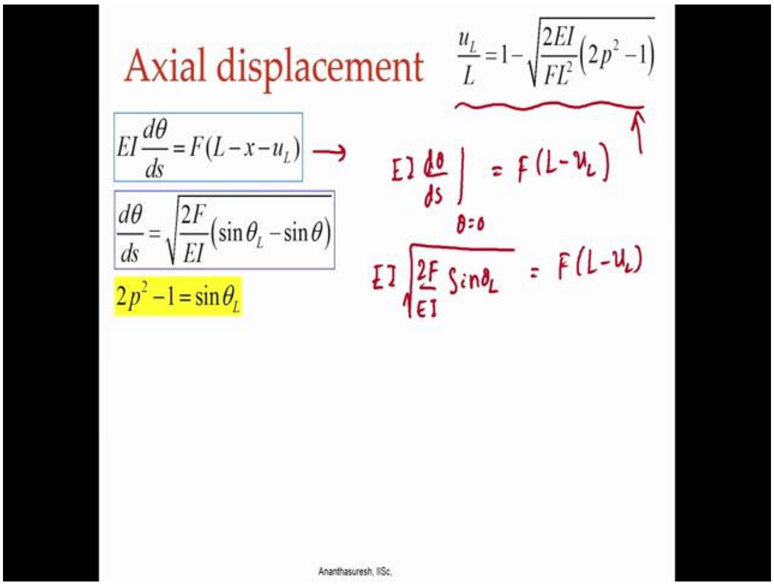

So, variable to do that w L there in order to axial displacement that is u L we had written that expression earlier, that basically comes from this equation that we can do at theta equal to 0. So, if I do that EI d theta by d s at theta equal to 0, I will have F times L x is 0, we need to find u L and then we see what is d theta by d s theta equal to 0 which we have sin theta is 0 then I can put that as EI into square root of 2F by EI into sin theta L because sin theta is 0, that is equal to F times L minus u L.

And then we further know that theta L is in terms of this 2 p square minus 1, 2 p square minus sin theta L we substitute we get what is written there.

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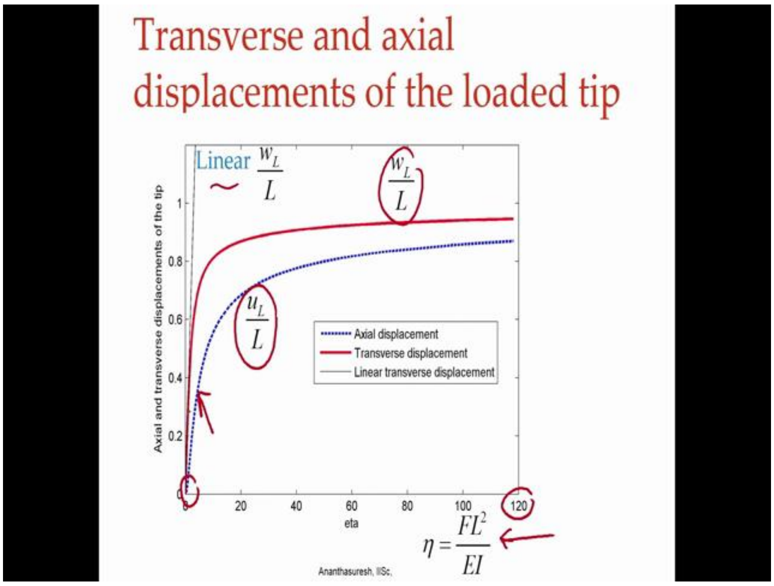

So, we get u L as well here. So, we can plot them. So, we have just non dimensional quantity eta here and on the y axis we are showing w L by L also u L by L, the transverse and axial displacement of the free end or loaded tip and we are also showing the linear one here that is a line that will be this like a force displacement rather we have inter change the axis displacement here and force on the x axis, we have a straight line that is a linear one and this eta has gone as far as 120.

And you can see how w L by L, u L by L vary and you may wonder whether the slope of this curve that doted blue line how come it has a finite slope. In fact, it does not it has a 0 slope at the horizon here because linear analysis will not show you any u L.

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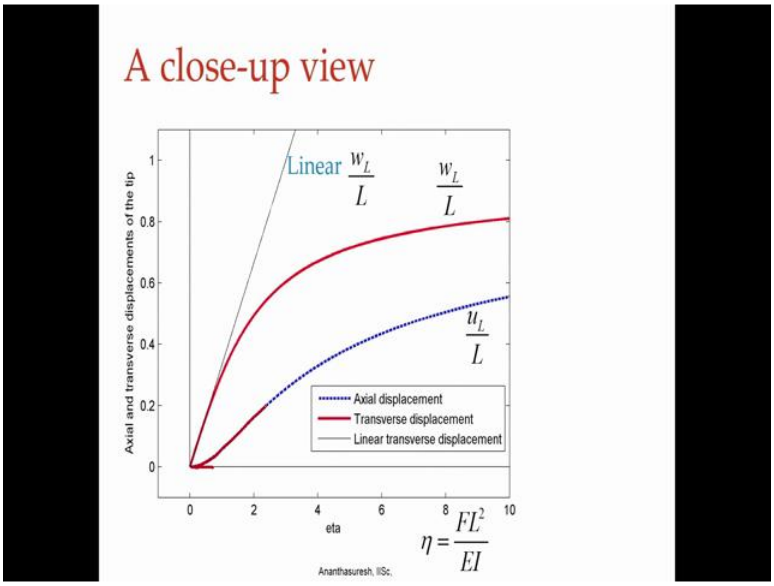

So, it should have 0 slopes coming there. In fact, if you take a close up view of f that, we can see that that actually has a 0 slope and then it terms like that that is taken care of alright.

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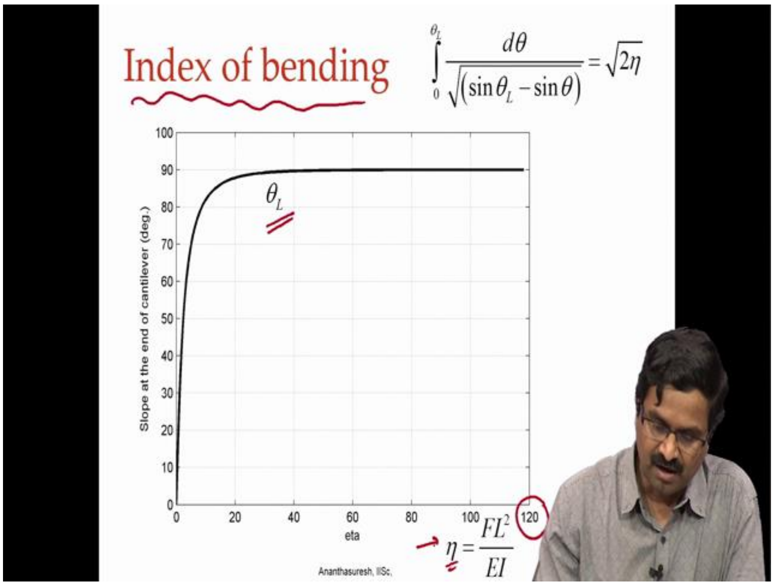

Now, let us also look at how theta L varies with this eta and as you can see as eta increases theta L is increasing approaches 90 degrees in the limit. So, if we take a cantilever beam and apply a large load, eventually it can go to almost 90 degrees that was you shown, that for what need you to go for very high value of f, FL square by EI together let us call this theta L, then index of bending the larger theta L is a larger the value of eta is, the larger theta L is it indicates how much the beam has bend.

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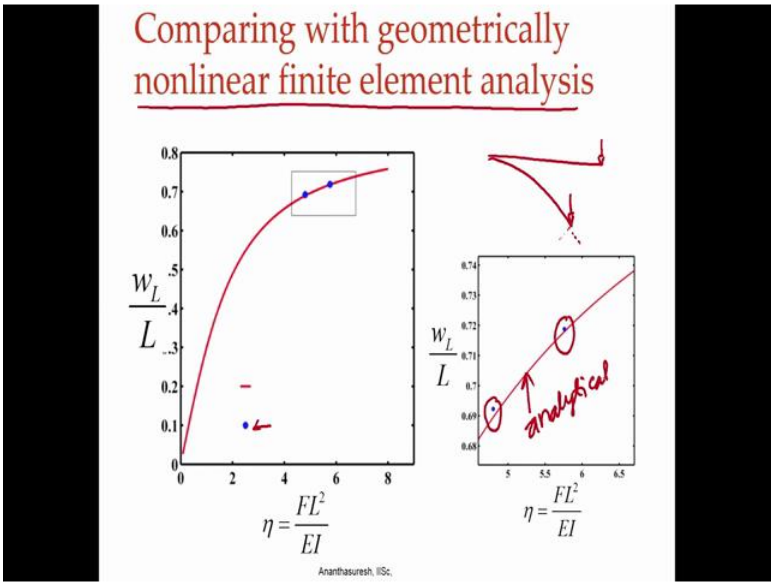

Let us also compare this finite element analysis because we discussed non-linear finite element analysis at length last week, let us see how close it is it look the 2 dots, the blue dots we have for 2 things this simply lying right on the curve.

But if actually take a closer look at that, and then actually there not there is a discrepancy. So, which is more correct clearly non-linear F E A is more correct than analytical one because analytical one may had made an assumption that the neutral axis does not stretch, the beam does not stretch, but in reality it would because if I take a beam and when it deforms where my loading is vertical like that, there is a component along the beam not only in the beginning it is not there that is we linear does not show, but it deflects load continues to be the same direction there is a component their component here, 1 bends other stretchers a little bit of course, axial stiffness will be very high compare to bending stiffness the beat of stretch which is not taken in to account the analytical one and hence analytical one which is our curve deviates a little from the finite element analysis.

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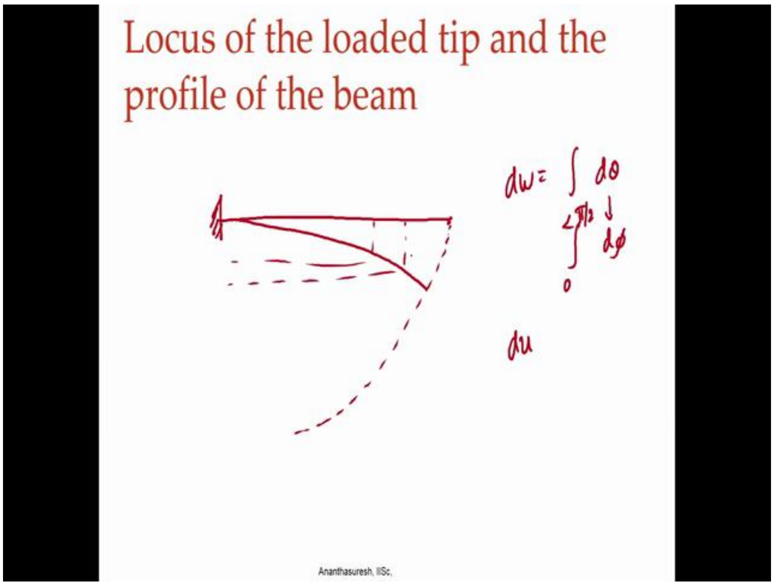

Now, if I want to plot the loaded tip, let us I have a beam like this, we calculate w and u L, for varying values of forces I can get this locus and we will talk about this later on and if I want to get this deform profile, this method does allow. If you remember we had this d w and then integral in terms of d theta which you convert later to the d phi, we did that.

So, we did this from 0 to L completely, but if you do partially then you will get that displacement if you change 0 to instead of pi by 2 something less than pi by 2, if you do then you will get this transverse displacement everywhere, similarly you can do that for d u also. So, you will know this value at different points. So, you can get entire beam different things that I leave it as an exercise for you to do.

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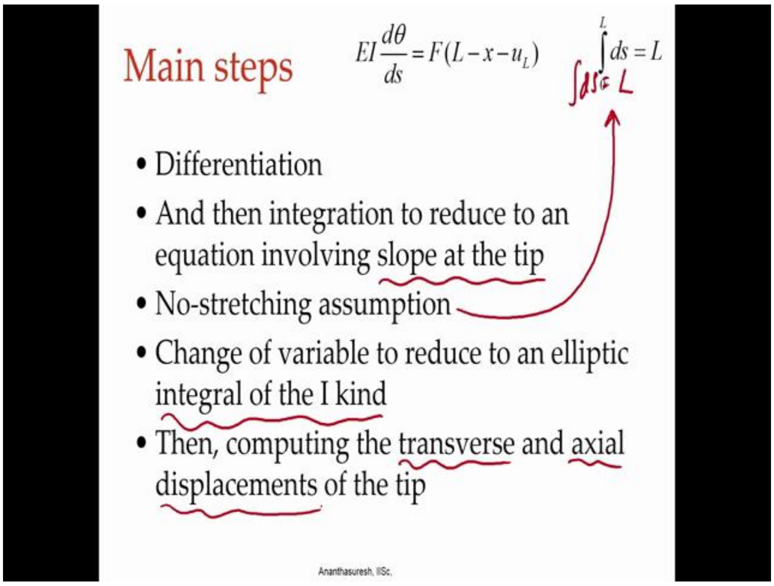

So, let us now capture the important steps in what we discuss today we started with this equation and also we had that d s equal to L, so that the important equations, we First differentiate it and then did an integration after little manipulation, to reduce to 1 equation involving only slope at the loaded tip.

And then we did this no stretching assumption, which is that to solve for the theta L and then we reduce to it to elliptic integral first kind and that we assume that this elliptic integrals cannot be solved in terms of eliminatory functions are analytically or a numerically you can do that there are lot of routines in math lab and other software to get this. So, we are reduce that tip numerical solution in 1 variable involving elliptic integrals and finally, we also saw how to get transverse and axial displacement of the free tip and also we can get for any point of the profile of the beam.

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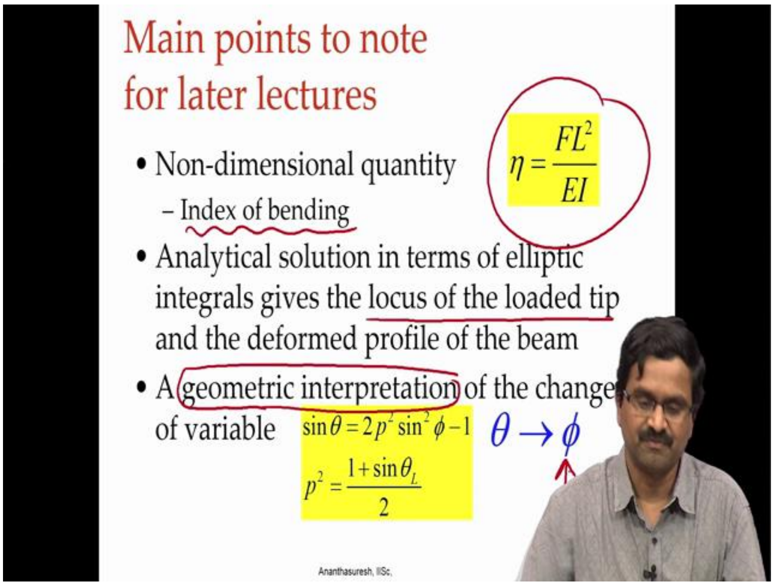

So, you can get the entire deformed profile and some more thing that we need to note for later lectures is that this non dimensional quantity eta has a lot of significance, we call it Index of bending and then the analytical solution that we got let us has draw this locus of loaded tip, which has very important implication in what we will discuss in later part of this week's lecture and one more thing that we will also look at is the geometric interpretation of the change of variable, it is not just a mathematical convenience this phi plays a very important role in solving other beam problems, cantilever beam with multiple loads are vertical beam like a column buckling and also beams under different boundary condition fixed and so forth,

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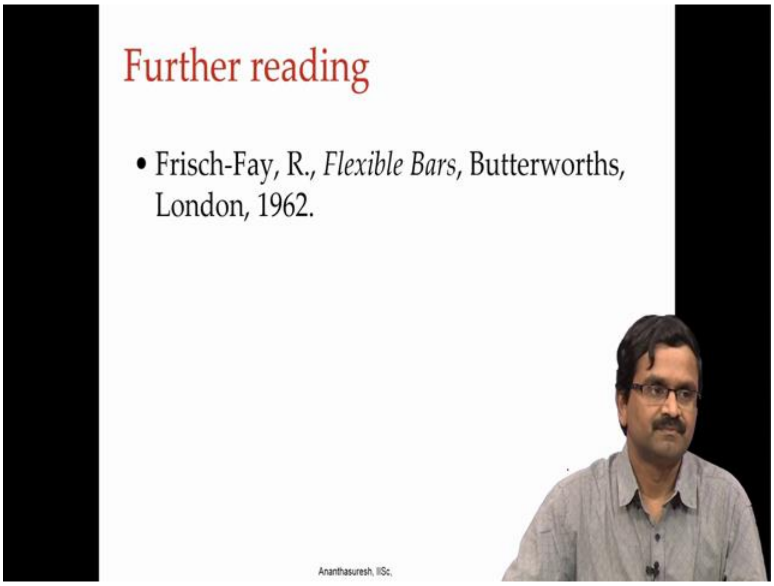

This will revisit by describing this geometric interpretation of the change of variable.

Thank you.