Complaint Mechanisms: Principles and Design
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Lecture – 11
Linear and non-linear finite element analysis using continuum elements

Hello, in the last few lectures several times we alluded to whether the analysis that is finite element analysis that we do is linear or non-linear for the complaint mechanism that we were looking at, today we will try to clarify what is linear finite element analysis.

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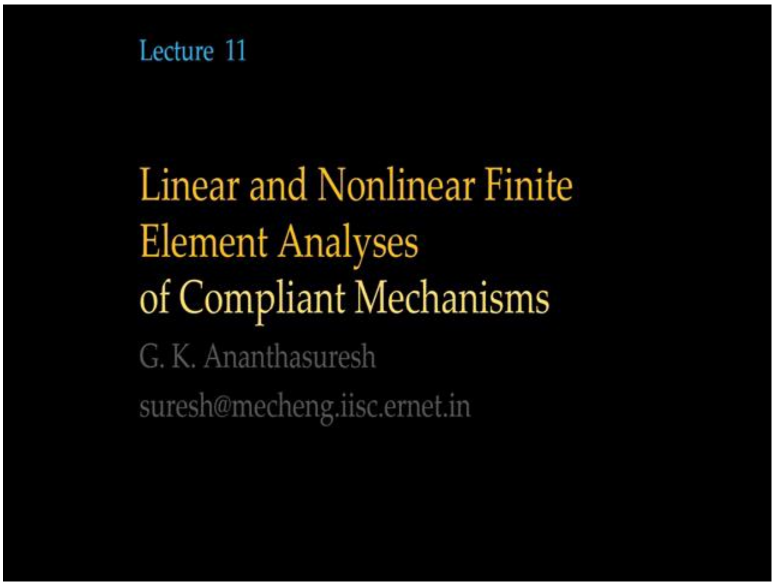

And what is non-linear finite element analysis in the context of complaint mechanisms. So, we all understand that complaint mechanisms unless they use elastic pairs they would undergo very large displacements deformation sometimes even large strains. So, we have to distinguish between large displacements, large rotations, large strains and whatever else that is going to lead to non-linear analysis.

So, let us look at some are those concepts. So, that we are clear that a non-linear FEA is different from linear FEA and how it is different; obviously, non-linear FEA is going to be closer to reality than linear FEA, when the mechanism or a consideration undergoes this non-linear behavior.

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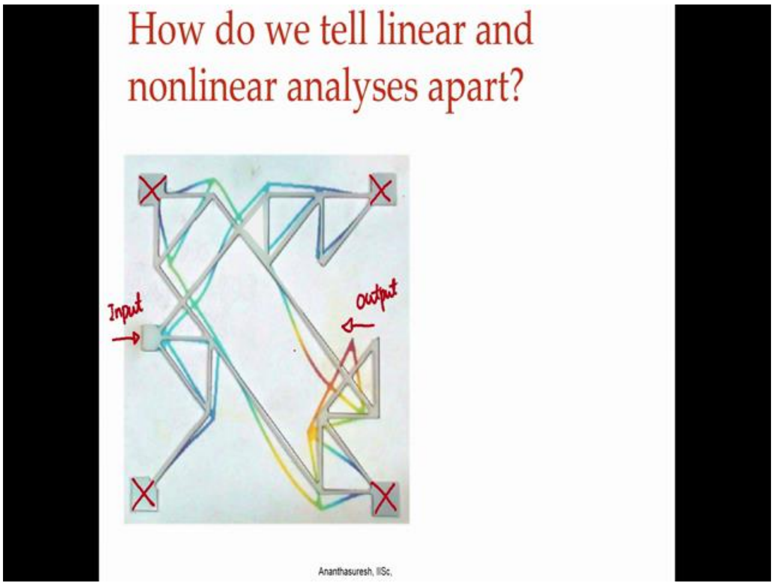

So, let us look at one mechanism on the slide that to we had considered in the one of the last lectures. So, here is one of the mechanisms that we had considering in one of the last lectures, where we are showing a complaint mechanism that is undergoing substantial non-linear displacement and deformation for convenience we did FEA got the result printed on a piece of paper and stuck the mechanism over it, this mechanism as you might call it is fixed at these four corners, and there is a force applied here this is our input and that made this point go this way that is our output.

So, let us say that somebody does not tell us there is a show us this FEA result and ask us to guess whether it is linear FEA or non-linear FEA, how do we tell these to apart in this particular case since we have pasted in a piece of paper and we showed that when the mechanism that is pasted on to it is deforming it exactly co insides with what is printed from finite element analysis here, we do know that it is non-linear FEA because non-linear FEA is the one that is going to be close to the realistic deformation of the mechanism, but if somebody does not tell us that just shows two results, one done with linear FEA other done with non-linear FEA can we tell them apart is our objective of this lecture. So, let us understand, what is linear analysis? What is non-linear analysis and then go from there?

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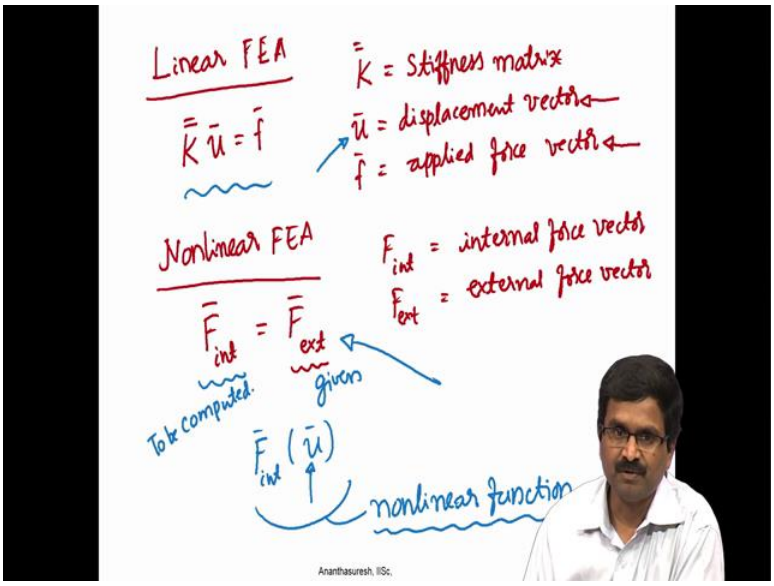

So, let us just first begin with what we consider as linear F analysis, linear finite element analysis.

So, here we would have after discretization done k u equal to f where we call k to be this stiffness matrix is a matrix. So, we put two bars over the letter and we write and u is the displacement vector or array say column vector and f is the applied force vector since, we are representing the discretized system as a linear system of equations that is when we have matrix equation because matrix are essentially linear. So, we recognize that it is a linear analysis right, how to get this stiffness matrix is what you learn in a finite element course or when you read a finite element analysis book.

So, displacement vector will contain all the degrees of freedom that are unknowns in that analysis and correspondent to each degree of freedom there will be a force vector, and that is how we get k u equal to F that is linear FEA, when you consider non-linear finite element analysis we will not have this form of k u equal to f, instead what we will have is internal force vector again discretize, because its finite element is equal to external force vector. So, again lets write F internal is, this is the internal force vector and F e x t is external force vector, here we are not going the entire theory of finite element analysis linear or non-linear which is giving you a quick feel for what is linear or non-linear. So, when we say something is linear we say matrix equation here, I have F internal conduct F external F external is something that most often is given to us.

So, this will be given or specified in analysis this we have to compute. So, this should be to be computed when you do non-linear of analysis, when you say we compute is not like a numerical value because, it is a numerical method it will be numerical value, but what we will see is that what we want to find or the displacement eventually is F internal will be a function of the displacement vector. So, just as we have displacement vector here we will have the displacement vector here, except that it will be closer to reality in terms of displacement.

So, F internal that is internal force vector we have to compute using this u that is equals to F external and this function if you called this a function internal force vector, function which depends a displacements this will be non-linear function of the displacements u that we want to find, hence if you want to solve a non-linear problem you are do iteratively if you want to solve the linear problem k equal to f we can directly solve it by just solving the such a equation 1 where to do is just inverse the matrix u is equal to k inverse F will give you the displacement vector u, where as if you want to solve this problem when external force vector is given to you because, internal force vector is a non-linear function of the displacements we are do it iteratively sometimes we also do incrementally that is external force you do not apply all at once you apply in small increments and each increment you would do iteration and do that that is the essential difference between linear and non-linear FEA, but what brings this nonlinearity into the problem is what we should understand.

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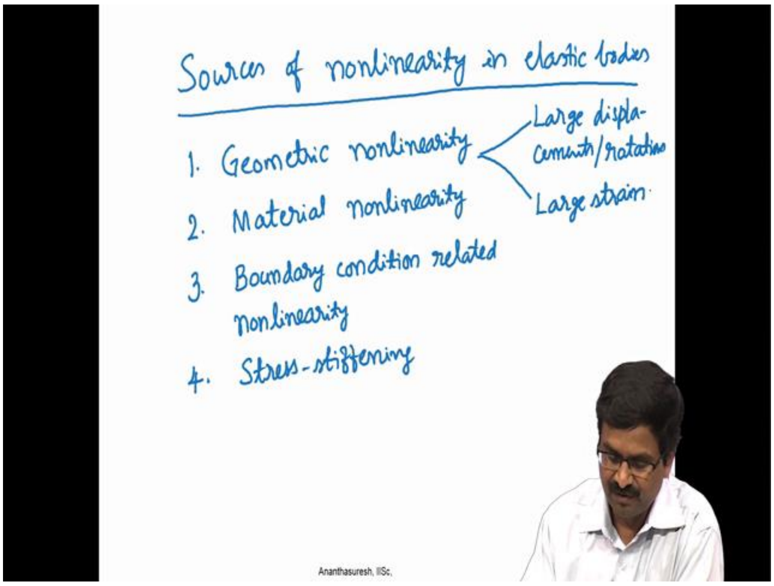

In elastic structures nonlinearities can be due to several sources let me list them as 5. Sources of nonlinearity in elastic bodies which are complaint mechanisms, the first 1 will call it geometric nonlinearity. So, this may becomes because of geometry, second 1 is material nonlinearity, third 1 will be what we can call boundary condition nonlinearity, boundary condition related nonlinearity and a fourth 1 which is more specific to beam elements which is called stress stiffening. I put all these in the contest of complaint mechanisms, but not in general because we concerned with complaints mechanism here, I have listing all the sources with reference to complaint mechanisms, although it could be generalized as well.

So, I said 5, but I have written only four that is because the first 1 has 2 parts, one is called large displacements slash rotations and the other is called large strains. Both are geometry related the strain is essentially a geometric quantity that we define lets understand each of these step by step. So, geometric nonlinearity has 2 parts large displacements or rotations or large strain and then will go to material boundary.

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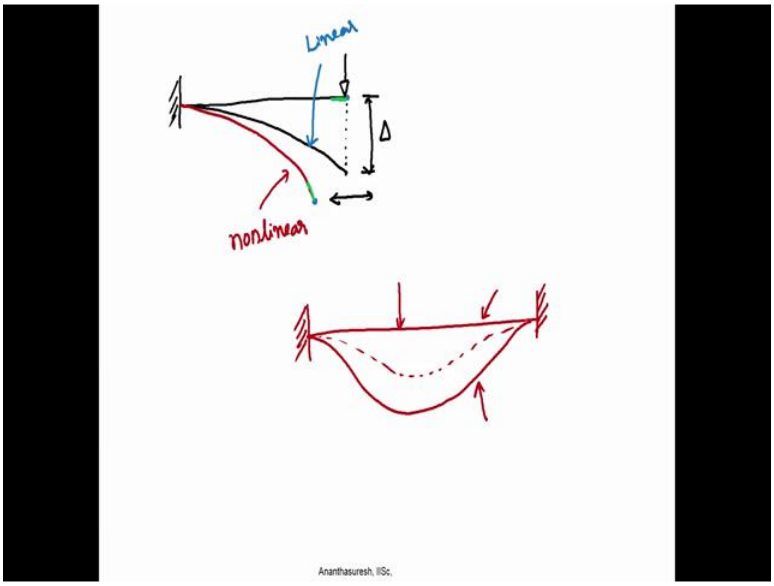

And then stress stiffening, when I say there are large displacements one easy way to understand this is by considering beams itself, because most of the complaint mechanisms that we are going to deal with half beams in them. So, if I take let us say cantilever beam and apply force in the transverse direction, when you consider linear analysis or what we study or what we teach in mechanism material course in undergraduate engineering, we never talk about this point actually moving in the axial direction we only talk about when it deforms like this, how much is this deflection is what we worry about you do not really think about whether this point will move in the axial direction this way.

In reality if you imagine how a beam would deform, it will not deform like this it would deform if we apply large force something like this because, its length whatever that is there will be same here and not more see for example, if I have this one if you consider linear axis that I have drawn first definitely is longer than that beam it will stretch a little bit, but not as much as this will show it will bend more like this.

So, this will be non-linear and first 1 that I have drawn will be linear. So, this is what we learned in the context of beams linear analysis. So, large displacement we can see because this point that was there has moved over here it is a large displacement, but the other thing we said is large rotation that is if I want to take an element here, let me use a different color for this let say I take this element that has gone here not only it has displaced, but also has rotated there is a large rotation this rotation could be a rigid body rotation or it could be actually the element bending at that point both can happen.

So, there is from when something goes from here to there could be rigid body rotation or there could be actual bending at that point, that we have to take note and subtract what we called the rigid body rotation. If you want to put a block there, a block would have simply rotated here that would not necessarily deform in the sense of bending the whole beam is bending that is mostly due to this large rotation that is happening. So, relative to each one of the, if I take another element next to it that also will undergo translation and rotation these are the large once, if you consider why it is important for us to look at this large rotation and large displacement we have to look at how a beam actually would deforms.

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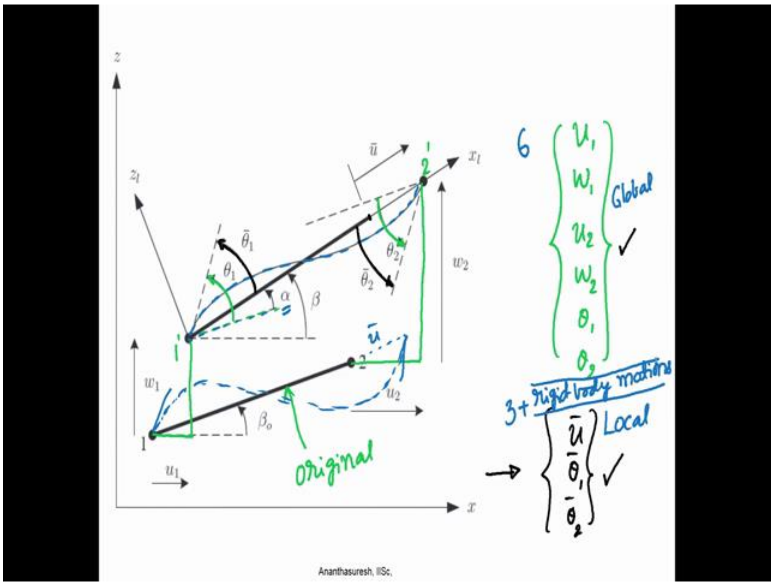

So, if let us look at this particular one here the original beam is here. So, this is the original beam segment let say we have an elastic segment in a complaint mechanism we have isolated one of them there are 2 nodes 1 and 2 this is the original configuration, after the complaint mechanism deforms let us say this original 1 2 has gone to let us say 1 prime and 2 prime, that is node 1 has gone to this node 2 has gone to that point.

Now, if we note is how much deformation it has undergone, we can described this with two different set of variables the first 1 is to say that this particular original element has moved by this u 1 and w 1 translation. So, this is u 1 and this is w 1 so, I can write u 1 w 1. So, there is a displacement in the x direction displacement in the z direction as it is chosen here, and there is also that u 2 that is indicated here, then w 2 that is u 2 and w 2 there is also the angle. So, if I take the original one and show this dash line parallel to the original element 1 2 with respect to that when the beam actually deforms, which is shown by this curved line that is how beam would deform a complaint mechanism we have seen several of those already.

So, we said nodes 1 and 2 have moved to 1 prime and 2 prime the beam itself has deformed like this blue dash line ok. Now, in this one if we see there is theta 1 and theta 2 these are the rotations at that point, if I were to draw a tangent to this curved beam here that will be give me theta 1 and theta 2 relative to the original element 1 2, that is why we draw the parallel line over here measuring theta 1 there and then theta 2 there. So, these are the 6 variables that we can define for this beam another set that we can use, which is more local to the beam element which is to say that there is a stretch u bar that is indicated here.

So, it was where the original length you can see how much it has gone there that has stretch to 2 prime that is u bar, and that is all we want to know how much the beam element has stretched another thing we want to know is relative to that there is a theta 1 bar. So, if we see this which I am thickening in black line itself there is theta 1 bar and then similarly, there is theta 2 bar over here entire tangent related to this as oppose to the original one that is drawn the dash line, we have theta 2 bar.

So, we can describe this beam deformation with respect to the original coordinate system using 6 quantities 3 at each node or only 3 for the entire beam element. So, what this second 1 does not have is rigid body information in the sense that this curved blue line I am at as well put it over here also. So, that curved blue line that I have there I might as well put it here, that is if I know this u bar how much it has stretched, how much it has deflected there, how much it has rotated over there node 1 and node 2, which is shown with these tangents there. So, this basically gives you kind of local information about what is have with beam element this gives you more about how it has displaced and rotated there has to global coordinate system, which is its exact system here so, if you look at these another quantity called alpha.

So, original beam that is there I can get this blue 1 that describes how this beam element is actually deforming, but I need to place it in the global coordinate system that is beam elements, because everything is connected to one another when this deform an element that is one place would not only stretch and bend it will also undergo 3 rigid body motions one is translation in the x direction translation of y direction, but also a rigid body rotation. So, here we have only 3 things you will have 6 where does the 3 go they are the rigid body motions. So, what differentiates linear or non-linear geometry or nonlinearity are these rigid body motions that is what is here has moved this 1 has moved to 1 prime and then it has also rotated, because this line here is parallel to the original element. So, 1 after the deformation is done that is you affected u bar theta 1 bar theta 2 bar to get this blue curve there is u bar, here and theta 1 bar theta 2 bar and after that you take 1 and translate it by this u 1 and w 1 to take it over there and then you also have to rotate by this alpha that is when this original 1 2 goes to 1 prime and 2 prime, it has translated in x and y direction or also rotated those 3 rigid body motions are the once that characterize this large displacement and large rotations.

These are important because, if you do not take out this rigid body motions you could think that there is a lot more deformation than there is otherwise, if you were to simply define strain or strain using the displacement vectors when element is here to there if you just say this has gone from here to there. So, I will take derivative that is a strain that would not be enough. So, we have to take out this rigid body motions that is what differentiates the linear analysis and geometric non-linear analysis with the reference to this large displacements or large rotations. So, this matters a lot in our analysis.

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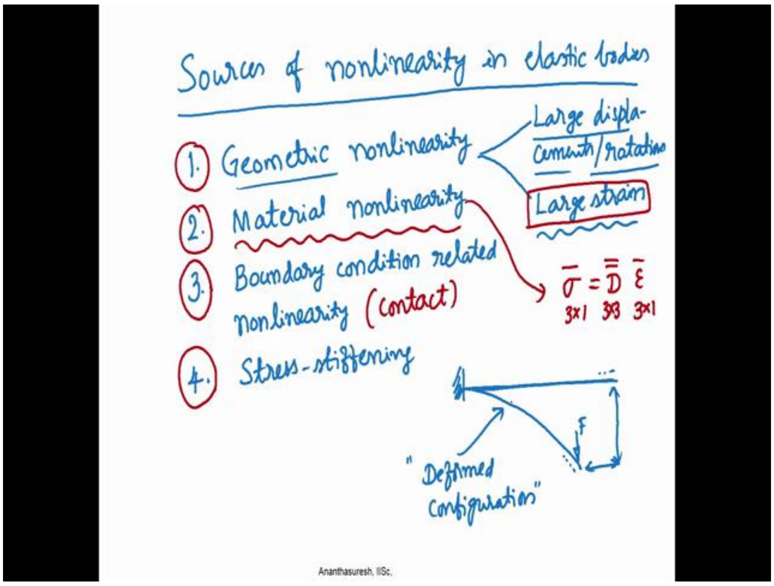

Now, going back to this our original thing so, large displacements, large rotations we understand, why is this important that is actually look at it if I take again the cantilever example, when I say geometric non-linearity what I would do is right the equilibrium not in this original configuration we are applying force I should not actually show the force there, because if I show the force that is good cause this deformation rather I should show the force in the deform configuration here, is where the force is this one has the linear one does not have force sorry, that un deformed one does not have force that is why it is where it is. So, we need to there is no force there. So, let me just get the pen so here is the force it has deformed.

Now, in non-linear analysis I would write the equilibrium equations in the deformed configuration not in the un deformed configuration, when we learn linear analysis we actually if we recall when you write the sheer force diagram, bending moment diagram you would actually consider the original beam and write it there you do not considered the new one. So, deformed one there is large displacement such as this translation and also as I said rotation something that is there has rotated to something like this. So, those rotations and translations have to be accounted for when you write the equation which will be done in the deformed configuration that only we look at geometric nonlinearity in the contest of large displacements and large rotations that is what considered the essential difference, there is other one which is large strain, large strain comes.

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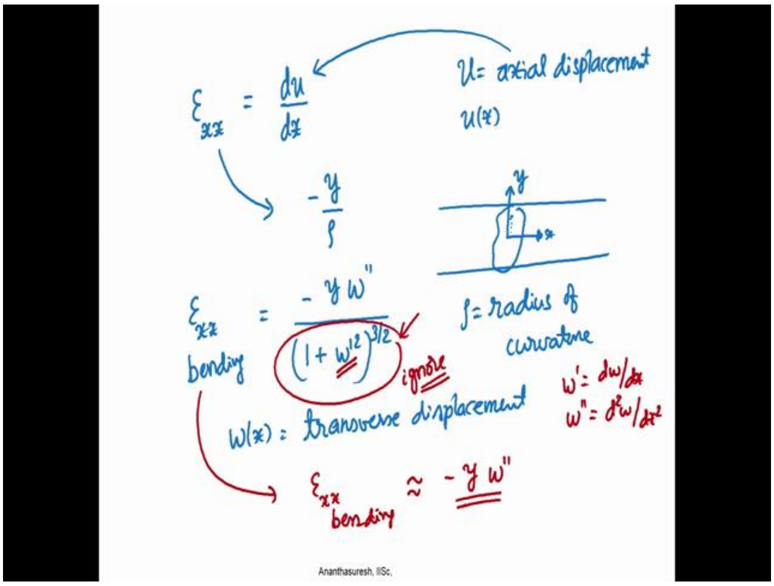

So, if I what to write let say strain here so, if I write a epsilon x for the x x I can say normal strain, if I know the displacement I could write this as d u by d x, where u is the axial displacement of the beam, a beam can stretch we said, we can stretches if I know that u as a function of x u will be the function of x, I can derivate that becomes the axial strain due to stretching, but we also know that there is another component to this which is on a beam bends that will also cause axial stress and also axial strain. So, that strain if you recall that epsilon x is what we say minus y by row, where y is if the beam is like this in this direction in the depth direction, this will be what I have indicated there as z, but in general report y and this is x if there is a cross section here any point I take there will be a y value there and this row is radius of curvature when this beam bends.

And we know that this epsilon x x due to bending, this is what I have written is due to the axial displacement, this is due to bending there is another component that we would have the bending 1 is minus y and radius curvature 1 over rho is curvature that will be w double prime divided by 1 plus w prime square raise to 3 over 2, where w of x is transverse displacement most often if you recall we would have assumed that, this 1 is approximated which is epsilon x x due to bending is approximated as minus y w double prime so if w transverse displacement w prime. So, this w prime is d w by d x and w double prime is d square w by d x square we approximate it like this.

Essentially, what we are doing when we do this is this is negligible compare to 1 that we are adding this 2 that is we have small slopes, that is again small rotations if a beam is there when it is deform in little bit the force is small or it is very stiff for the applied force it will deform a little bit then w prime will be small w prime square will be much smaller compare to one you can neglect it. So, this one you do not consider so, with this we ignore right that becomes this we get a linear differential equations when we get the beam. So, if you recall we relate this w w prime to bending movement or w fourth derivative to the q that is transverse load that is applied these are the other source of nonlinearity what will be doing later on is not to ignore this and consider it because, becomes important w prime that is the slope if w is transverse displacement w prime is the slope at every point of the beam then we cannot ignore because it is large. So, that is the other source of nonlinearity.

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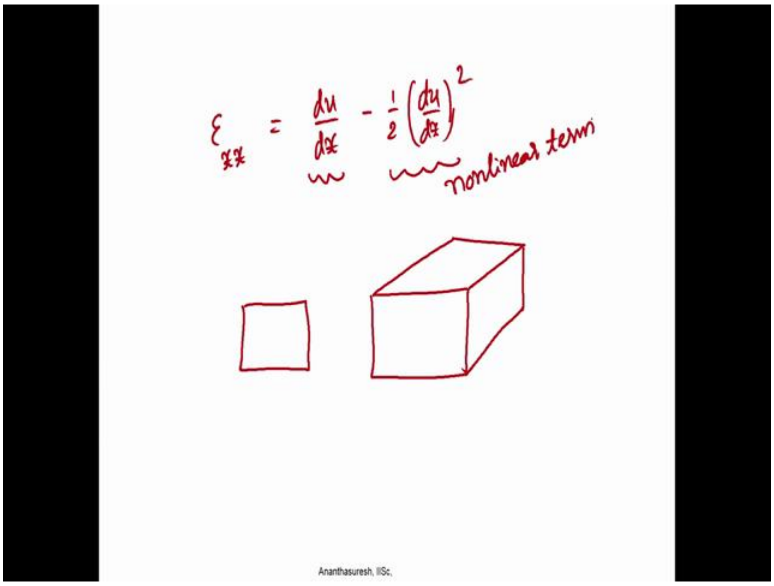

We can also look at in general, when we actually look at the strain that epsilon x x that we right normally for a given problem. So, we have written that d u by d x, but if I want to include large strain there also I could include this second derivative of this. So, I can put d u by d x that already have then I will also put second derivative, which will be half d u by d x square we can do that for a general element also if I have like a 2 d element or a 3 dimensional parallel piped for this one from here it go somewhere you can general notion there is this non-linear term. So, this non-linear term we neglect; that means, that we are assuming the strengths are small when including them that become a large deformation problems.

So, going back to our list so, now, we understand what it means to have a large strain definition that it will be in terms of displacement axial transfers and so forth, in addition to these two that both come under this we have material nonlinearity which is not. So, relevant for complaint mechanism what this means is that the stress strain relationship if I have let say 2 d problem or 3 d problem if I have a stress vector how is it related to strain vector. So, if I these db by a matrix if it is a 2 d problem, I will have sigma x x sigma y y sigma x y, so it will be a 3 by 1 vector and same thing 2 normal strain when shear strains 3 by 1 this will be a 3 by 3. So, this is linear whenever use a matrix; that means, its linear, but if the relationship is non-linear that becomes a material nonlinearity that we do not consider, because we do not want to use complaint mechanism where this non-linear stress actually matters because, by the time when you do once it goes non-linear it may not be repeatable many times which is what we want, but that is a material nonlinearity which we do not consider.

The third 1 is this boundary condition nonlinearity, what we can call contact. So, a complaint mechanism can touch a rigid surface that is there or touch itself that is like a self contact when that happens things change that gives you a non-linear condition that is this contact linearity, which will see a later with regard to beams there is a another one which is stress stiffening see if we look at cantilever beam that we have seen is fine, but let us look at a fix fix beam, let us look at a fix fix beam, which is fixed this side and the other side now there is some loading acting on it will deform something like this clearly here the length of this is much larger than the un deformed one.

The cantilever to there will be a slight stretching, but in the case fix fix beam or other boundary conditions is very clear that unless it stretches, it cannot deform to this when it stretches there will be in additional stiffness that is coming and that is call the stress stiffening as the beam goes from here to here, as it stretches you can imagine that if I have a musical instrument wire when apply more tension and when you apply transverse force it will be much harder when there is more tension that is stress stiffening that is another kind of nonlinearity that comes because, of the stress that will be there when from that condition let say intermediate condition with a slight case smaller force. So, you can see how stretching becomes more and that also leads to a non-linear conditions in beam like elements.

So, we have this 5 source of nonlinearity, so keeping these in mind when we look at couple of pictures where somebody has done finite element analysis and given us let us see whether we can tell them apart.

Thank you.