Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello today we are going to talk about constrained minimization in the last lecture that is lectures 4a and 4b we had looked at unconstrained minimization in finite number of variables we had said that was a small detour from calculus of variations this will also be a small detour extension of the detour where we are going to talk about constrained minimization in finite number of variables.

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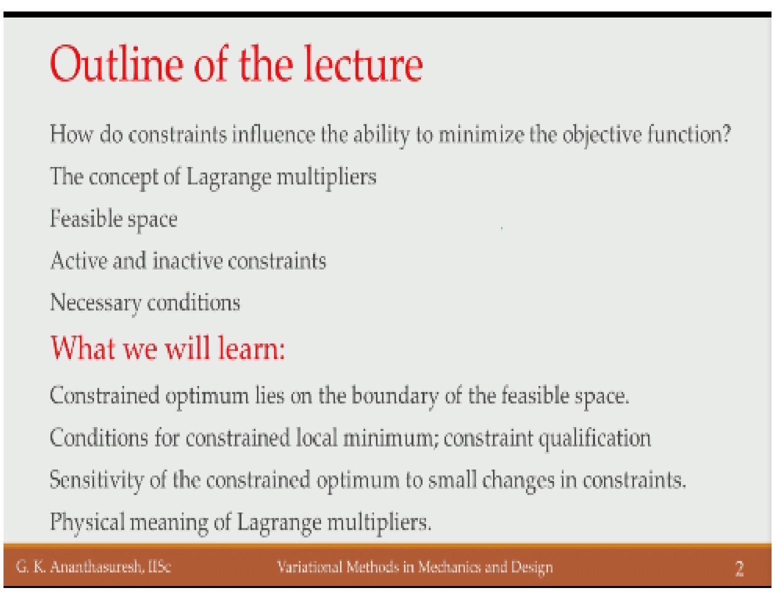

This is the outline of the lecture first we will discuss how constraints influence our objective function in terms of minimizing it and then the concept of Lagrange multipliers which is very important aspect of constrained minimization then we will learn what is feasible space when there are constraints and constraints can be equalities and inequalities then this question of whether any equality is active or inactive also needs to be dealt with finally we will establish the necessary conditions for a constrained minimization problem.

So as part of this lecture we are going to talk about constraint optimum when it lies on the boundary sometimes it is called a boundary optimum because now the optimum where is a minimum or a maximum lies on the boundary of the feasible space we will derive the conditions for that and then talk about this local minimum and another concept called constraint qualification and we will also see the sensitivity of the optimum point with respect to small changes in the constraints and then finally the physical meaning of the so called Lagrange multipliers.

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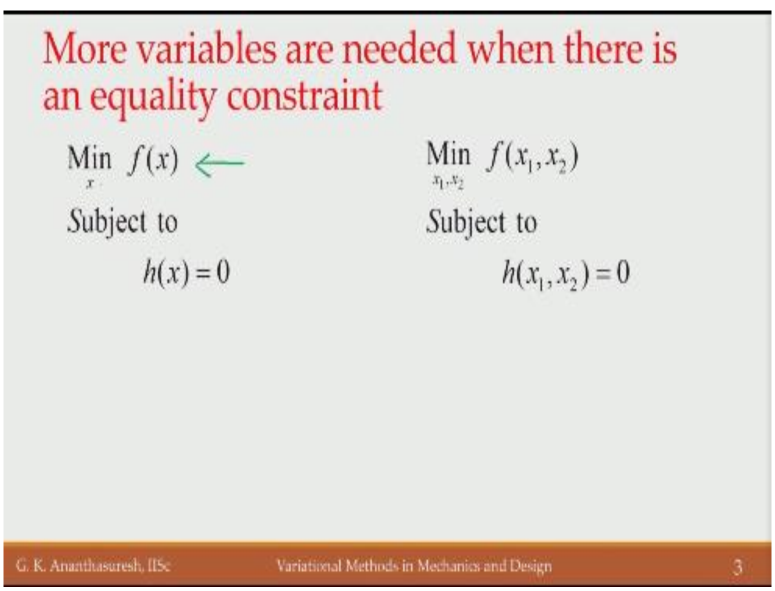

So let us look at a two variable problem that is minimize f(x) as we see here we are minimizing a function f(x) with respect to x subject to an equality constraint is this a properly posed problem that is a question we ask does it make sense okay let us look at that let us say we plot our equality constraint.

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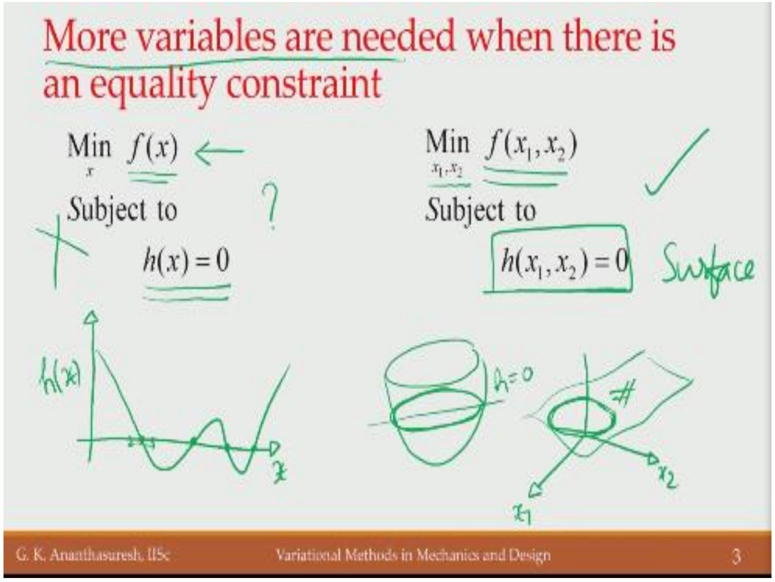

Llet us say I plot here h(x) and this is x okay it is inequality right so that means that if I have h(x) something like this let us say but how many points is it going to satisfy this equation h(x) is 0 here 0 here 0 here0 here four points that is all so our feasible space here is for discrete points so we cannot talk about minimizing f(x) now because if I look in the vicinity of this point okay let us I am looking at the vicinity of my first point which is feasible meaning it has satisfied h(x) equal to 0 there is nothing in the vicinity then there is nothing in the vicinity there is no way we can say it is a minimum or maximum that is if you are running a race and only you are doing it you cannot claim to be the first to come in the race right.

So we only compare with what is surrounding that point and then talk about whether it is a minimum or a maximum so here such a thing cannot be said when you have an equality constraint one variable because the space is not continuous you now to talk about a minimum when you talk about continuous space that does not exist here so this problem is not properly post so what do we need we need two variables more than one variable if we have one equality constraint ok now h(x1, x2) will be a surface right so this will be a surface which is set at zero value let us imagine in we have two axis let us say we have here x1 and x2.

Now h(x1, x2) will be some kind of a surface okay some kind of a surface will be there right this surface has to be equal to 0 so this surface is going to intersect this plane where x1 x2 will be some values when it is at zero value there will be a closed curve open curve something will be there so on that point then you have continuous set right imagine a surface like we have talked about this bowl for a minimum right let us say I cut it here which corresponds to that h value being equal to zero everywhere as there are different values over here we will get a closed curve on that closed curve we have one degree of freedom left that is we had two variables x1 x2.

Now this equality constraint makes that two variable space to a one variable space like that are in general they will be curved like this can be closed or open but there is going to be something on that we are talked about minimizing this function f okay this makes sense when you have any quality constraint then you need to have more than one variable so more variables are needed when there is inequality constraint so the simplest problem that we can think of in constrained minimization is.

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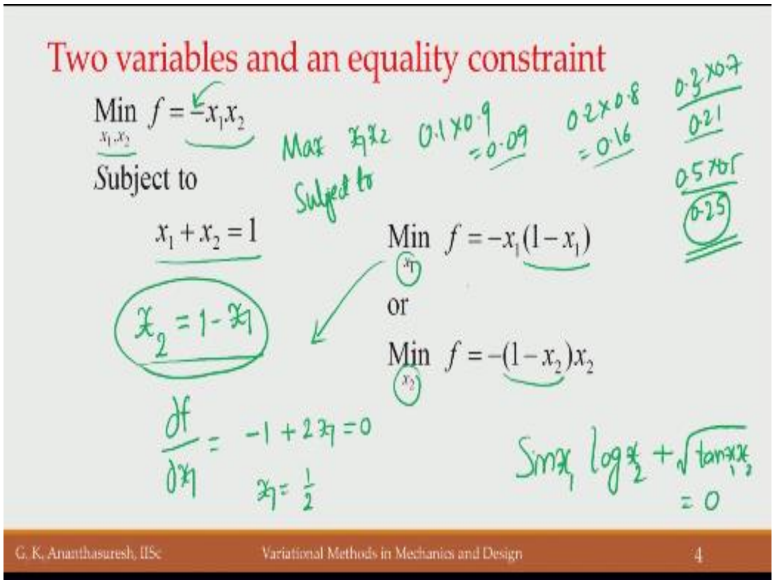

One that has two variables here we are taking an example that I want to minimize with respect to two variables x1 and x2 a function which is defined as shown here - x 1 times x 2 subject to an equality constraint which is x 1 +x 2 = 1 okay this is a simple problem so we are saying that find two numbers x 1 and x 2 such that the sum is equal to 1 and the product is actually maximized so if you notice we are minimizing -x1 x2 so that minus sign basically says that we are trying to maximize the product x1 x2 that we pose as minimizing negative of that product of x1 x2.

Okay essentially this is maximized x1 x2 subject to some people say such that both will be st x 1 + x2 equal to one if you pass and think about what solution it would have you can try it out you can say x1 is 0 x2 is 1 because they satisfy this constraint then the product will be 0 because x1 is 0 you make it point 1.9 then it will be 0.81 and then so far sorry 0.81 it is 0.1 times 0. will be 0.09 right and then you can do 0.2 times 0.8 because you have to always make sure that this is satisfied that will be 0. 16 and 0.2 times 0.3 time 0.7.

So the sum is 1 this will be 0.21 you see this is increasing so first 0 and 1 was 0 this slightly more slightly more likely more and then you do 0.4, 0.6 that is = 0.24 larger and then finally we have 0.5, 0.5 that takes a 2.25 right and then we will come back to this right so now 0.634 will be the same 0.7 0.3.8 29 .1 and then 10 right so among all of these you see that this is the largest right.

So we can see that there is a way to solve the problem but then we are trying all possibilities instead we can eliminate one of the variables over here so I can say I want to eliminate x2 in terms of x1 right if I do that I get a problem such as this right let me erase some of this yeah so we have so we can eliminate x2 by writing it as 1 - x1 then there is only one variable unconstrained or I can eliminate x1 in terms of x2 the variable will be x2 so by eliminating one variable is the help of equality constraint we can make it unconstrained and solve the problem okay.

If you now want to solve it then we will immediately see let us say this problem I want to solve if you recall from our previous lecture on an unconstraint minimization if this is F then the necessary condition is $\partial f/ \partial x$ 1 which is in this case equal to we have x1 - x1 that will be -1 minus of minus plus that is $x1^2$ that will be 2x1 that should be equal to 0 we get x1 equal to ½ which is what we found out right if x1 equal to 1 based on this relation of x2 is also this is x1 equal to ½ x is also equal to1/2 right so it works out.

When you can eliminate one variable in terms of the other we can reduce the size of the problem from two variables one variable we can solve it but this type of elimination is possible only in some cases if the Equality constraint is complicated let us say I give something like sine x into you know log x + √ tan x or something like that I may not be able to solve it when I will say x 1 this is x 2 and this is let us say x1 x2 something like that if that is my equality constraint I may not be able to express one variant of the other so we would not be able to do what we have accomplished here that is eliminating one variable okay so we will discuss how we do it when that is not possible.

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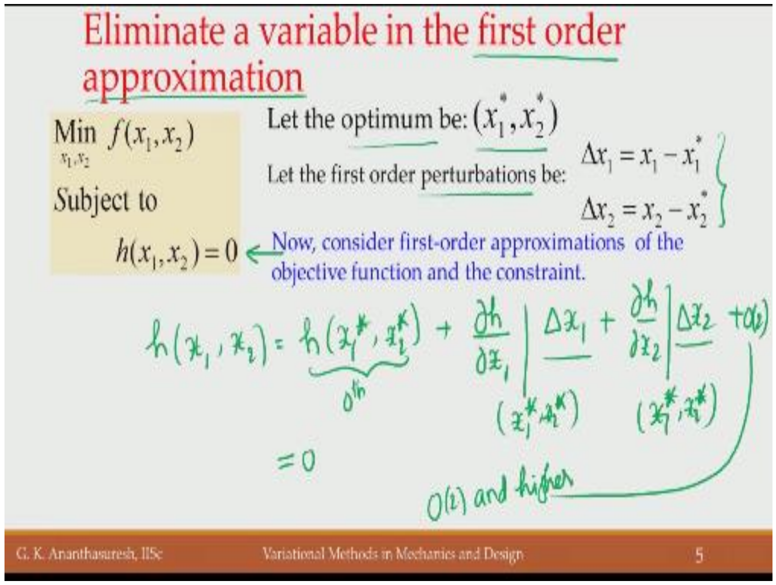

Okay let us say we have a constraint which is not going to let us eliminate one variable in terms of the other then what do you do in that case we go to the first order approximation okay so we think about this small perturbation so let us say that optimum is given by let the optimum be x1 * x 2 * at that point we will consider this perturbation okay that is Δx1 we say x 1 - x 1* Δ x 2 is x 2 - x 2 * so that the v x1 after perturbation is Δ x 1 + x 1 * and x 2 will be Δ x 2 +x 2 * when we have that then we can consider the Taylor series approximation of this function h how do we do that if I have h.

Let us say I want to do at x1 and x2 okay which are given by this then it will be whatever h is at x 1 * x 2 * okay this is 0^{th} order term right so this is 0^{th} order term right then will be first order term so we have to perturb bit by this much then what we say is that partial derivative of h with respect to x 1 evaluate that at x 1 * x 2* okay we are taking their with respect to x 1 partial derivative but resulting expression we have to evaluate at x1* x 2 * because that is what we said is optimum and then this will be multiplied by Δ x1 and ∂h / ∂x2 again evaluated at x1 * x 2 * into Δ x 2 and there will be order to unhide okay so this will be order to and higher okay.

Why do we do this because we say that when we take the approximation of this equality constraint which is not amenable to eliminating one of the variables then we take up to first order and make it equal to 0 that is if you ignore all these higher order terms then you make it 0 and this will be linear equation in terms of the perturbation Δ x 1 Δ x 2 so we can eliminate one perturbation in terms of the other what we did the whole variable in the previous slide now we are going to eliminate in terms of the perturbation.

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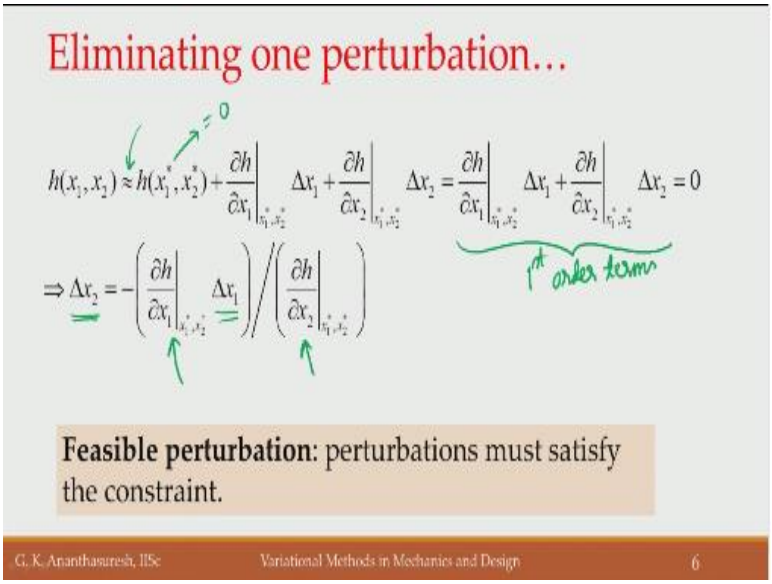

So what we do is eliminate one perturbation in terms of the other either Δ x 1 Δ x 2 so that after perturbation our equality constraints remains again valid meaning to be equal to 0 what I wrote by hand we are approximating that is a symbol for approximation h x 1 x 2 we are approximate is 0^{th} order term and then the first order term that we wrote ∂h/y 2 x 1 x Δ x 12 h /∂ x 2 term Δ x 2 and that set the approximation for first order we are stopping there and then say that that should be equal to 0 that is after perturbation you should be 0.

Now this is0 anyway because if x1 * x 2 * is an optimum it must also starts with equality constraints so that is 0 so what is left with what we are left with are these two terms the sum of these two terms should be equal to 0 these are the first order terms in two variable case that should come to 0 therefore by using this relationship we can get Δ x2 in terms of Δ x1 and some information that partial derivative with respect x1 evaluated at that point partial derivative of h with respect to x 2 evaluated x 1 * X* okay.

So now we are eliminating perturbations rather than the variable itself is an approximation which is fine because the local minimum is always defined with respect to a local vicinity so taking first order perturbation is perfectly fine with that so the feasible perturbation is something that satisfies the constraint even after perturbation.

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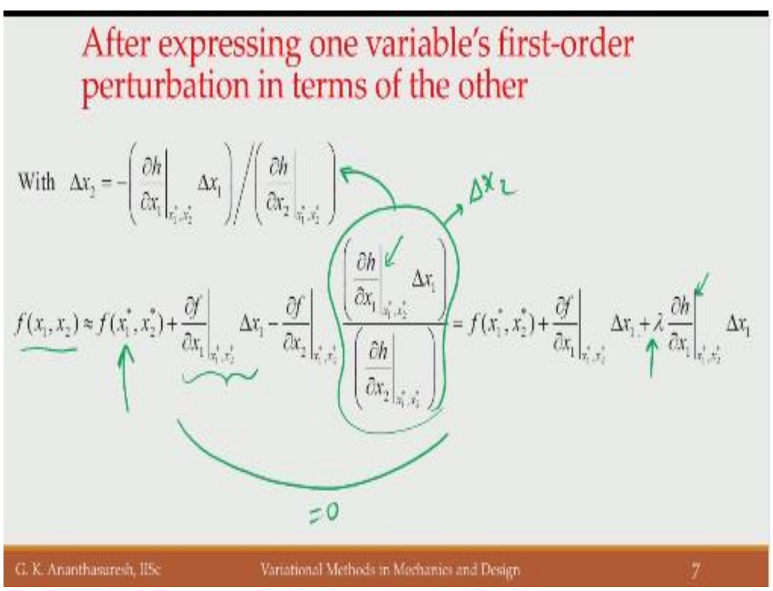

That is what we are doing we got $\Delta x2$ in terms of $\Delta x1$ now we consider similar expansion for the function so we have F at x 1x * some value and then we have the first order term the first one as it is second one what we have done is the $\Delta x1$ here which would be basically this right that is what we have from here so that one we have replaced this is nothing but $\Delta x2$right that is what is here we substituted there okay so that should be something that we should look at what we if we say that this function here has to be a local minimum.

Then to first order we do not want any change in this right whatever we have put here that should not change the first order because as we had discussed earlier for unconcerned minimization you have $\Delta x1$ that $\Delta x1$ may be passed you may be negative so sometimes f of x1 x2 may be larger than F of x1 * x 2 * something may be smaller so first order we do not want the term to change second order will be there that will be for sufficiency first order we say that should be not changing that gives us a condition to say that.

That should be equal to 0 this first order term should be equal to 0 before we say that we denote something that we have here we have defined something called λ here se $\partial h / \partial x 1$ this term is here as it is but what we have is $\partial f / \partial x2 / \partial h/ \partial x 2$ so these two we have defined as λ along with the negative sign.

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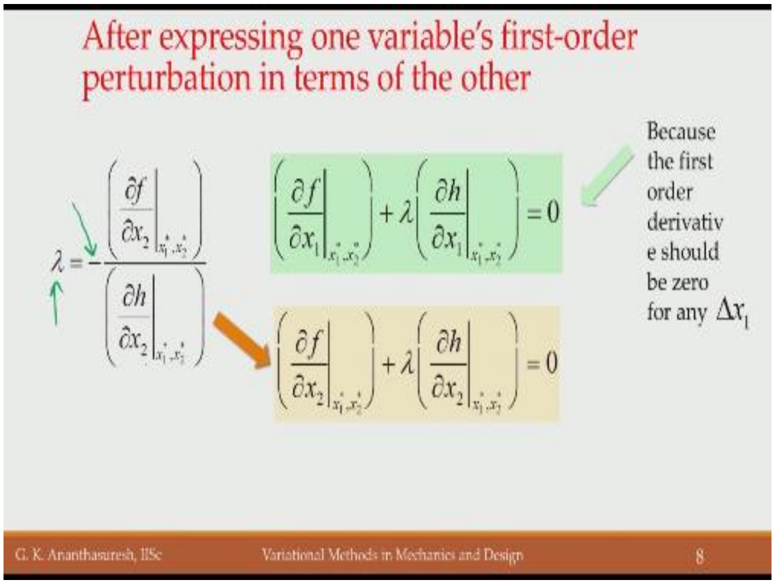

That means that we have defined this λ 2 B minus so notice that negative sign ∂ f / ∂ x 2 divided by 2 h / λ x2 both evaluated at x1 * x 2 * now again go back.

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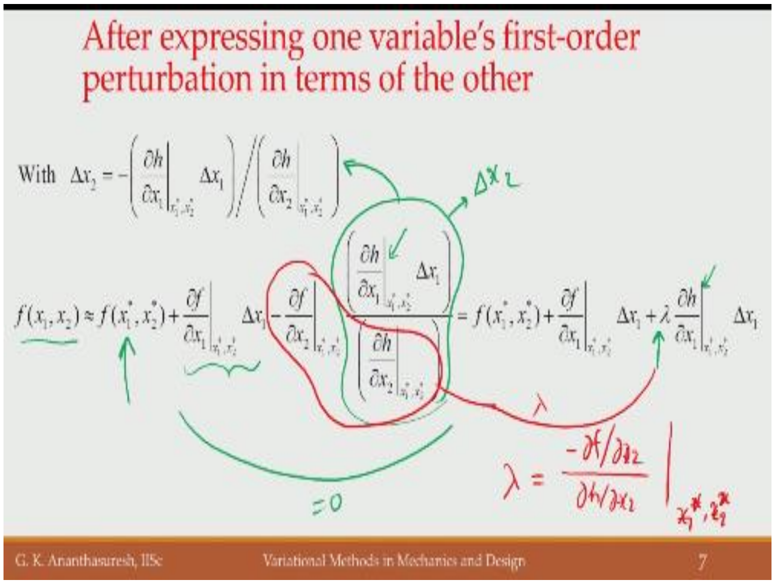

What we have defined let me change the ink color important so I will put it in red so along with the negative sign okay so this whole thing is this λ that way okay that is λ right so λ is minus ∂ f/ ∂ x 2 divided by ∂ h /∂ x 2 so let me write it so here you see minus ∂ f / ∂ x 2 divided by ∂ h by ∂x2 both of which are evaluated at x1* x 2 * so this whole thing is evaluated at x1 * x 2 * okay so that is what we have λ.

λ is defined that way and then we said that∂ f by here so this term here so we said that the first order term this whole thing here okay that must be equal to 0 for because for any perturbation Δ x1 we do not want the function to change so what we get here is that ∂ f / ∂ x 1 +a λ times ∂ h / ∂ x 1 equal to 0 that is what we have shown there look at the previous one but the way we have defined λ over here that leads to this equation okay that leads to this equation right so if you look at this was the necessary condition.

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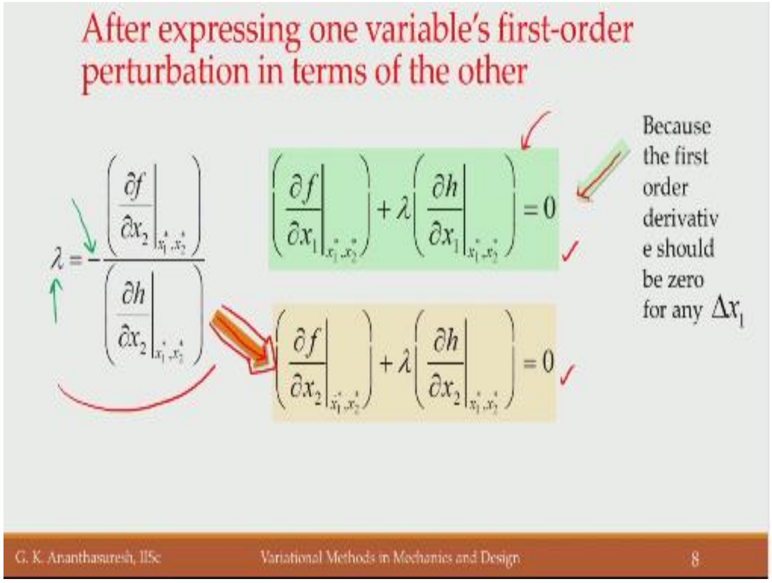

When we eliminated Δ x1 and had only Δ x2 and had only Δ x1 this is the way we defined λ so these two equations if you see they have a pattern this is ∂ f / ∂ x 1 is ∂ f / ∂ x to this dough edge by ∂ x 1 this is no h by 2 x 2 both λ have same λ so we got now two equations okay which are identical with respect to x1 x.

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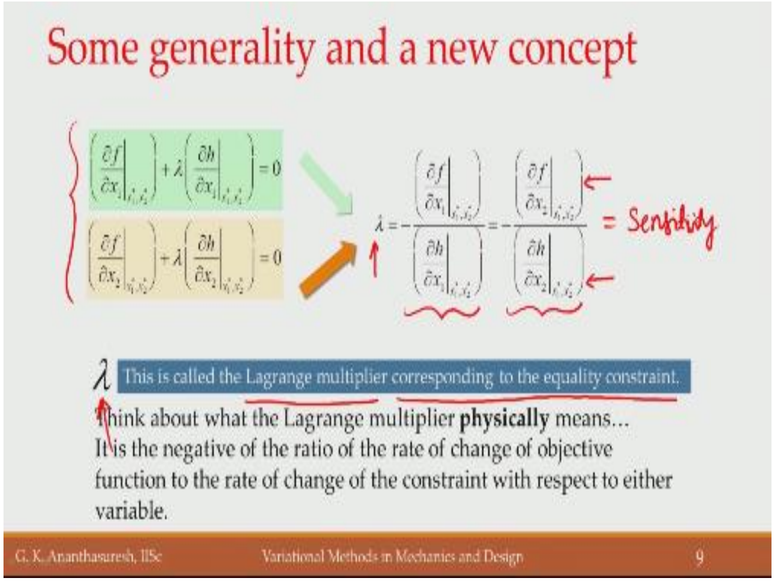

Now this gives us a way of thinking about the necessary conditions ∂f/ ∂ x 1 plus λ x 2 h x 0 x 1 equal to 0 ∂ f /. ∂ x 2 plus λ 2 h by2 x 2 equal to 0 in both cases it is the same λ the wave is defined λ is minus ∂ f / ∂ x 1/2 h / 2 x 1 R minus of ∂ f / ∂ x 2 divided by 2 h / ∂ x to look at the same λ for both and this λ is called the Lagrange multiplier corresponding to the Equality constraint we have an equality constraint correspond to that there is a λ which is defined in this manner which gives us the necessary conditions also okay.

And think about what this Lagrange multiplier means it shows that this is nothing but sensitivity what we mean by that it shows that if there is a small change in the constraint it tells you what change is going to be in the objective function with respect to that variable for that matter any variable because this is valid for x1 x2 when you have more variables the same trend come continues right it is a sensitivity is the negative of the ratio of the change of objective function rate of change of it if I with respect to a variable compared to the rate of change of the constraint that is what is the Lagrange multiplier a very important concept okay.

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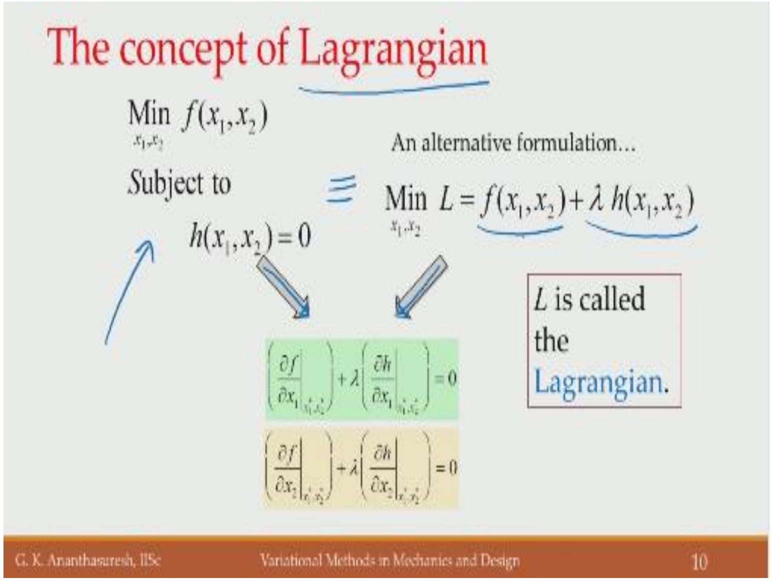

And it is also useful to define what we can call Lagrangian okay Lagrangian isdefined as the function plus λ times the constraint okay when we do that it becomes unconstrained problem in a way because this is the original problem these are original problem which had an equality constraint now that problem is identically equal to just this minimize Lagrange respect x1 x2objective function plus λ times the Equality constraint okay if you were to write the necessary conditions for this problem you would get exactly the same equation Lagrangian derivative respect to x 1 ∂ f / ∂ x + λ time to h /∂ x 1 the same to the second equation also so we arrived at these equations in this way now we are saying that we can also arrived at it in a different way called using their concept of Lagrangian okay.

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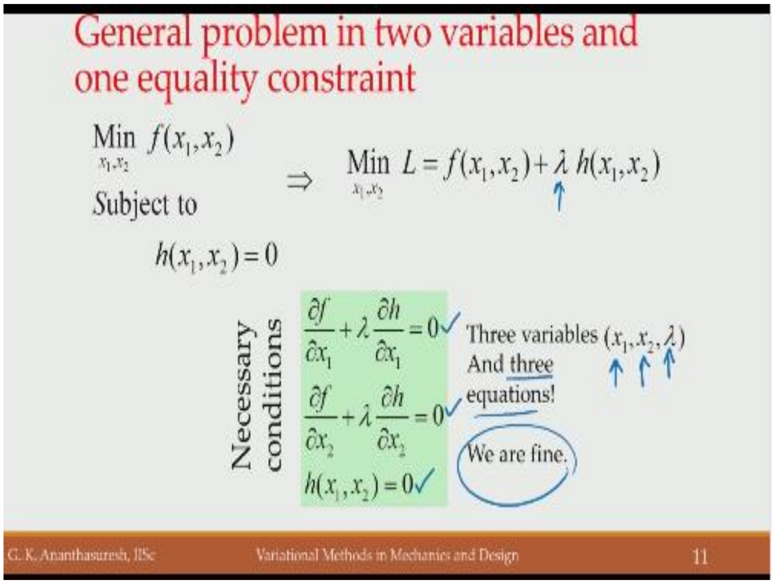

And these are the necessary conditions now the moment we say this Lagrange a minimize respect x1 x 2 and there is unknown multiplier so then we get these two equations necessary conditions and then we have the Equality constraint anyway with us hx1 x2 we have three equations in three variables are variables are x 1 x 2 and λ and we have three equations that we have in front of us.

So we are fine we have as many equations as unknowns and hence we can solve for them okay this is how we deal with the constraints equality constraints there is a geometric interpretation to this.

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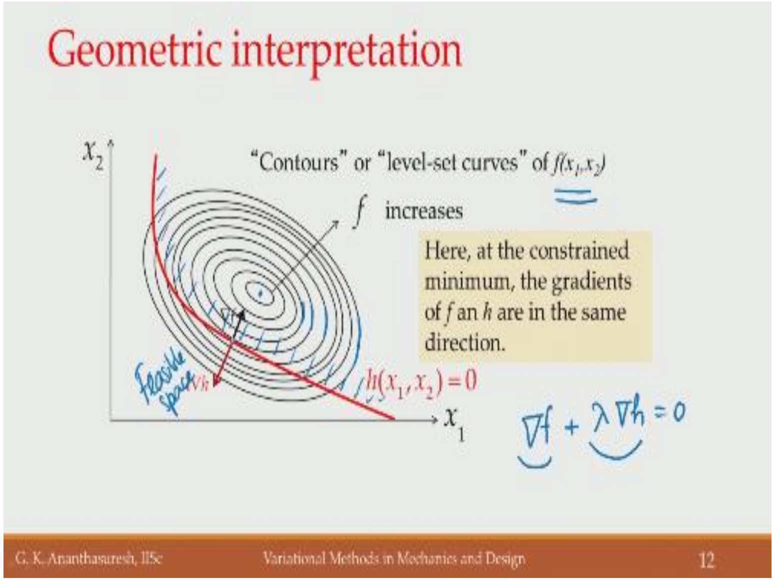

Let us say we take this two variable problem we had Fx1 x2 there will be contours as we have shown here in black lines that is for different values of F of x1 x2 we can draw this contours so f value on the contour on anything you take will be constant okay there I so objective function values all right now h x 1 x 2as we said will be a curve it can be close to our open curve which is shown here as a red one okay.

So in this case where is a ma because we say that constraint is such that everything you know because f increases from here it increases this way this is the minimum but let us say constraint is such that we cannot go on this side of the surface yes you have to be only on this side this is our feasible space feasible space okay then unconcerned minimum is not accessible to us we will keep on increasing creasing increasing when you come to a particular point we realize that that is the smallest F because F increases as you go away from here.

So that is the smallest if we move a little bit further into feasible space then I would increase my objective function okay so geometric interpretation here when we wrote that gradient of F plus λ into gradient of H equal to 0 what we mean to say is that the gradient of F is compensated by this is on the negative side but λ times gradient of h okay so geometric interpretation is that the contour of the Equality constraint is going to touch here it is a point that touches the contours of the objective function okay that is a geometric interpretation of the Lagrange multiplier and the associated equality constraint.

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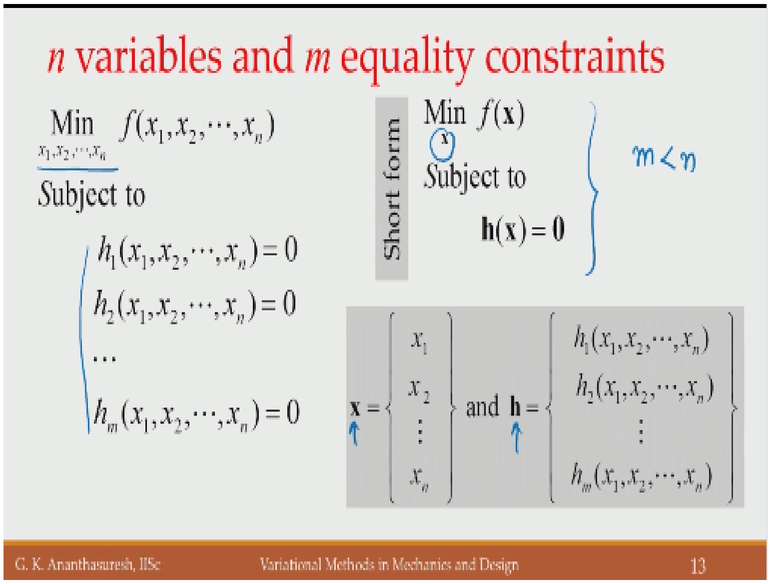

What if there are n variables in EM equality constraints right we have n variables x1x2 xn and there is h1 h2 hm we necessarily want em to tan n right if mis equal to n then as we had said in the case of one variable one function we only have a few points even if at all they exist then we cannot do optimization so you need to have freedom so that is why we want em to be less than n so in which case we will raise it okay when we say we minimize here I am just calling x capital X meaning that that capital X is x 1 x 2 xn and h is h1 h 2 hm if I want solve a problem like this I can solve only if this M is less than M so that I have some freedom to search okay.

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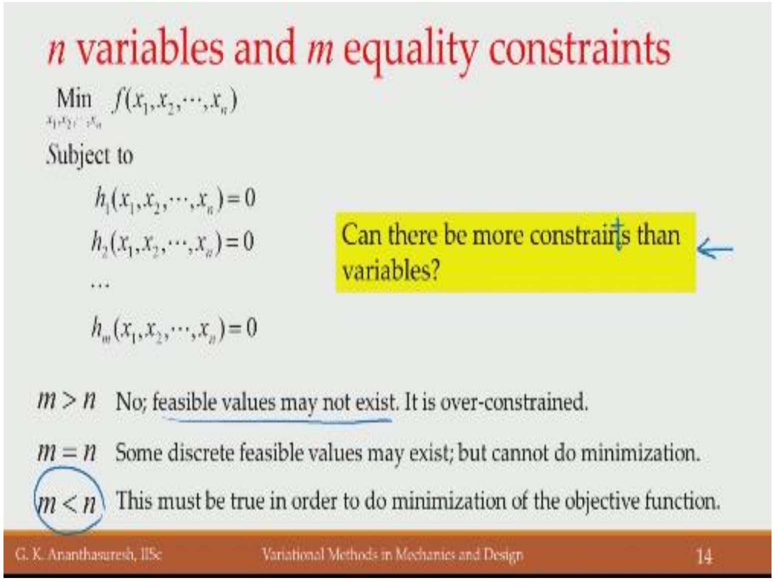

So here we are asking question again can there be more constraint than variables the answer is no we cannot have this is constraints we cannot have more constraints than the variables if m is greater than n feasible values may not exist even if they exist it will be a few discrete value so it is an over constraint problem we cannot optimization when m equal to M some discrete values may exist but you can still not do minimization in order to do minimization we need to have em less than m okay that is what we send now when you have such a thing.

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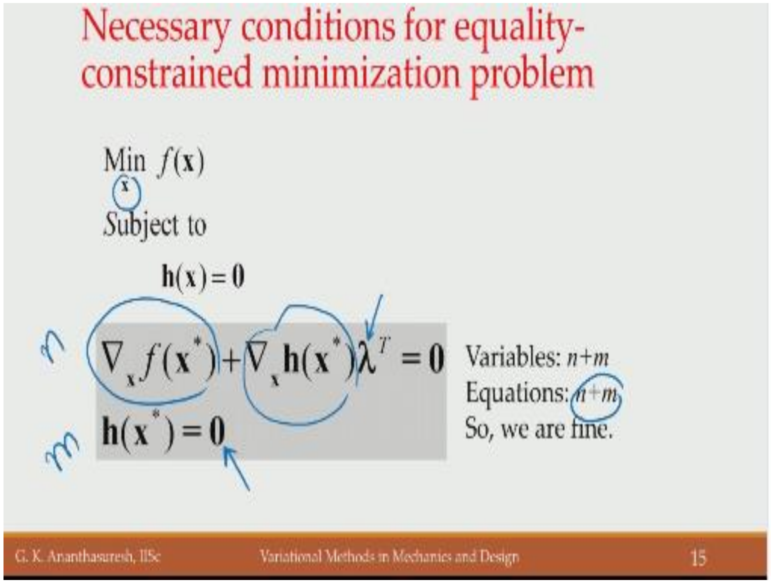

What we did with the two variable case necessary condition we can add a general case now I have x like upper case that means that i have x 1 x 2 x n and then h also em soh1 h2 h3 up to hm then the necessary conditions can be written like this there is something called the gradient of the objective function and gradient of the constraint equality constraint and then we have this Lagrange multiplier vector because respect to each of these edges h 1 h 2 h m there will be a λ 1 λ 2 λ M.

So and then we have the Equality constraints itself so again if you count how many equations are there are n equation s here there are M equations here total we have n plus some equations how many variables will we have we haven variables in x and then M variables for λ 1 λ 2 λ 3 λ M right we have n plus M variables and press M equations again we are fine so here also if you think of geometric interpretation let us say we take a problem here are taken the same thing but I am saying that instead of having only.

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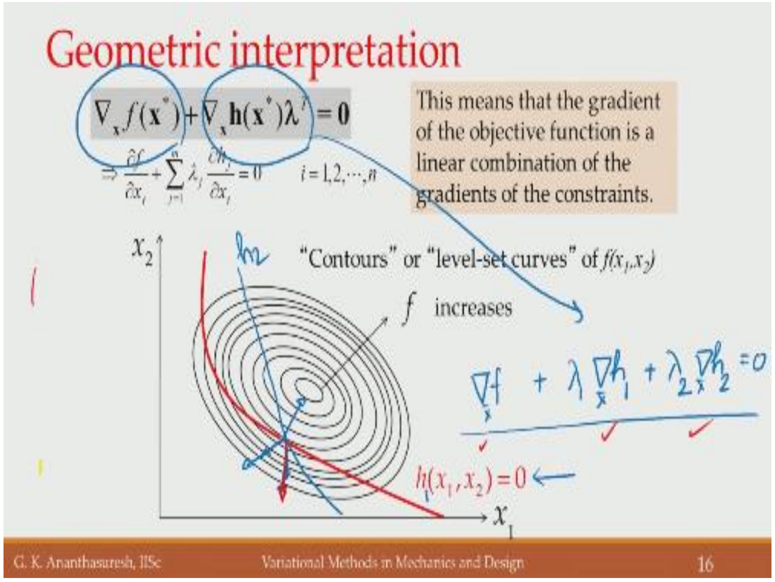

One equality constraint that is shown here let us say there is another equality constraint something like that okay over there what happens is the gradients so this is h2 this is h1 right let us call this h1 that is h2 then what we have here is that gradient of the objective function is a linear combination of this what does it mean it means that we have gradient of the h1with respect to that x their times there will be λ 1 there will be another 1λ 2 for the second constraint and the gradient of the objective function all of these respect to this x here they should sum to zero so if there is the gradient over here for the objective function like that this red one may be like this blue one maybe like that okay.

You have to draw the gradients properly so there will be some of all the gradients should be equal to zero okay here we did not draw the thing properly actually the blue one gradient will be like this red one gradient will be like this let me change that to red color okay so let us say that gradient is like this and the objective function will be some of these two so the gradients are the two objective function this and this will be in the opposite direction to the gradient of the function that is how we interpret in multiple variables if there are there n variables and M equality constraints m should be less than and we will discuss how we can derive this in the next lecture thank you.