Indian Institute of Science

Variational Methods in Mechanics and Design

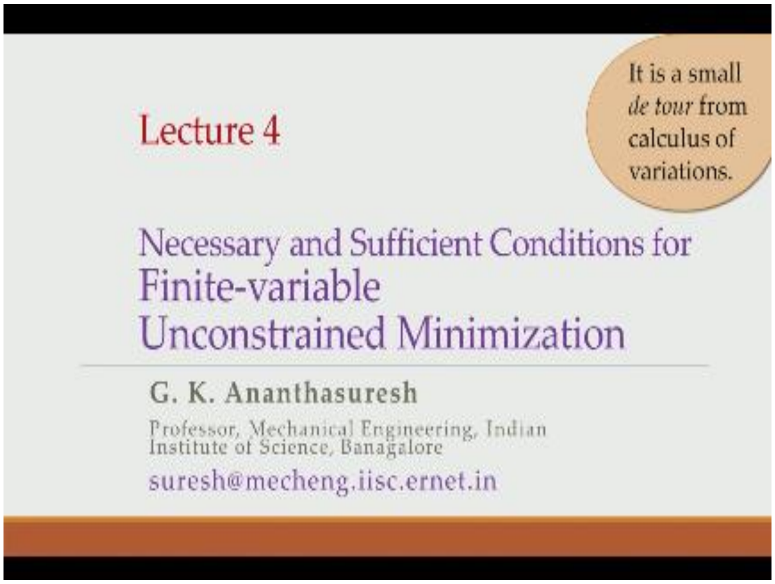
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering

Indian Institute of Science, Bangalore

NPTEL online certificate Course

Hello we continue our lectures on variational methods in mechanics and design and today we are going to take a small detour from calculus of variations.

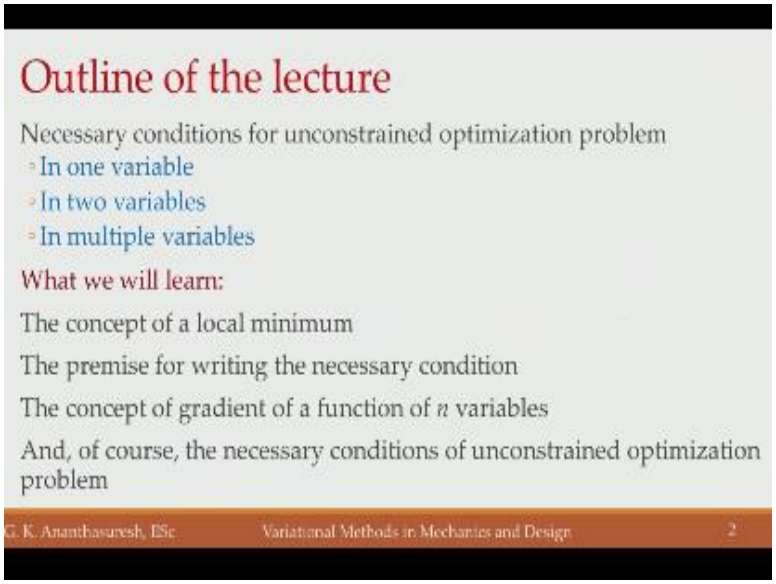
(Refer Slide Time: 00:28)



As we have discussed in the last three lectures calculus of variations is crucial for variation methods whether it is applied to mechanics or design but and calculate variations as we have discussed is an optimization problem where the function itself is an unknown but finite very optimization is different in that there are finite number of optimization variables x1 x2xn and so forth okay so today we take a small detour to talk about finite variable constrained minimization that is it is an optimization problem without any constraints we just have an objective function that we minimize with respect to finite number of variables meaning there are x1 x2 x3 and so forth up to X n where n can be just one variable.

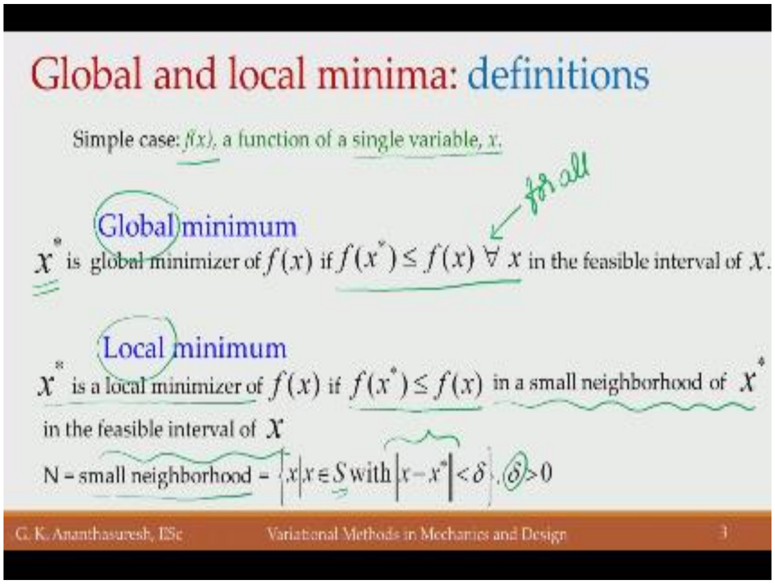
And can be to two variables and can be any number but finite number for such a problem we will discuss today necessary and sufficient conditions for finite variable unconstrained minimization.

(Refer Slide Time: 01:41)



Here is the outline of the lecture we will start discussing the necessary conditions and sufficient conditions for a problem with only one variable and then move up to two variables and then jump to multiple variables they can be n where n, n can be any number can be 100,000, 100,000 does not matter we will discuss necessary and sufficient conditions for such a problem first what we will learn will be the concept of a local minimum which is very important for optimization it is only a local minimum that we have control over in terms of how we find it there is also the concert of global minimum we will discuss the difference between the two.

(Refer Slide Time: 02:42)



And then we will also discuss what is the base what the basis is for writing the necessary conditions and then for that we need the concept of a gradient and also something called a hessian for sufficient conditions so let us look at a very simple case of a problem which is stated here just f of X okay so we have this function f of X a function in one variable X okay a single variable X that we have for such a function if you think about a minimum we want to have definitions first what we call a minimum there are two kinds of minimum as indicated here global minimum and then there is local minimum okay.

Global minimum is defined as it is shown here if for f of X star if X star is a global minimize meaning that X star is a particular value at which point this function f of X has the largest value when is a global minimum largest value that satisfies this definition that is it is X star is a global minimize of f of X if this condition is satisfied for all X this symbol as you know this symbol stands for, for all okay so for all values of x f of X star is less than or equal to f of X that is you have some domain for the function f of X in that domain whatever value you take whatever value take for x then f of X star will be smaller than f of X smaller than or equal to greater this less than or equal to such a minimizing point X star will be called a global minimize.

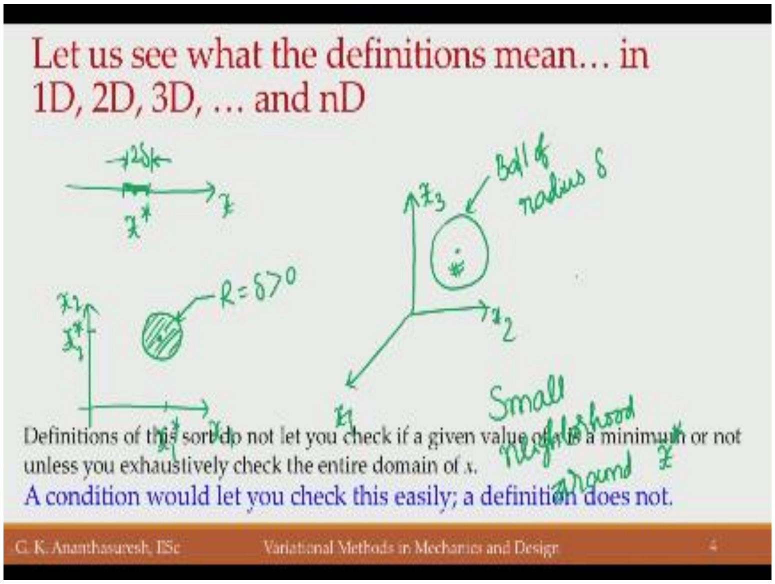
That is clear so if you say that you are you have the smallest weight in a group of people then your weight should be smaller than everybody else's wait then it will be a global minimum right as opposed to that local minimum is defined differently as you can see the second line here X star is a local minimizer off of X EF similar condition we have f of X star is less than or equal to f of X but this is not for all X in the domain but only in a small neighborhood of X star in the feasible interval of X so X has some values that are permissible what we call feasible values.

But then we are not taking all the values all we are saying is that X star has a neighborhood that we can define in that neighborhood in that local neighborhood that should be the lowest value okay so if you instead of think of a minimum if you think of a maximum let us say you are a school tougher you are a local maximum because compared to the other students in your class you are the top right but then if you consider the entire state or entire country or world if you are still higher than any other student of your age or your class there will be a global maximum so global maximum has exactly that thing in a local minimum or maximum.

We have to look at a small neighborhood that is around you, you take a small neighborhood you and your class then it will be a local best local minimum if you are the best in the largest group that is there then you will be global minimum or a global best okay global optimum okay let us understand these two definitions first these are the definitions and definitions are well before we go to that let us see this small neighborhood let us actually define that also okay small neighborhood it is that neighborhood X where X belongs to some set s okay.

With this condition satisfied so we have defined this Δ which is a small positive value X minus X star so we are talking about neighborhood around X star right so the distance between X star and any other point X you take okay that is why we take this as later on we will call it a norm but right now think of it as a distance, distance from X star to any other value if it is less than Δ that will constitute the small neighborhood around X star okay let us understand this figuratively okay.

(Refer Slide Time: 07:20)



Are using a figure so if I have a one variable problem okay let us say I am like this and somewhere here so let us say the X goes like this there are lot of different values of x now let us say there is a particular point which we call X star all right now for this one-dimensional problem single variable problem I can define a small neighborhood okay small neighborhood so that I can say is from here to here around our point X star I can say this is two times some Δ okay that is Δ on the right side Δ on the left side that is the small neighborhood.

In that neighborhood if this X star has a least value that we call a local minimum okay we go back to the definition here that is exactly what you are saying the small neighborhood now instead of taking a single variable let us take two variable problem let us say I have two variables X 1 and X 2 now I take a particular point let us say this is x 1 star and this is x 2 star okay so this coordinates are X 1 star x 2 star then I can define a small disk around it okay of radius for that radius equal to Δ okay here it was a small line that was the neighborhood there.

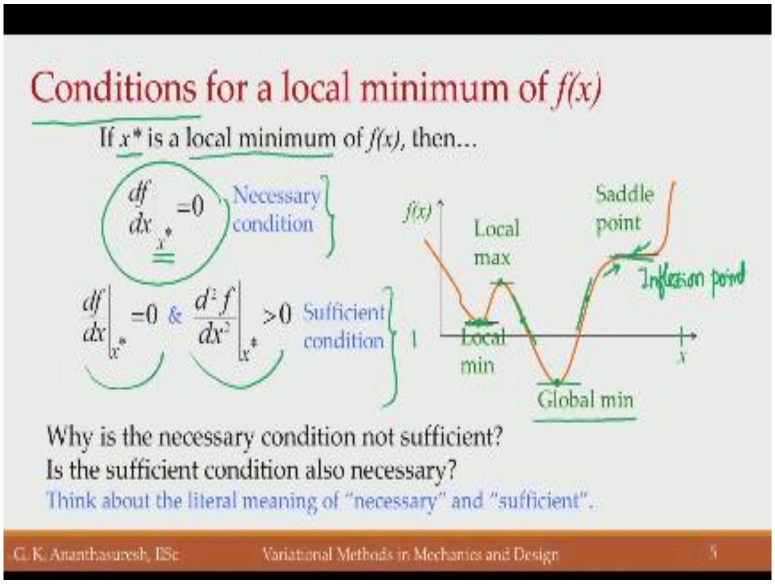
But here we have this whole disk as the neighborhood within that neighborhood around X star X star here has two values it is a vector now x 1star x 2 star 2 values around that if you define a disk of radius Δ which is positive which is greater than 0right in that neighborhood if this has the lowest value we call that a local minimum now if I take three dimensions okay let us say I have x1 x2 and x3 okay now I have point somewhere which has coordinates x1 star x 2 star x 3 stars then I can define it was just a little one-dimensional domain two dimensional it became a disk in three dimensions it will be a ball okay.

Now this is a ball of radius Δ again small neighborhood so we are just defining our small neighborhood, neighborhood around our putative optimum X star okay I will put a bar so it can be x1 x2 or x1 x2 x3like a vector that is a small neighborhood okay within the small neighborhood if you have this point to have the least value then it is a local minimum otherwise if it is for the entire domain if it is the lowest then it is called a global minimum and that is exactly what these two definitions tell us okay.

So these definitions are good but they are not practically useful okay somebody you see a problem and says okay here is a local minimum or here is a global minimum right how will you check you cannot check this entire neighborhood right for example you claim you go to a house and say on the school then your neighbors have to verify that there to come to the school and ask marks of every one of our classmates so they have to make sure that everybody is asked which is very difficult if it is a large school.

If you say I am the country first they not to ask every kid of your age to ask how many marks they got right in order to confirm that you are the topper similarly here if you say this point is a local minimum there to check around this neighborhood exhaustively search and find which is not practical so definitions are not enough we need conditions so we have a condition you plug some values into it and then verify it is a local minimum or a local maximum or global, global maximum.

(Refer Slide Time: 12:12)



Let us see if such conditions exist okay pictorially let us first see how they look like right so here we are showing a point which is a global minimum with that we have indicated right this point is a global minimum because in the domain that I have shown let us a domain from here to here of this X here this is a global minimum that is the least value every values above it at account of definition the global minimum is that but you look at this point that point is a local minimum meaning that in the small vicinity if I take right is a one-dimensional problem in that small vicinity of X here that is the least value locally to the least value okay.

Similarly one can talk about global maximum and when can talk about something all day inflection point or as addle point sometimes you Corazon saddle point sometimes we call it inflection point here looking this side it is a local maximum that is looking at coming from this side it is local maximum coming from this side it is a local minimum right such a thing an inflection point where it has a special thing that when you draw a θ the curve is on both sides whereas if you draw a θ here curve is only one side above for a minimum or local level global minimum.

If the θ here for a the curve is below the point okay where the θ is right so these are the graphically the concept of a local minimum local maximum global minimum I can have a global maximum also in a problem right now definitions we have learnt graphically we understood what they are now let us find conditions for one variable problem it is very simple all we say is that if X star is a local minimum such conditions which are operationally useful can be written for local minima but not for global minimum okay.

Plural of minimum is minimal do not say minimums that will be incorrect grammatically minima multiple local minima okay so global minimum there will be one local minima there can be several but every one of those if X star is a local minimizing point then the condition that it should satisfy is written here okay it says DF by DX that is the derivative of X, X with respect to X is equal to 0 when it is evaluated at X star okay that is clear from this graph so θ DF by DX is a slope, slope at this point is 0 okay.

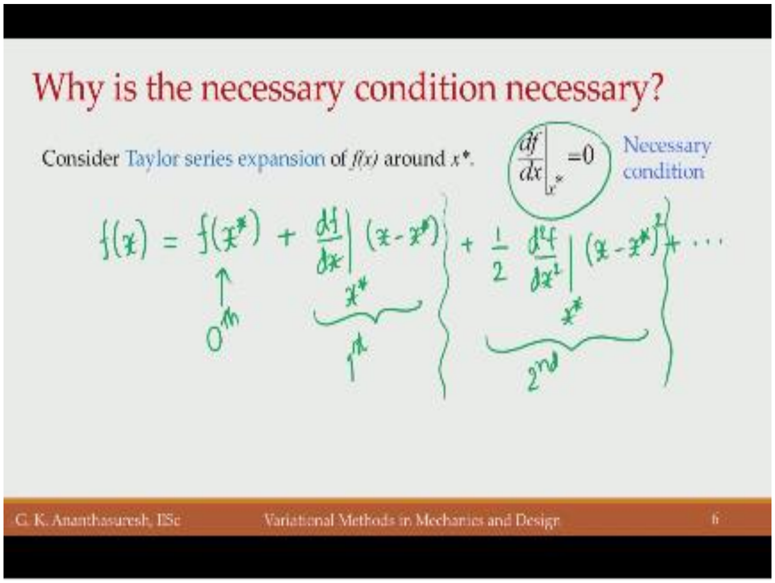
Slope at this point is 0 if I take a point like this, this is not a local minimum or local maximum because slope at that point is not zero right are this point not zero slope at the point where as here slope is 0 but neither a minimum or a max or is it is both okay but in any case the local minimum has a condition that slope at that point is equal to 0 okay that is a necessary condition the slope being equal to 0 at a point is necessary condition for that point to be a local minimizer.

We have a necessary condition you has a sufficient condition if there is such a point which satisfies the necessary condition that is DF by DX at x star equal to zero then if this is true that is d square f by DX square evaluated at X star if that is greater than zero that is sufficient okay so now we have necessary condition and then sufficient condition okay these are conditions so if somebody can see a, a function and use a value and say that this function has a local minimum at that point then immediately you take derivative substitute. That value that X star that is given to you if the derivative is equal to zero then you can say yes your point could be a minimum because it is a necessary condition and then you find the second derivative d square f by DX square if there is greater than 0 at that point you say it's a charge for sufficient condition necessary sufficient.

Both are satisfied then you say okay what you're given me is a local minimum right so here you just need to substitute into the derivative first derivative and second derivative at that value and you are done unlike exhaustively searching the entire space entire domain of the function okay. That is what conditions are useful operationally as opposed to definitions, definitions are important to understand the concept conditions are important to solve problems okay now we have defined two words necessary and sufficient you have to say why is the necessary condition not sufficient they're two different words right we say condition is not necessary we are not saying necessary condition is sufficient and we are saying sufficient condition why is it not necessary okay let us look at these two terms necessary condition is not sufficient, sufficient condition is not necessary okay.

That due to different words while these words ness sufficient are very simple words in English language in mathematical sense for optimization we had unless and their meaning but they do not suggest anything other than what they actually say necessary means it is necessary it is essential it is necessary and sufficient means is adequate it is sufficient they do not say anything more but we should not get confused okay.

(Refer Slide Time: 18:22)



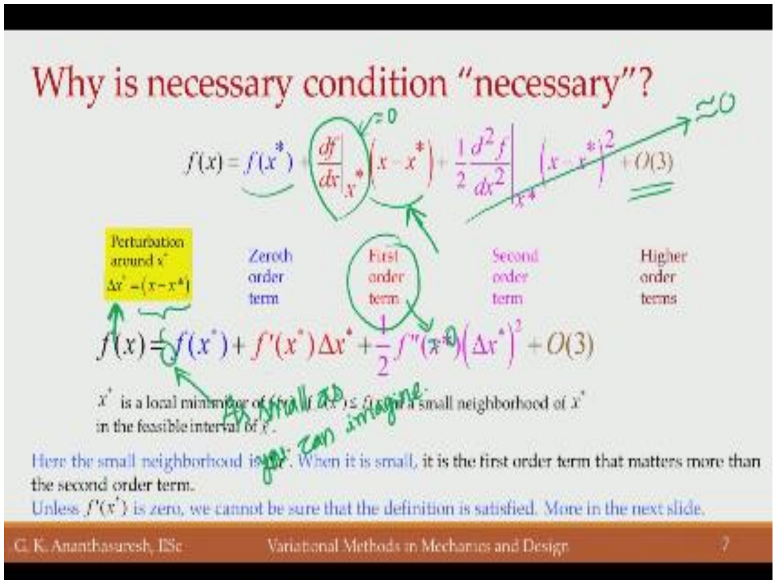
So let us ask this question why is the necessary condition necessary first and then we will say why is it not sufficient first land understand necessary condition being necessary so for that let us consider Taylor series expansion of the function f of X around this point X star so that we will say this necessary condition is actually necessary okay how do we do the Taylor series if you remember so we have if I want to expand the function f of X around a value X star okay so f of X star we know the value of the function using that information.

You would like to approximate the function expand the for not approximately equal to now so I will have several terms right first term what we call zero order term and then we will take the derivative of this F at day with respect to X and evaluated at the point around which we are expanding and then multiply by the distance of that X from X star okay and then the sec this is a zero order term this is zero order term okay.

This is zero order term this is the zero order okay and this is first order there is second order also where you take derivative with respect to X twice evaluate at X star and then we'll be square of this distance X minus X star square and there will be more and more so this thing is second order third order and so forth if you write up to antiradar as, as many times as you want your function will be more and more closer to what to be for f of X so X minus X star should be small if you say I know the value of F at X star if you want estimate for f of X.

If you know the first derivative and second derivative you will get some approximation we call it first order approximation means that if only up to this point you take that is a first order approximation if you take up to this we will call second order approximation and so forth that is you are approximating the value of the function using the derivatives of the function at a given point okay this is Taylor series which you all know there is also remain reform which we do not need right now okay.

(Refer Slide Time: 21:11)



So this is what we just wrote in the previous slide right there is zero order term so we have perturbed perturbing around this X star Δ X so whatever that we have X minus X star we can call it Δ X star as it is indicated there okay then we have the zero order term which is in blue color and then we have first order term which is in red color and there is a second order term which is in this magenta color and then there are higher order terms o 3 that so we write that is order3 and higher you can have third order fourth-order fifth order you take as many derivatives also when you do that okay.

Now if you look at this let us ask the question why is that necessary condition we said DF by DX at X star should be equal to 0 is necessary condition that is this thing over here should be 0 for the necessary condition why is that we said for a local minimum X star in the neighborhood, neighborhood is defined by this Δ X star okay this Δ X star that we have that is our neighborhood right X minus X star if I keep this what we called Δ earlier if I keep it around that point X star okay.

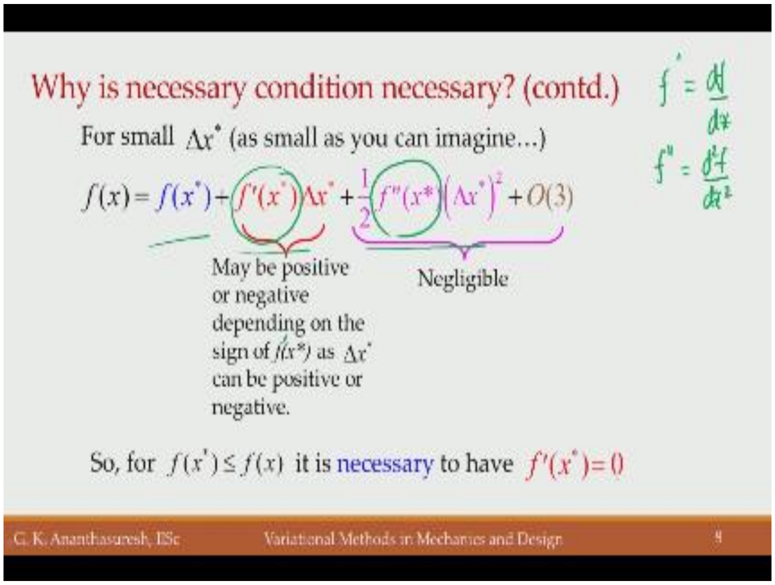
And I can make this as small as we can imagine so this one you imagine as small as possible as small as you or one can imagine okay very small in that small neighborhood we want f of X to be larger than f of X star rather f of X star should be smaller than f of X right if you look at that let us say we make this Δ very small very , very small so small that all of these okay second third everything can be assumed to be 0because second RW x minus x star square third will hex minus x star cube and so forth if Δ is very small as small as you can imagine right so if it is very small then you can neglect all of them then it boils down to only the first order term this is 0 that will be numerical value once F is known and X star is known this is a numerical value.

This is the one that varies as we make this Δ assume smaller and smaller values then it will all depend on this first order term right now we do not know the sign of DF by DX at that point X for Kim positive or negative let us say it is positive it is positive and X minus X star is also positive then f of X will be this is positive this is positive it is fine f of X is large and f of X star but what if I take x minus x star or this Δ to be negative right then f of X will be smaller than f of X star because it become f of x star minus this value which are simply be positive.

This will seem to be positive and if this value is negative then overall it is negative value function value will be smaller for some other point X as compared FX star that is not allowed because we said XR is a local minimum so because this term that is this term can be positive or negative we want this to be equal to 0 so to first order the fun value does not change if it is zero first order term goes to 0 right so as you make Δ smaller and smaller this f of X will not be different from effects star okay that is a necessary condition without that we cannot say because we are imagining Δ to be as small as one can we want.

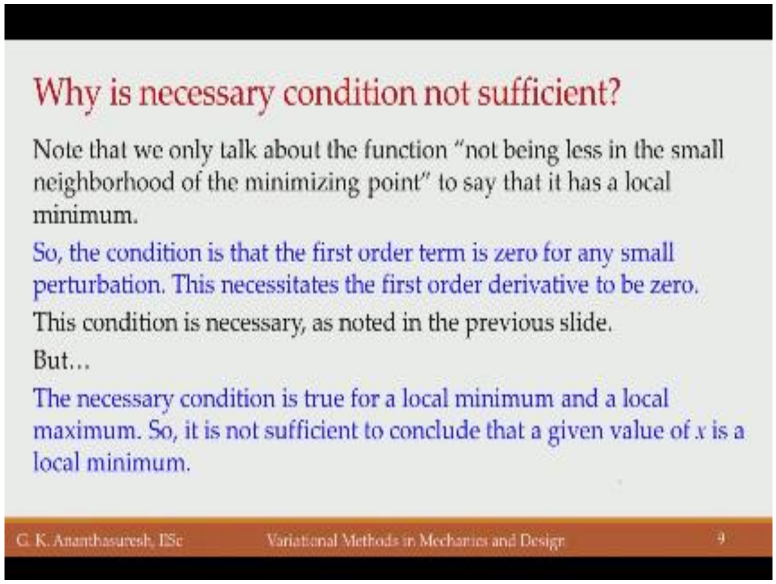
The first order term to go to zero okay there are rigorous proofs for this but this intuitive proof is good enough for this d tool that we are taking okay so that is why necessary condition is necessary all right so okay so let me erase some of it are okay it is okay I will explain what I have written over so just that some notation is there okay so let us go to the next one and then talk about notation.

(Refer Slide Time: 25:38)



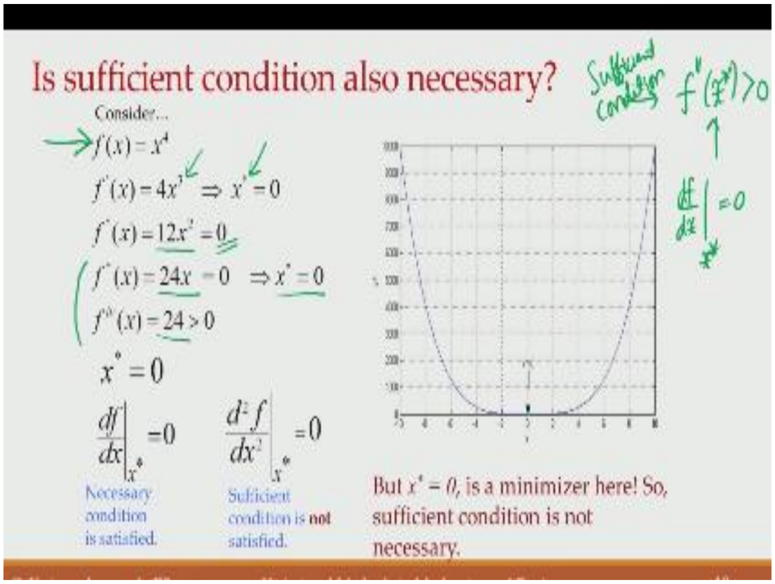
So this thing here f prime is first derivative DF by DX so when I write F prime what I mean is DF by DX similarly we have F double Prime okay that is second derivative d square f by DX square okay f of X equal to f of X star 0 the other term first order term zero order term first order term and then second order term and higher order terms right so now whatever I said you know this may be positive or negative depending on the sign f of X star sign of F prime here f prime x star that can be positive or negative right hence we say that should be equal to 0 that is unnecessary if you want to have FX starless than equal to f of X is necessary to have that equal to 0 okay.

(Refer Slide Time: 26:43)



That is our condition now why is the necessary condition not sufficient okay now we said in the small neighborhood it should be a minimum and hence the first order term should be 0 right shorter term is 0 for a minimum as well as for a maximum which we know the slope is equal to 0 for a minimum local minimum as well as local maximum in that sense necessary condition is not sufficient for a point to be local minimum okay because both maximum and minimum satisfy the same condition okay.

(Refer Slide Time: 27:22)



Now we also said sufficient condition what are sufficient condition sufficient condition is in our notation now f double prime X star should be greater than 0 when I say X star that automatically means that DF by DX evaluated at X star is equal to 0 okay at that point we want to be greater than0 right now this is a sufficient condition right that is a sufficient condition that sufficient condition is not necessary that should be greater than 0 is not necessary we can understand that with a simple example so consider this f of X to be X raise to 4now f prime x is 4x cubed right take derivative respect to X now account to necessary condition that should be 0 so4 x cube equal to 0 gives you that X to be 0.

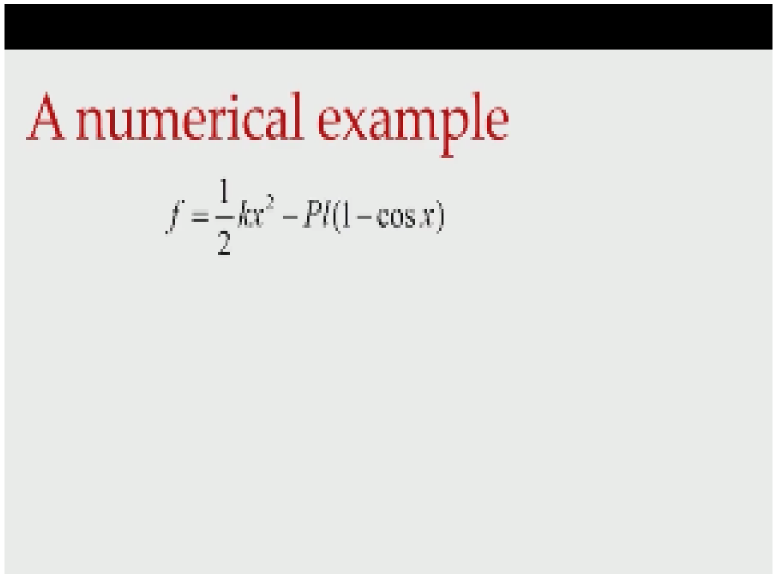
So that is that is our X star now consider second derivative f double prime which is 12 x square substitute x star equal to 0 there you get 0 right it is not greater than 0 right but look at this thing here all right this has definitely local minimum over there right yet our sufficient is not satisfied right so suffer condition now we see is not necessary if sufficient where it is satisfied but it is not always need okay so it is not necessary now for this problem you take third derivative which is 24 x that is equal to 0 also gives you x star equal to 0 you take fourth derivative that is 24 that is positive.

So you have to go for the pair of odd and even derivatives we're even derivative is not 0 to conclude whether it is a minimum or a maximum or that inflection point okay anyway with this argument we understood necessary condition is not sufficient because that can be maximum or a minimum or inflection point likewise efficient condition is not necessary based on this example all of these can be rigorously proved but here we are getting intuitive understanding because it's a detour that we are taking right.

(Refer Slide Time: 30:01)

So now we have to understand the terms which simply mean what they imply an English necessary and sufficient but we should not be confused about these two terms we should be really clear necessary is absolutely necessary sufficient is just sufficient okay now from definitions we went to operationally useful definition or a condition right so given a minimum minimizing point and a function we can verify now by substituting at the same time we can also solve for it okay.

(Refer Slide Time: 30:35)



We will solve this numerical example when you continue the next lecture okay thank you.