Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online certification Course

Hello we looked at calculation problems in geometry now we look at that in mechanics.

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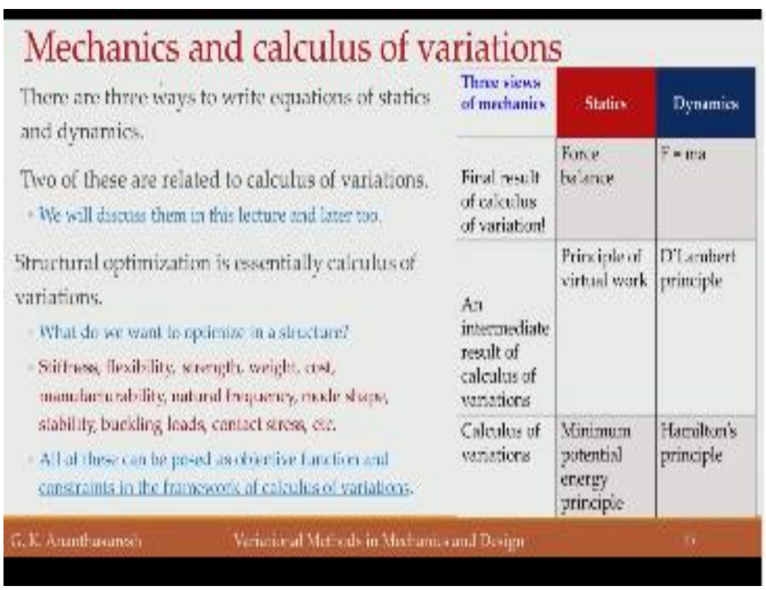

So we already discussed there are three ways to look at mechanic static and dynamic equilibrium not just force balance and Newton's second law other ways today we look at how these things come about from calculus of variations okay.

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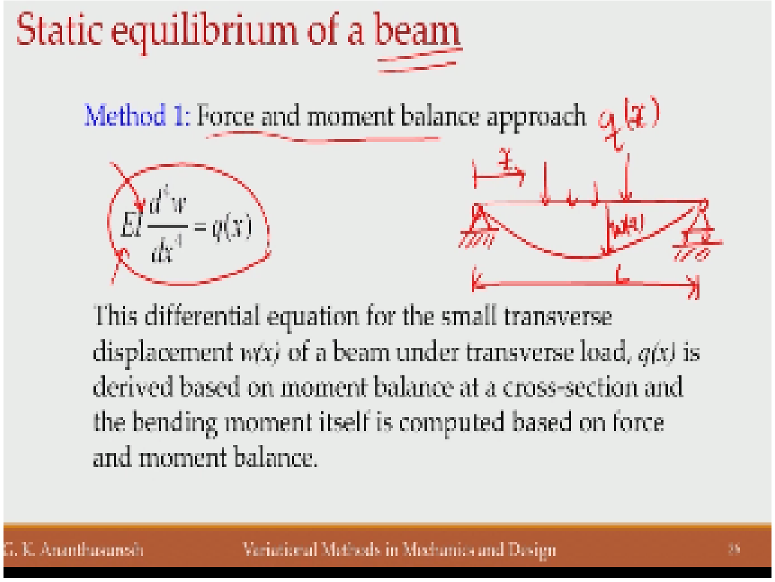

Now let us look at a mechanic's problem which is a beam let me get a pen it is going back okay let us look at a beam problem okay a beam that you are all familiar when you have a beam let us say I take a beam simply supported so there is a hinge there is a hinge now if there is a force acting on it our forces does not matter any number of forces okay it is going to deform in some way right.

So now we call this function W of X transverse displacement where X goes like this X equal to 0 to the L so let us say this total length is L, now this beam can take a number of shapes as deformation but it takes a particular one which satisfies this equation which is a differential equation where the loading is Q of X and there is Fang's modulus e there a second moment of area I and W is our unknown function which is the deformation of the beam.

And this one from your engineering mechanics courses you would have gotten it by doing either force or moment balance okay that is a method one that we all learn that when forces are balanced you get static condition now we will pose this problem differently using what we will call method two.

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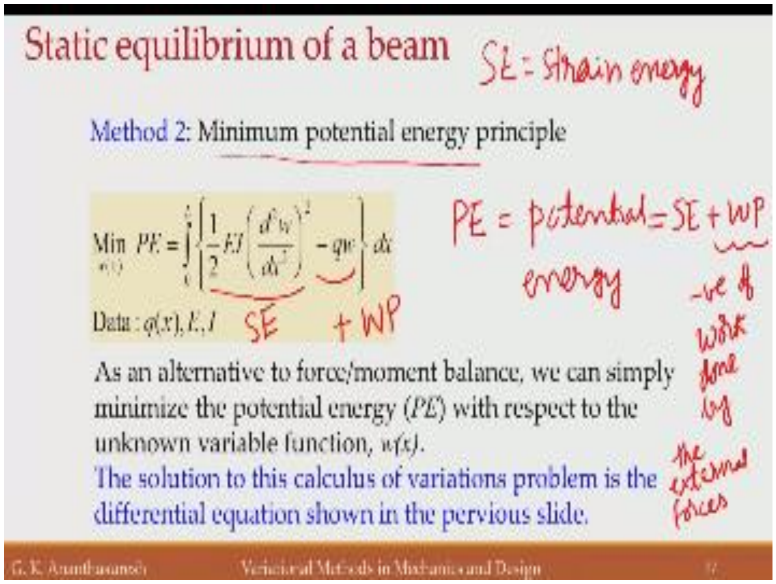

Which is minimum potential energy principle so what does that mean we have something called potential energy which we write pe4 as is potential energy potential energy okay this potential energy is a combination of two things and one is the strain energy ok this is the what we call denote by SE which is the SE is strain energy that is the deformation energy stored in an elastic body strain energy and then we have minus work done by external forces and normally we write it as plus work potential okay.

So PE equal to potential energy which is equal to strain energy plus work potential where this work potential is the negative of work done by only external forces work done by the external forces okay not internal forces and that is the important distinction here we do not need to worry about forces that are inside an internal forces we only look at external forces and how much work they have done negative sign that becomes work potential strain energy which is basically area under the stress strain curve for a body okay.

That in the case of a beam is given by this expression and we get a functional as you can see from 0 to L that is 0-2length of the beam L we have a functional which is 10 energy functional and then work potential functional both are put together if minimize foundation with respect to W and that gives you the star equilibrium condition which we had in the previous slide so this is obtained with force and moment balance.

Whereas we can get this same equation by minimizing the foreign energy with respect to W of X this is what me no potential principle says in other words while the beam has a number of okay so let us draw some of them this beam can actually do form like this right instead it is deforming a particular one because that particular one minimizes the potential energy as you change w of x this expression is going to change this w here and secondary to of w here right.

There is w here the secondary wand as you change w the value of our energy is going to change that w will be chosen by the beam which minimizes this potential energy that is what this second method says.

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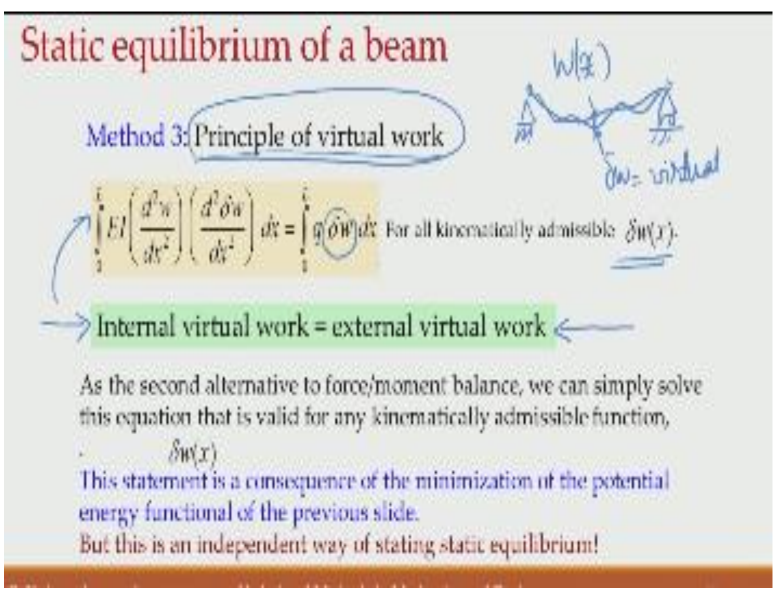

There is a third method now first one was force and moment balance second is minimum potential energy third one is what we call principle of virtual work virtual work is not real work as it is implied in its name what it says is what is put here in words okay simply says that internal virtual work is equal to external virtual both what is internal virtual work it is the work done by the internal forces when you imagine a virtual displacement okay.

What you do is let us say we have a beam let us say that our beam that is pinned at both ends and fixed at one end and then what we call simply supported condition is a slider condition there let us say this beam has taken this particular shape under some loading okay now what you do is you imagine a virtual displacement around it okay small perturbation what we call variation earlier okay this color has not changed okay a small perturbation around it okay.

So that is the virtual displacement that is distance between whatever that is thereto here that is our virtual displacement which we are denoting as λ W okay so what will displace meter vertically that thing okay so this is the virtual displacement it is virtual meaning it is not real you just imagine is a thought experiment you know that the beam has already achieved equilibrium what do you want to find what that is so imagine that around that the virtual displacement then you calculate the internal virtual work and external virtual work.

Internal virtual work is the work done by the internal forces over the virtual displacements and external virtual work is the external forces work over these virtual displacements now when these two are equal that that is what this thing says principle of virtual work and that also gives you an equilibrium equation which looks like that okay which says that this side is in internal virtual work this is external virtual work and this is true for any λ W X okay.

That is a perturbation that we considered right the virtual display something to ewe imagine that can be arbitrary that means our imagine can be arbitrary for any of them if this holds good then the W that we have taken the original W of X that we have taken that is the static equilibrium solution so that is what this principle of virtual work says okay, it is a very profound principal and Bernoulli was the one who thought of this way of doing than force and moment balance and later sure other people have developed.

So this statement which is very simple internal virtual work every external virtual work gives you the equilibrium question which is same as what we have written over here with the force balance or what we will get if you were to minimize g or what we will get if we solve this problem such that λ w is arbitrary even then this equation holds that is internal virtual work is equal to external virtual work so these are very important consequence of calculus of variations okay.

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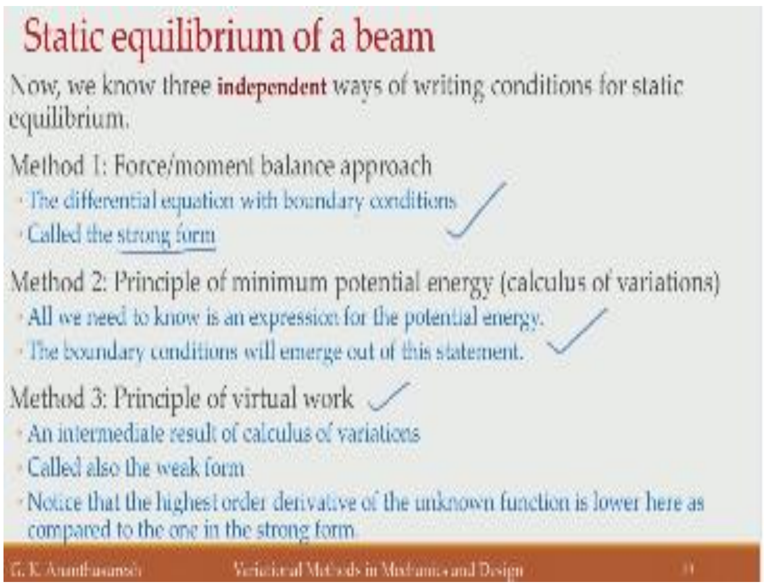

Again to summarize there are three methods force and moment balance and then principle of minimum potential energy and then the principle of virtual work normally when you do this with the first method you it is called the strong form okay, strong for meaning that the differential equation that you get for static equilibrium will be in the strong form meaning that you have to use you will have higher derivatives than in the other forms may higher derivatives you require certain strong conditions on the variables involved.

You say that it has to be differentiable twice differentiable four times in the case of a beam if you recall let us go back a few slides.

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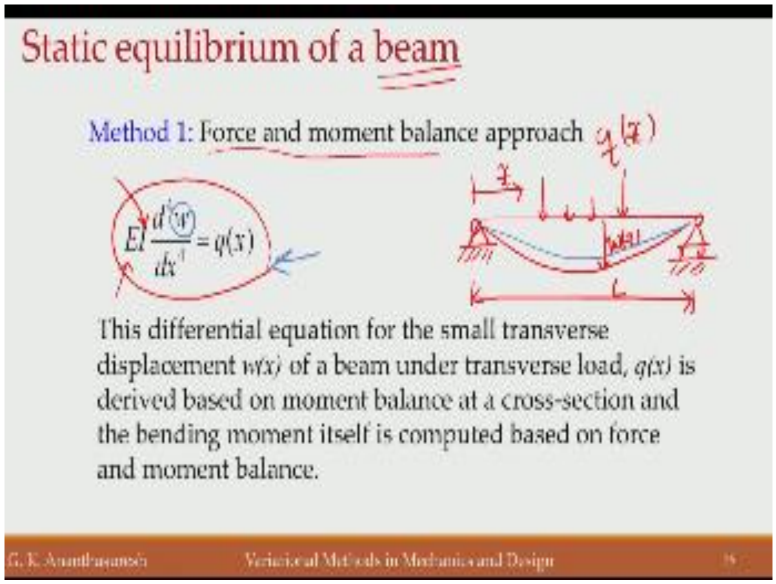

Look at this w over here we are taking for derivatives due to the4 W by D X to the 4 that means that this w has to be differentiable four times okay.

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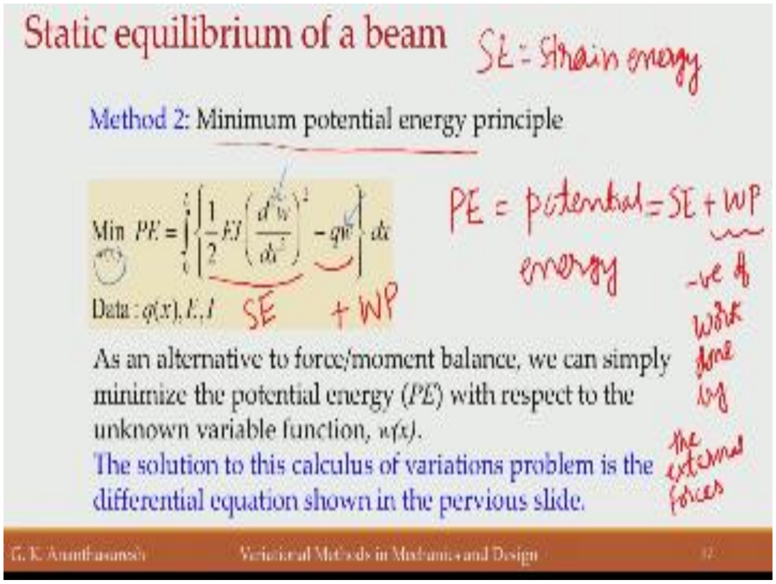

Whereas if you look at the petitioner G you have only d square w DX square only twice and if you look at the third method that is also only twice so as supposed to the weaker conditions of lower equilibrium which we call the weak form here we have it in strong form and you need to have boundary conditions also along with that when you are doing force and moment balance whereas those boundary conditions will come as part of the solution in the other two methods.

That is a principle of min operation energy as well as principle of virtual work okay the weak and strong once again are a matter of the order of the derivatives that exist in thee xpressions involved.

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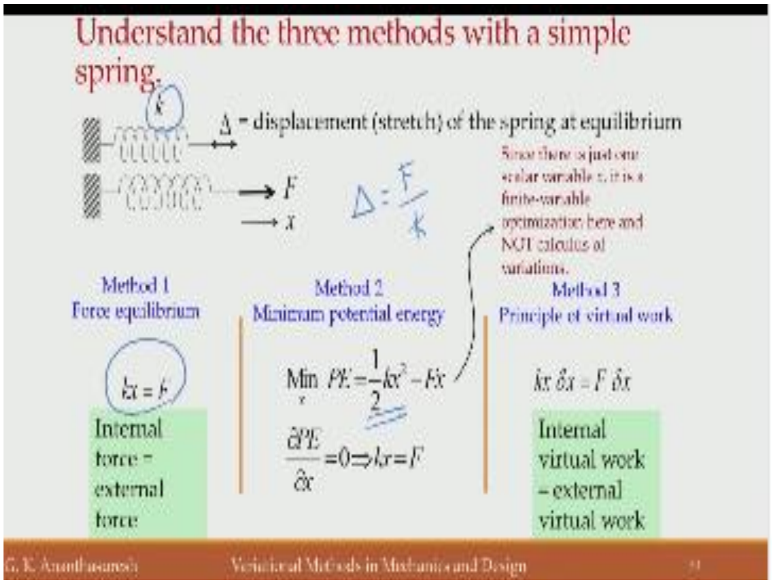

In order to understand the three methods let us take the simplest problem that one can imagine elasticity which is basically a spring a spring of spring constant K and we apply a force F and it is going to undergo a displacement λ everybody knows answer for this is that λ that we have our X that we have an equilibrium is f / K so basically if I call this delta and unknown X this is what you have that is basically balancing force external force F is equal to internal force applied by the spring when you pull it has to pull you back that is KX is a force balance.

Internal force equal to external force that is what is the method one that we saw in the case of a beam now method to method 2 is a minimization of the potential energy so potential energy if you recall is strain energy that is 1/2 KX^2 when that is displacement X of the spring and then there is negative of the work done by the external forces which you call work potential that is f times X okay that is been that is the potential energy when you minimize it we are take derivative of that with respect to X that gives you half KX square gives you k x and minus FX gives you minus F which other side you get KX equal to F which is same as what we had with the force balance.

Here we do not do force balance really we rather minimize petitioner do is equivalent to balance in the forces for the necessary condition of minimizing potential energy this is what is method to which we saw for the case of a, a beam in the by using unknown function WX now there is method 3 which is a principle of virtual work what we do is we already know that KX equal to F there is internal force external force or balance now we imagine that this spring has a little let us say it has come here but as it moves a little bit this side okay that is ΔX so after already from here to here we have X right now we move by additional amount which is Δ X this additional Δ X that is from here to here is purely imaginary that is it is not real.

It is virtual that is the thing so what is the principle of virtual work say it says that internal virtual work is equal to external virtual work this is already F we match that there is Δ X the external was a force does external virtual work f times Δ X and internal force KX does k x times Δ X this what it does now Δ X is arbitrary okay Δ X is arbitrary that is you can imagine any displacement that you wish and for all of that this should be this should hold good that gives you again KX equal to F because if it is arbitrary then k x must be equal to F and that gives you the same thing right.

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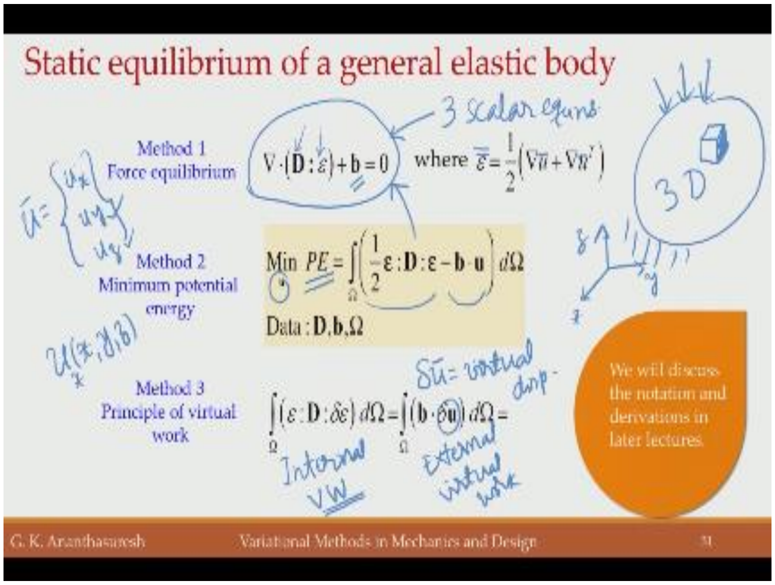

So the three methods give you the same equation e chickens use to solve method one force and balance method to manipulation energy method to the principle of virtual work now we considered a beam and then we can state a simple spring to understand the three methods now we can jump to make a general elastic body let us say I take the continuum mechanics potato fix some place and there are some boundary forces which are called tractions there could be body forces let such as gravity or centrifugal force and so forth but a 3D body okay.

Continue okay it is not a be more plate abstraction for that for those of you are familiar with the notation this is the differential equation okay where del dot we have D which is the stress strain matrix and then we have the strain Saudi basically use you stress when you multiply with strain okay in the tensor notation we put this way D colon epsilon that gives you stress together and divergence of stress del dot Sigma plus the body for such as gravity centrifugal force and so forth which act at every point in the body that is a force field team okay.

If you were to take a differentially small element a little cube here I am drawing a very big cube here but imagine this cube to be as small as one can imagine and then look at various forces that act what are the forces external forces be that acts on it and then you have stresses, stresses x corresponding areas or which let us say you take the stress acting on it normal and shear stress what total force will be there like wise on all the six faces of the cube when you balance it out you get this vector equation there are three scalar equations here okay.

Three scalar equations because you have that many function that you need to solve that is at every point there will be U which is like a vector actually sir vector and this will be the tensor strain tensor that U when I put like a vector what I really mean is that it has three components okay there is you x and then UY and then you said all of these will be functions of X Y Z because we have taken a three dimensional system if I say this is x y and z so U that we have written here that u is a function of UX is a function of XYZ and so is UY and so is UZ okay.

And in order to solve for the three unknown functions ux uy uz you need three equations which are over here that comes from the force equilibrium okay if we take small cube into the force field that is what you get and the strain is given by this expression for linear small strain but the same holds for if you have a nonlinear strain here you can get a force equilibrium which will look like this okay with small changes in the stress and strain corresponding stress that you need to define okay that is one way to do it for a general continuum you can also do much simpler by simply minimizing potential energy where the strain energy for the 3d body in tensile notation is written like that.

And what potential of course is just body force vector dotted with displacement vector that is that u that we have here is same as that okay minimize potential energy it will find that vector function you with a three component you xuy and use that when you minimize the respect to these three the potential energy you get back or differential equation are you can go to the method 3 where you can write the internal virtual work which is given by this, this is internal virtual work and this is the external virtual work external what you will work which I have indicated two letters VW so this should be true for arbitrary virtual displacement so Δ W Δ u here is the virtual displacement that is imaginary displacement consisting of three components if you has you x UI user the Δ you will have Δ u X Δ u Y Δ user okay.

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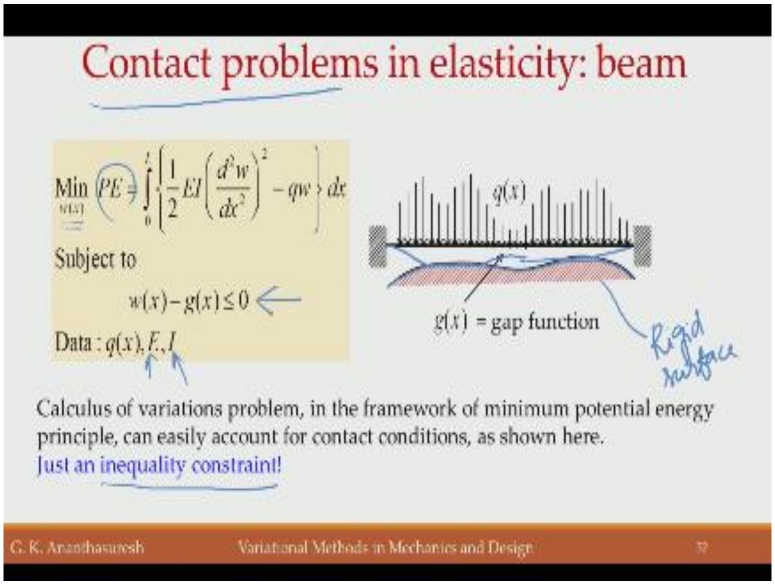

See one for general elastic solid you can write the starting equilibrium in three different ways okay and in mechanics you can a solve number of problems one of them is that is a contact problem let us say I have abeam fixed on both sides there is a loading and below that there is a rigid structure so there is a rigid structure of certain shape okay rigid surface I can call it rigid surface that is this boundary okay now there is load applied this beam will deform and might contact this surface at different points depending how much load you are applying and then deform like this where the contact acts are contact happens you do not know a priori then what you do is to put a constraint like the Chatterer's problem we had a constraint that curve has to lie below the river equation here we are trying to minimize the potential energy with respect to W of X.

But we are also including a constraint that W of X should be less than GFX whatever what we take that the gap that we have GFX is here it should be within that way that the gap function is defined by the GF X here what your deformation cannot exceed the gap that is what we have here as a contact constraint now when you minimize it you have to also put a constraint here like this but that is a calculator variation because unknown is the W of X okay. And the given loading Q of X and material property is known second moment of area is known and so forth okay now we have an inequality constraint referring to the contact condition.

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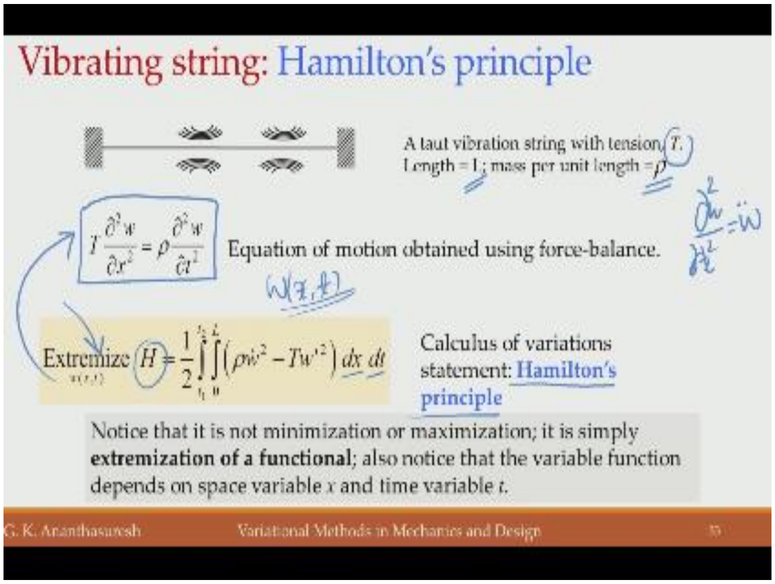

Now moving on to dynamics we discuss the statics mechanics now mechanics also has dynamics where things are not at rest they are moving okay if we take a vibrating string again a simplest problem on it one can think of in dynamics a string that is kept taught between two points let us say the spring length is l and the mass per unit length like mass density is denoted by Rho and there is a tension T here.

Then in order to find the dynamic equation which is shown here this basically comes by equating the forces that are acting when this is vibrating is free vibration but there is going to be inertia force that is Rho W double dot so this ∂ square W by ∂ T Square is w double dot ok Rho W double dot is inertia force is equal to the stiffness force. That is there T into ∂ square W by ∂ X square okay so this is the force balance but you can get it in a different way by using what is known as Hamilton principle where we have something called an action integral.

Which is denoted sometime by A center by H where there is a functional if you notice this functional has two variables X and T we are indeed with respect to X and T because our function. w here is a function of X and T so we have an extremist functional in two variables X and T which is given as Rho W dot square minus TW prime square integrate from 0 to L over the space and t1 to t2 over time if you extremis it again notice the word. That we are not saying it is minimum or maximum which is saying we are eternizing we are getting optimal but we do not insist on that being a minimum or a maximum okay when extreme eyes is functional you get back the equation that you would get when you do force balance inertia forces brought in here along with the stiffness forces and damping if may have and that is called the Hamilton's principle.

So w(x,t) which this vibrating string takes it takes that particular shape which extremisms this functional that is what is Hamilton principle says and this is very analytical because you do not have diagrams anymore in fact you do not need diagrams. We just write down strain energy, potential energy, kinetic energy all of those energies and write this functional and say extremist to find w(x,t) function and you get the differential equation to solve for.

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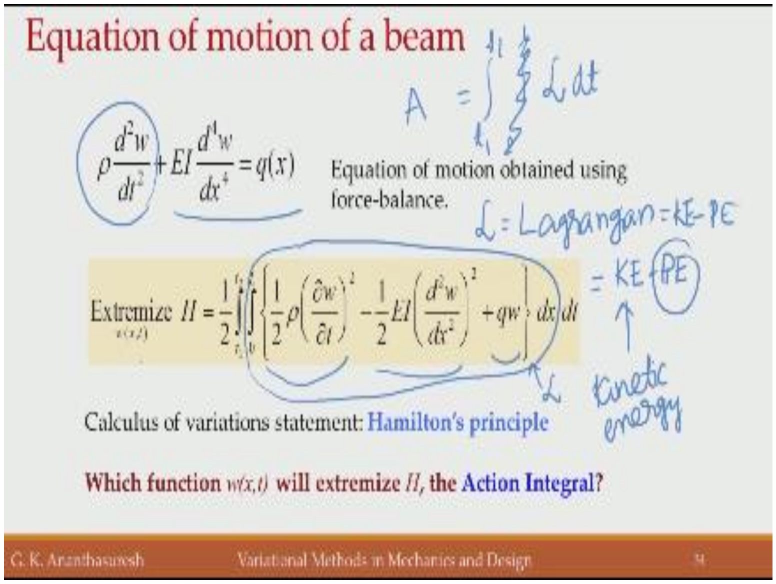

In the case of a beam if you earlier we had only these two terms now we have the inertia force also ρ(ẅ) rd^2/dt^2 okay, what is the function that we minimize or extremist there in addition to what we had earlier that is the strain energy and the work potential now we have kinetic energy also okay, so here what we have inside what we have we have 0 to l and t_1 to t_2 what we have inside actually there is a name for it called lagrangian okay, this L is called lagrangian this other kind of Lagrangian optimization will come to that later.

But now this Lagrangian is kinetic energy minus potential energy, okay so this ke is kinetic energy okay, potential energy we have already discussed, but notice that it is kinetic energy minus potential energy that is what we have that is why potential energy minus and this was minus of minus that is why became plus here we have kinetic energy. So this Lagrangian if you take integral over space okay and time so l itself will have the time part of it here this l that we write that involves already in fact I need not write this here.

Because L the way we write it will already include this integral over space okay, that is l what I have encircled now is your Lagrangian and then this t_1 to t_2 LTD is what is this action integral or what we call the Hamiltonian sometimes, if you extremist this action integrate with respect W(x,t) then you get the equation of dynamic equilibrium or what we call Newton second law, okay.

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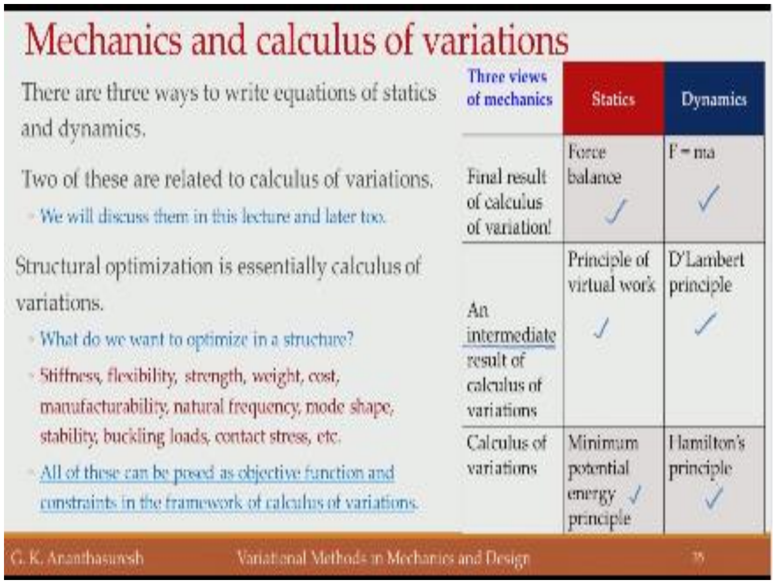

So now to summarize one more time we have force balance and Newton's second law we have principle of virtual work and we have not discussed this D'lambertprinciple in the framework alpha variation is the intermediate result which we will see later okay, and we discuss static case minimum range of principle then Hamilton principle which says we have to extremist what is called an action integral which is t_1 to t_2 LDT where L is kinetic nergy minus potential energy it is called lagrangian, okay.

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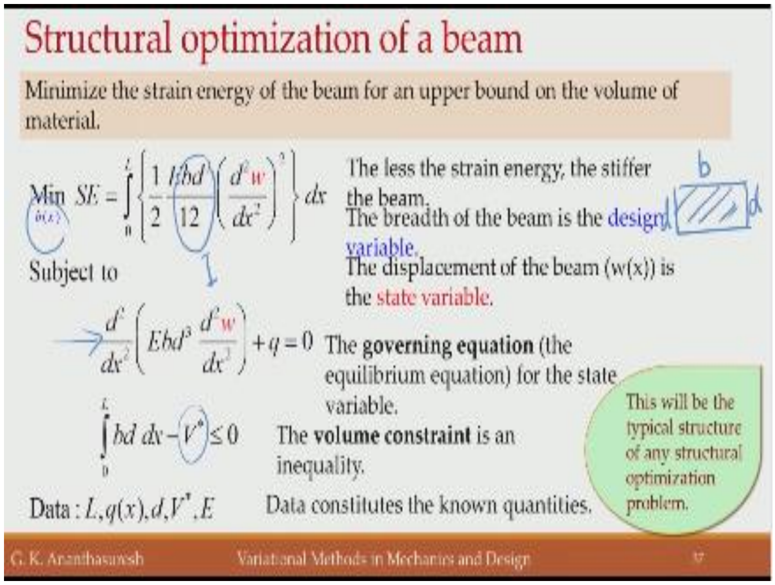

Now moving on to design or structural optimization okay, structural optimization is structural design that in this class we are focusing on mechanics as well as design using variational methods okay, when you think of a structure there are a number of things that we might want to optimize for you want to minimize the weight or volume of material or cost and stiffness you want to maximize most of the time you definitely want to maximize strength sometimes you want to have flexibility of to some extent and he is definitely want to be structures to be stable.

But dynamic considerations natural frequency mode shape will be there all of these things you have and you have to define your variables okay, the variables which define the geometry of your structure, okay. All of these can also be posed as calculation problems we look at a few examples now, here is a problem of minimize the strain energy of a beam with an upper bound of the volume of material that is you are given a beam let us say of rectangular cross-section so if I take a beam of rectangular cross section that is a width is b and depth is d okay, these are a rectangular cross-section and if I integrate rectangular cross-section 0 to l we are not saying that even though is rectangular cross-section the b and d here okay, this is the d, b and d are not the same along the beam.

So it can vary okay, sometimes constraint in profile can be small and somebody can be very large and then small again and so forth, okay. So if I integrate that bd over the distance 0 to l that you see the volume there should be less than or equal to that v* that is given you are given certain amount of material volume of material that is our resource within that you have to distribute along the length 0 to l sometimes you can make it bigger area of cross sections volume across section and so forth you change it.

And you would you can use either b or d in this case we have used in we have used b(x) that is the width profile of the beam as a variable we want to minimize the strain energy to make the stiffest beam and the strain energy is written here in terms of EI you recognize rectangular section $bd^3/ 12$ is second moment of area are called I and then $(d^2w/dx^2)^2$ will be involved in this strain energy expression and then w that we use here it has to be satisfied by it has satisfied governing equation which is equilibrium equation which is written there and then you have volume constraint and length is given loading q(x) is given, d is given depth of the beam and then v* and v* is a volume that is a load upper bound on the volume material that you can use and the X modulus, okay.

So with all this data we have to solve this problem as you can see it is a calculation problem because you have a functional which in integral and there is a differential equation constraint another functional constrain there on the volume of material.

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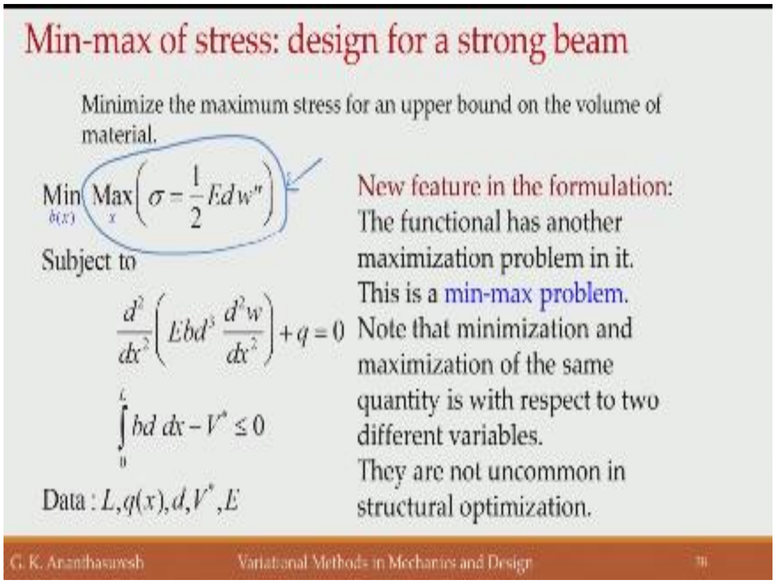

Let us say you want to worry about strength then you want to minimize the maximum stress that is there in a beam then you would say minimize maximum stress maximum over x that is 0 to l that you have the beam anywhere the maximum stress can be reached depending on the load and boundary condition. Now you want to minimize that maximum stress by adjusting the b(x) which is the width profile here, okay.

Then again we have the a functional now if you look at this, this is also a functional it is not a integrates not integral now but is a functional nevertheless okay, so in this lecture you have to focus on the types of functionals that is a different type of functional maximum over the spatial domain a quantity in this case it is stress and then there is a governing equation and there is a resource consent form of a an integral.

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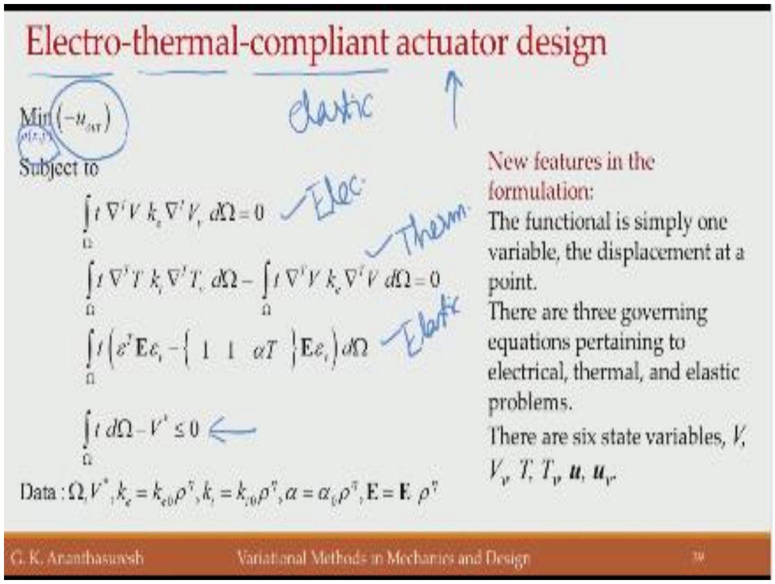

Let us consider even more complicated problem where we are considering electrical domain, thermal domain and then elastic or compliant okay, elastic or compliant domain an actuator electro thermal compliant actuator so there we like to minimize minus u out u out is output displacement, when I put minimize -u out I am basically maximizing output displacement which is what we want to do when we have an actuator I want to maximize a stroke of the actuator and there is a variable we do not have to worry about what that variable stands for now we just ρxy for a 2d problem.

There are these governing equations electrical this one is for electrical and this is for thermal this is for elastic so this is for electrical this is for thermal and this is for elastic analysis, okay. And there is a volume constraint like before this is also calculation problem. Now if you see this is also a functional that is displacement at a particular point functional can be just that also, okay.

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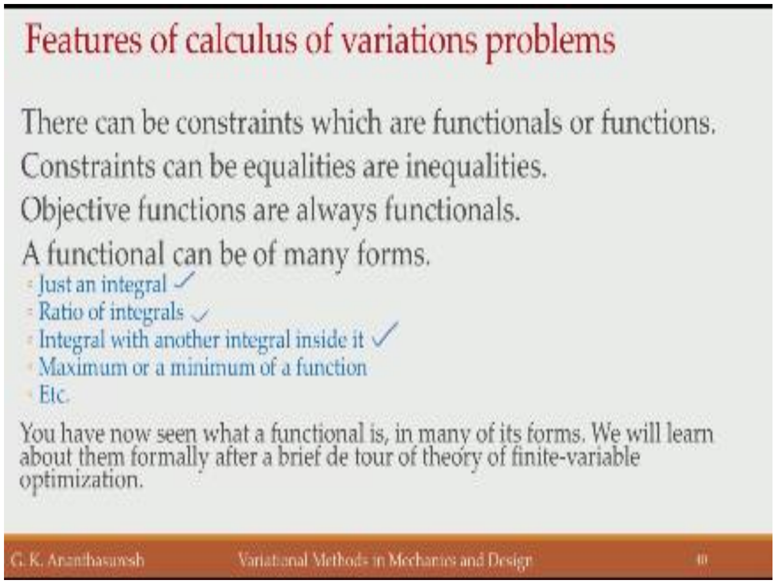

Let us look at the features of calculation problem before we end this lecture so it can be just an integral, it can be we have not considered that but can be ratio of two integrals it can be integral another integral which we came across in the case of optimal control the helicopter problem and you can be maximum, minimum of a function that you can study in the case of minimizing maximum stress in a structure, so by looking at all these examples we know how a functional looks like now.

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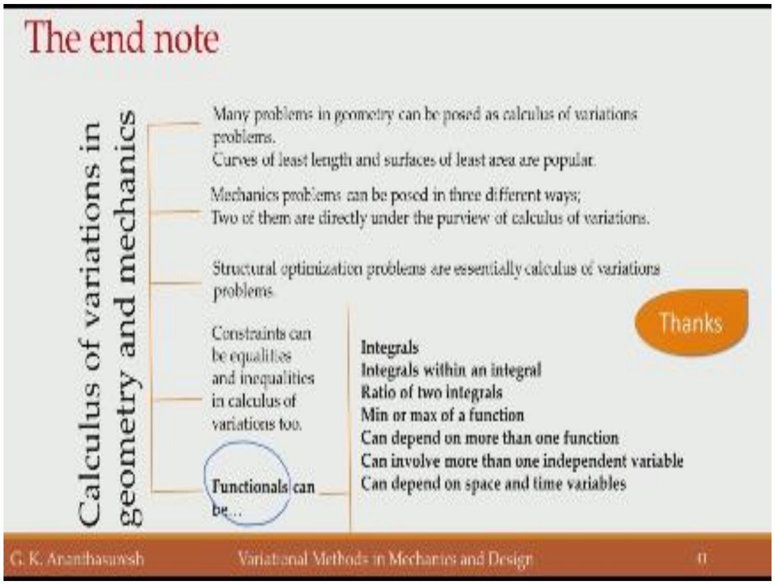

In the next, next lectures we in the subsequent lectures we will look at the nature of the functional a little bit more detail from the mathematical viewpoint. Now to summarize this lecture we have considered a number of problems both in geometry as well as in mechanics and formulated them based on what is minimized and what is to be probably maximized or minimized then you put a negative sign and then we also looked at constraints they can be constrained like a differential equation form or they can be integral form or just at one point, okay.

And again the nature of the functional a number of ways just an integral are integrals, integrals ratio of two integrals which we did not see today minimum, maximum function and it can depend on one function or more than one function like if you to 3d elasticity you have ux,uy,uz three functions that are unknowns and you can dynamics case it can depend on space and time variables and most important thing for this lecture that you need to remember is that there are three different ways to pose the static and dynamic equilibrium problems in mechanics and two of them are related to variational methods with the topic of our course, thank you.