Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello today we will talk about calculus of variations by looking at some problems which will serve as examples will not solve them today we only learn how to formulate calculus of variation problems in geometry first we will look at geometry related problems and then we look at mechanics related problems.

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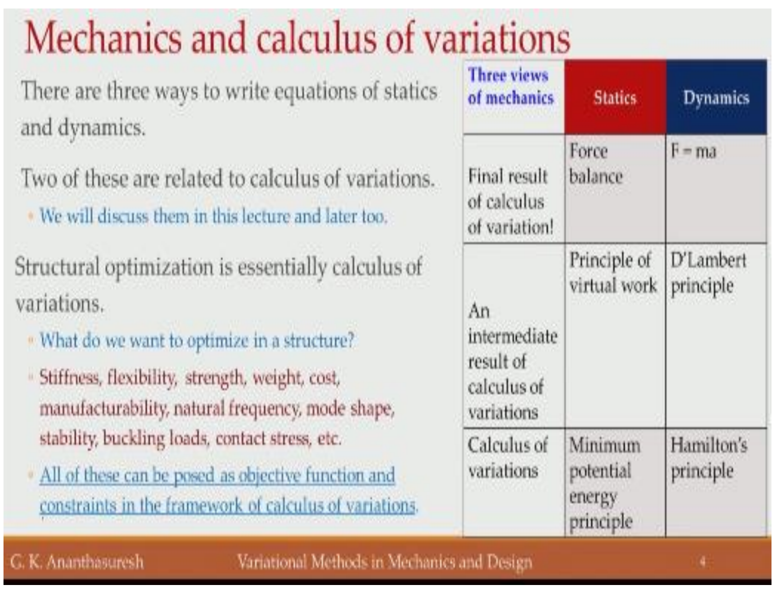

So here is the outline of the lecture first we will discuss the geometry problems where calculus of variations is convenient to post those problems there will be minimization maximization there but the unknowns will be functions which is the characteristic of calculus of variations and then we look at the role of calculus of variations both in mechanics and structural design optimization and what we will learn today is once again talk about what kinds of problems belong to calculus of variations.

We will get a few historical problems in calculus of variations in the last lecture such as a brackishtoken problem and firm ass problem for the fraction and so forth today we look at a number of other problems belong to geometry and mechanics that is what we will learn today and we will also look at how we can formulate them as the calculus of variations problem and look at the connection between mechanics and calculate variations historically how the field of mechanics a world or co-evolved with calculus of variations and then look at the connection between structural design optimization and calculus of variations.

In all of this will learn one thing which is about a functional we use that word in the last lecture also today we look at a number of functional so that we know how a functional looks like so in geometry there are a number of problems and all of them pertain to minimal or optimal curves and surfaces in the case of minimal curves are optimal curves we look at geodesics something that we had talked about in the last lecture also and the chain that hangs when you put it between two points under gravity what shape does it take that type of problems the number of lines where there are minimal surfaces such as what soap films do and a number of other things that we come across in geometry and mechanics.

So when it comes to mechanics and calculate variations there are something that we need to take note and it is very important and that is there are three ways to write equations of statics and dynamics that is static equilibrium as well as dynamic equilibrium and these are listed in the table here under statics we have something called force balance we all know that when all forces acting on a body are balanced and then the system will be at rest what we call steady state or static equilibrium and the corresponding thing and dynamics is F equal to ma Newton's second law.

These how we normally learn mechanics in high school and undergraduate courses as well but if you want to learn mechanics from an analytical viewpoint then you will realize that there are two other ways which are equally independent as the force balance and Newton's second law which will give you the same result but by different means and those are principle of virtual work for statics and D'Lambert principle for dynamics and similarly there is a third one which is a principle of minimum potential energy for statics and Hamilton principle for dynamics.

So you can take any one of these three and derive the remaining two so you can belong to different school the first school is force parallel Newton's second law second school is principle of virtual work and D'Lambert principle third school is principle of minimum potential energy and Hamilton principle so all of these is what we should appreciate in the framework of calculate variations which you can see in the first column where we have put this as calculus of variations.

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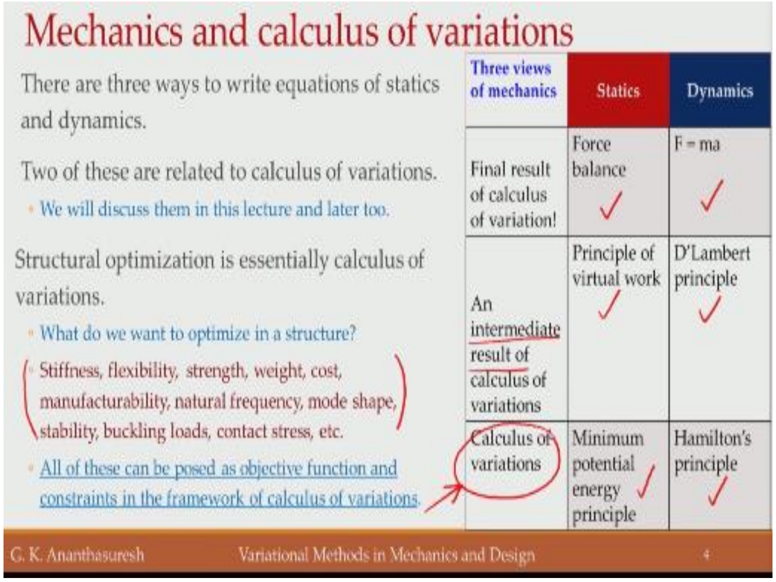

That basically gives you minimum potential in the principal and Hamilton principle and then we put something called an intermediate result and that actually leads to principle of virtual work and D'Lambert principal and the third one is a final result or rather where we write a differential equations and that gives you force balance and in the case of dynamics Newton's second law so all of this starts from calculus of variations okay and when it comes to structural design of structural optimization those problems are naturally calculus of variations problems and when we look at the unknowns it could be the shape of a particular surface or a boundary or the topology of a structure or the size profile variation of a structure and so forth.

Those are all the design variables are optimization variables and what do we optimize optimized for all of those are listed here when you think of a structure as a designer you would be interested in stiffness flexibility strength wait cost manufacturability natural frequencies mode shapes stability such as buckling loads and then contact stresses and things like that or for all of these things we can pose calculus of variations problems.

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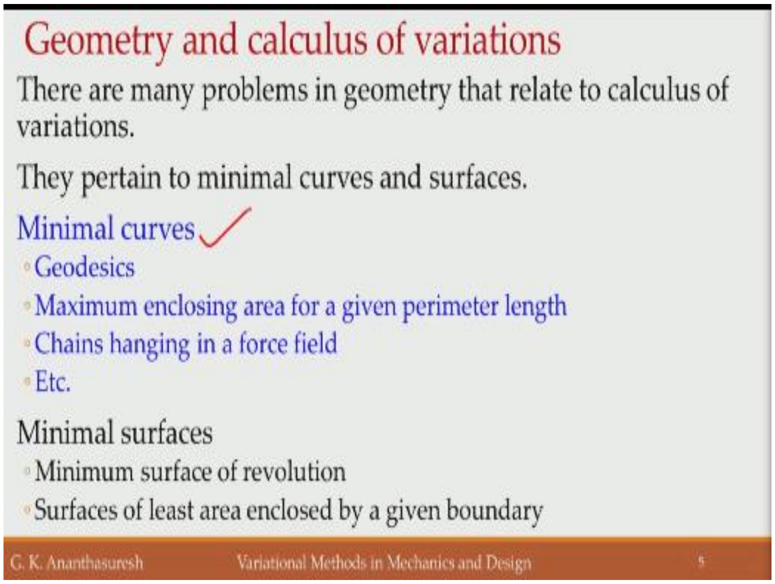

First let us look at geometry and calculus variations in particular let us look at minimal curves or optimal curves.

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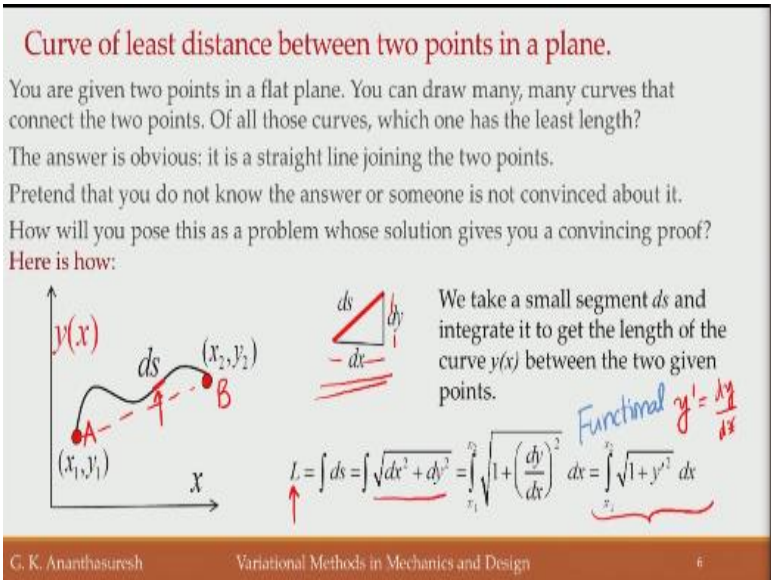

Let us start with a very simple problem for which we all know the answer and that is if you have a plane where we have indicated the x-axis and the y-axis and there are two points given let us call this point A and point B now in a plane you know that between two points if you want to find a curve of least distance the answer is the straight line joining the two points but let us pause this as a calculation problem.

And then see how the objective function of the functional looks like so for that we take a small element such as the one that is shown here call it ds and enlarge that over here so the ds and this is dx this is dx and this is dy okay so ds in that case is given by √ dx^2 + √ dy^2 we take dx out and write it as √1+ dy/ dx^2 are in our notation we use y' it is a symbol for dy / dx.

So what we get is this length of the curve or the distance of the curve between points A and B if you draw a curve like what we have shown the black here the length of that is given as an integral or what we call a functional which is from limits X 1 to X 2 √ $1+y'^2$ dx so this is called a functional so let us write that one more time so this is called not a function functional okay so if we minimize this functional that that is l is given like a functional in this case it looks like an in t it is an integral if minimize that with respect to y(x) because that is what we do not know or at least we are pretending that we do not know.

You minimize this function with respect to that function y(x) then we will get an equation whose solution will be a straight line which we know as the answer here okay let the simplest calculus of variation problem in geometry.

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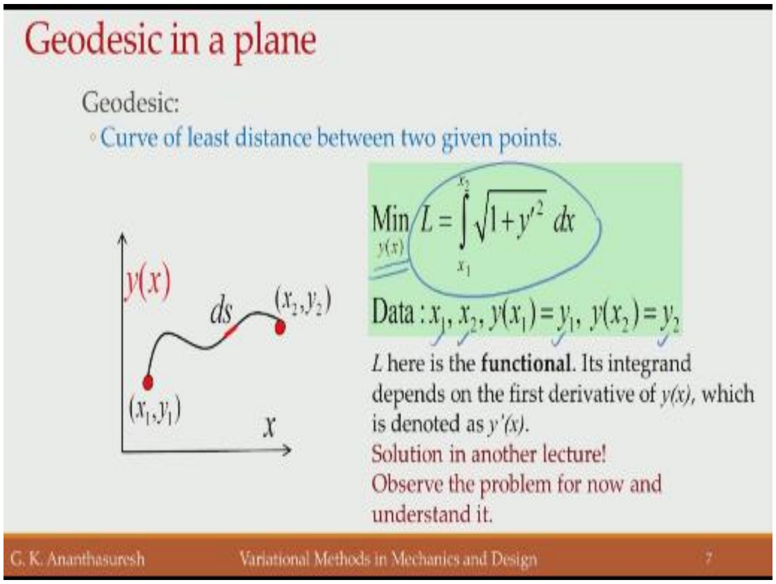

So we say is this L with respect to y(x) and what is given here the data the two coordinates x coordinate x1 x2 + y1 y2 that is this point at that point are given you need to find that curve that has the least length and that turns out to be straight line anyway we will look at solution much later let us first understand how these functional look like that should be our focus because we will consider a number of problems to look at what a functional looks like and how to formulate it.

Here formulation was quite easy there are no constraint you just minimize an integral x1 to x2 √ 1+ y’ ^{2} dx and no constraint.

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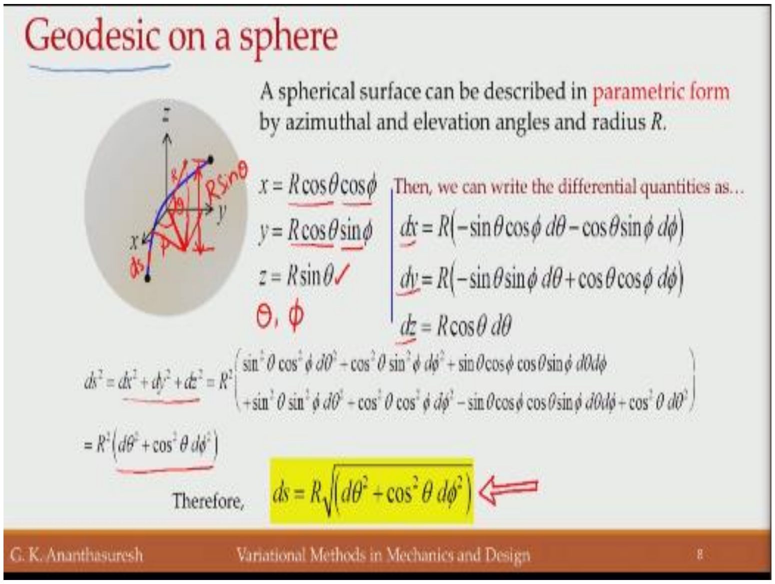

Now instead of those two points being in a flat plane if they are on a spherical surface such as what is shown here then there are no straight lines on a sphere that is what we had mentioned in the last lecture also and hence we had to find a curve which is called a geodesic is a generic term on any kind of surface you can find a geodesic now let us look at how we formulate the geodesic problems okay first we have to express curves on a sphere in parametric form meaning that when you are moving on a sphere you have two degrees of freedom rather you need two coordinates.

But now in a constraint to move on a curve you have only one variable and that is our parameter so we express first the sphere as two parameters θ and Ø and get xyz coordinates and find the differentials of those things so x equal to R cosine theta cosine π y equal to R cosine θ sine π and z equal to R sine theta so here θ will be the elevation angle Ø will be azimuthal angle meaning that if I take any point let us say a point over here let me change the color so that that blue and this blue do not mix okay.

If I have a point there if I project that point on to the xy plane I look at this so this will be the angle Ø that we are taking first of all we have this R from here to here that distance will be R and this angle will be θ what we call elevation angle θ is elevation angle as if the R that is here is risen up by rotating by an angle θ then it goes there okay so this gives the length of this to be R cosine θ and this height that is Z coordinate is R sine θ so from here to here is R sine θ which is what we have written here and then this portion will be R cosine θ and then we have to get the x and y coordinates.

So we have to look at how much is this how much is that then they will be whatever cosine θ that we had already and then I will take cosine Ø component because the Ø is the azimuthal angle and sine Ø for the y-coordinate that is how we wrote this xyz coordinates in terms of two parameters θ and π 2 axis are to denote any point on the spherical surface now what we are interested is occur with one degree oven to go along that before that we are taking differentials of all these.

So dx that we have θ and Ø variables we differentiate with respect to them and get these two terms and dy we get these two terms taking y equal to that and z R sine θ simply becomes R cosine θ d θ bigger depends only on θ z coordinate where x and y depend on θ and Ø okay now if I take any small segment on that curve that if I call that ds the ds or square of that is given by sum of all this Pythagoras as theorem applied twice and we substitute for dx, dy, dz these things over here then we get a long + times $d\theta^2$ times cosine square $\theta d Ø^2$ okay that is what we get as ds.

So the ds that we have here this is ds^2 √ R times all of this why do you need this because we are interested in finding a geodesic that is the curve of least distance so we take this expression that we have for ds and then integrate it to get the total length of any curve that is there on the spherical surface.

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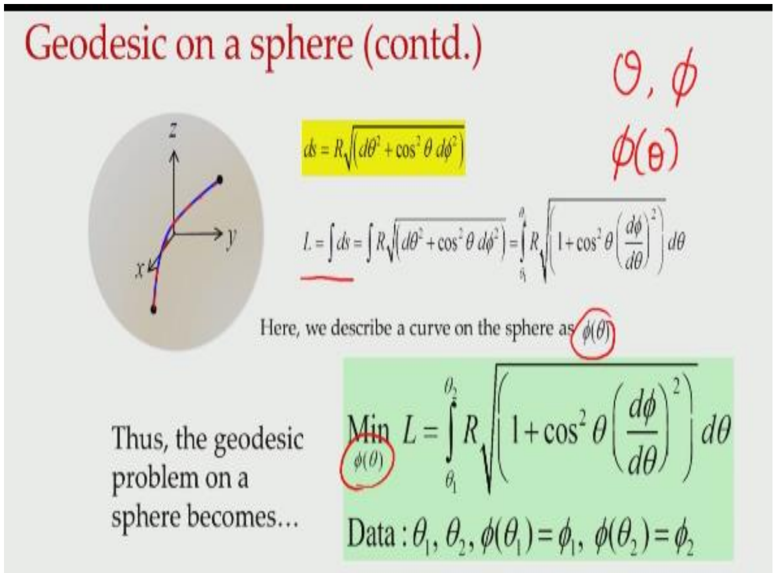

So in order to get that length will take this L and integrate it all along this okay that basically means that you go from if I take here $d\theta^2$ + cosine squared θ dp^2 we divided by dθ and take it out or rather d that is taken out that gives you 1+ cosine square θd π d θ^2 okay and this will be the length of the curve there and now if you prescribe a curve on the sphere as a function of Ø which is in terms of θ

Because we have two variables θ and Ø to reach any point on the surface of sphere if I consider that Ø is a function of θ then I have only one degree of freedom to move and that is θ okay that is when you move along a curve you only very one variable that if you take that is a θ then Ø will be a function of θ.

So that becomes our unknown function here then the integrand that we have here cosine^2 e Ø θ we put that in and try to minimize this integral subject to basically the what is given that is that I want θ 2 v1 and v2 that is store divine pointer given we need to find this function V θ to get the were scored a geodesic or minimum distance occur.

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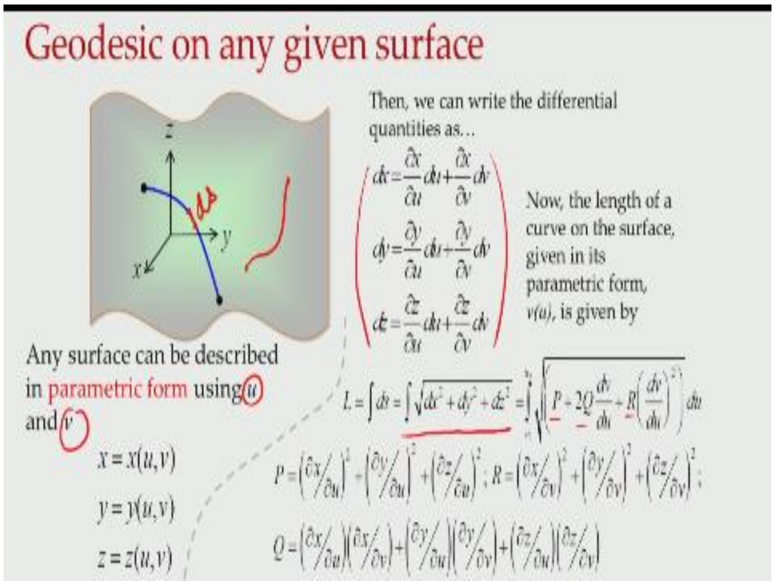

Now this geodesic actually can be drawn on any surface any three-dimensional surface here one is shown so if I take two points here and here there will be one particular curve which basically follows the geodesic or rather minimal distance curve and for a general one we can write for that also in this case just like before we had data and Ø we have taken two parameters UV because in a surface we can we have two degrees of freedom deviously an a spherical surface we took θ and Ø now we are taken UV so x is a function of U and V.

And so is why and so is that so now like before we take differentials DX ∂ux/ ∂u times Du + ∂ u X / ∂ u V times DV likewise for y and z once you are these quantities we can express that D as the small distance that you want as exactly like what we had before square root of $DX^2 + dy^2 + dz^2$ and substitute for DX dy DZ using what we have derived here if you do that you get some P Q and R if we are wondering what those are this or they look like p equal to something q equal to something nor equal to something okay.

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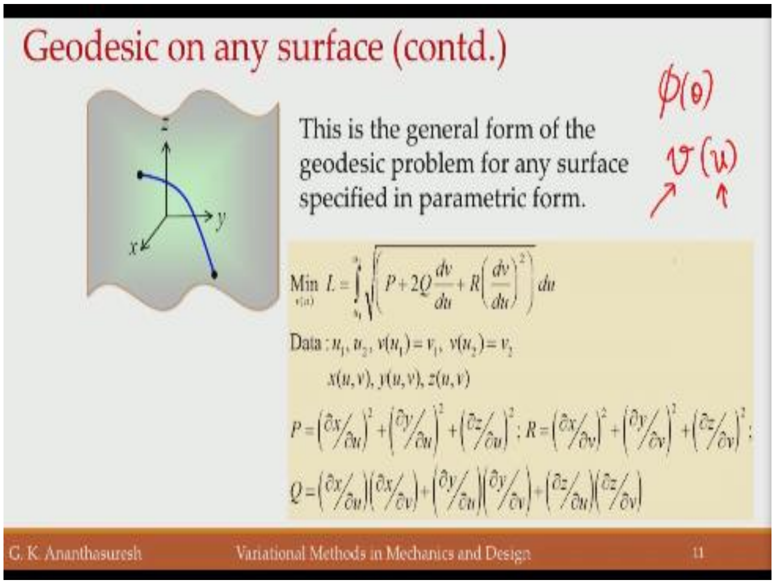

So how we get this integral now for finding the geodesic that you minimize this that is basically d s in terms of P Q and R which are known once you assume a parametric form for the surface okay then p and q and r are given as follows this what you do with respect to V as a function of you previous of a sphere we had considered Ø as a function of θ here we consider V as a function of Q because you are two degrees of freedom but you are constrained to move along a curve then one of them has to be independent other has to be made dependent here we are chosen to make you independent and V dependent on it we have to find V of you function then you would know where the points on the geodesic will be okay.

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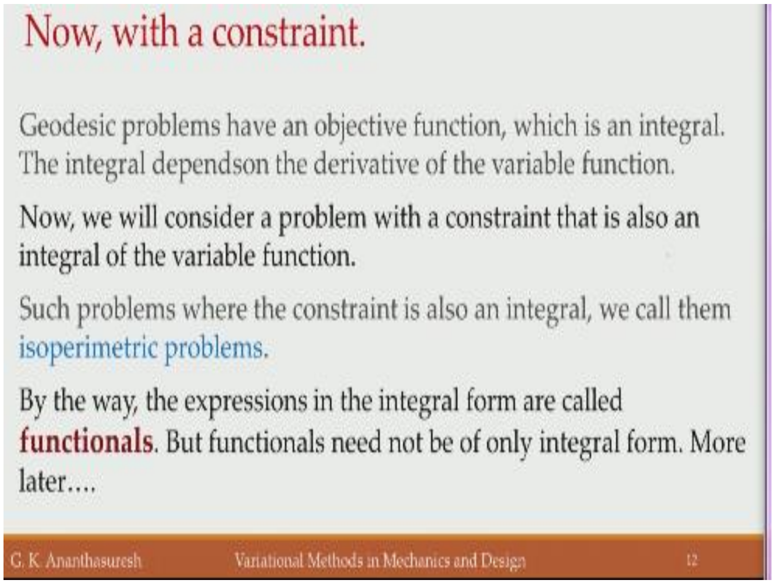

So far we did not have a constraint now we will try to have a constraint okay we look at the problems which are known as I so parametric problems mean that they have the same parameter okay now the integral form that we had we call it a functional so you know that if there is a functional as the objective function or in a constraint we call it a functional will learn more about functional later on okay.

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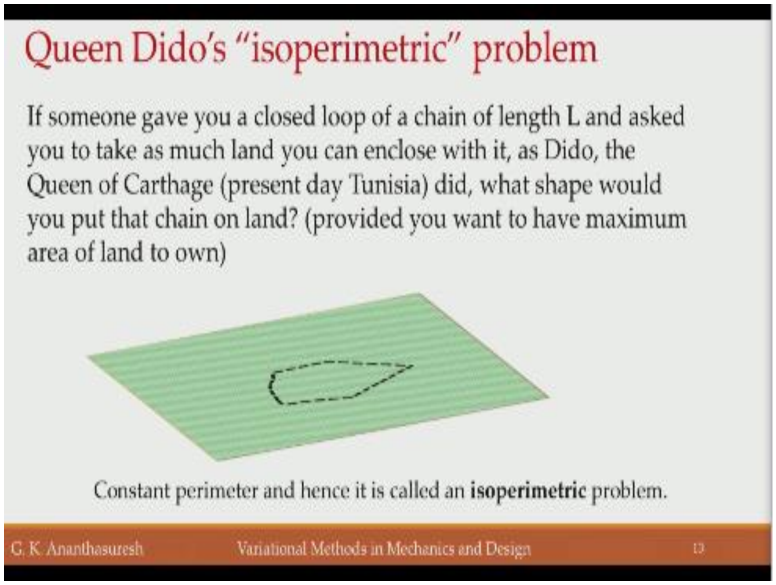

Here is a problem of calculus of variation historic problem where a constraint is also involved constraint is that the length of the curve if you are given a piece of string you need to arrange it on a plane in some fashion so that the area enclosed by it is maximum okay, now this is a classic problem called isoperimetric problem or queen gyros problem but how do you do that an answer for this at least circle. But we will learn how to pose it as a calculation problem first.

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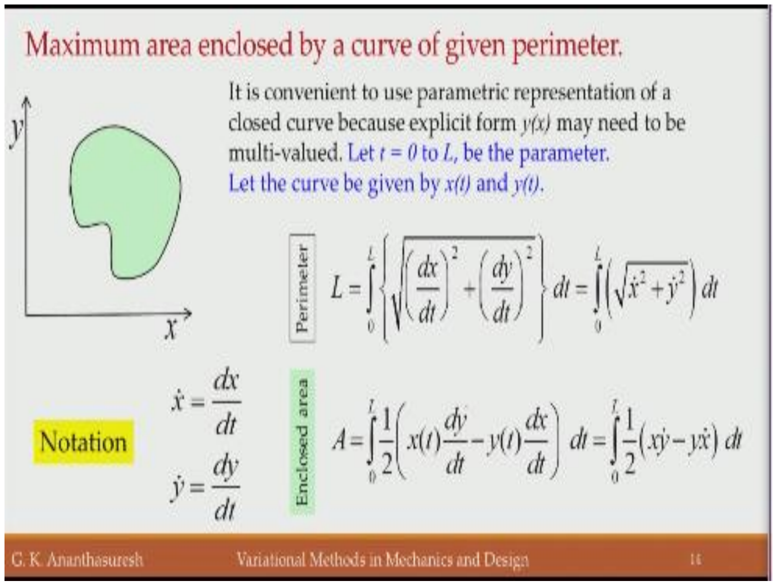

So in this case area enclosed budget by a curve which is what you see in this green one is given mathematically in this fashion okay if I take X and T as functions of a parameter T, X and Y as one functional for parameter T then the area enclosed by a curve if you give parametric form for the boundary area enclosed is given by this particular integral okay, pyramid other hand is given by this particular integral okay. That is something to remember when you have them then we can pose a problem.

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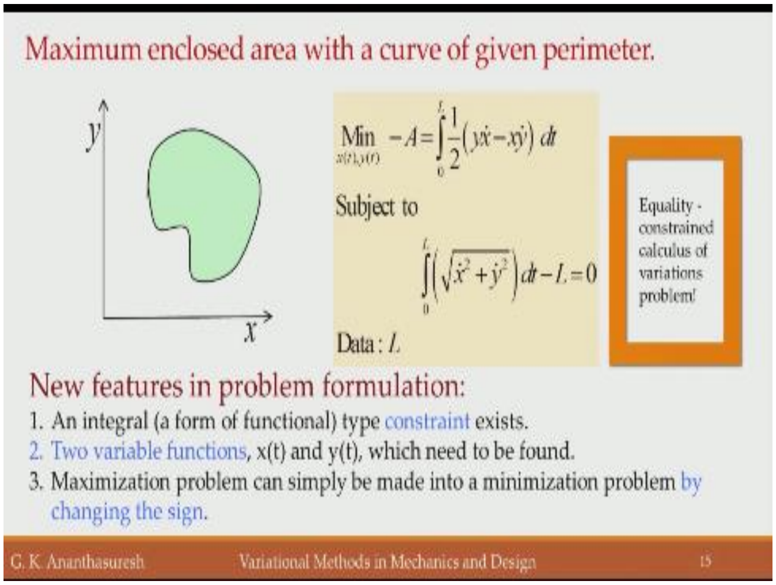

To maximize the area you notice that I put minimize negative area-A that is same as maximizing their here and what we are controlling are optimizing with respect to our these two parametric equations that define the surface X(T) and Y(T) subject of course a length constraint because that is given only a piece of length known length string is given to us we have to find Xt yt such that area is maximized.

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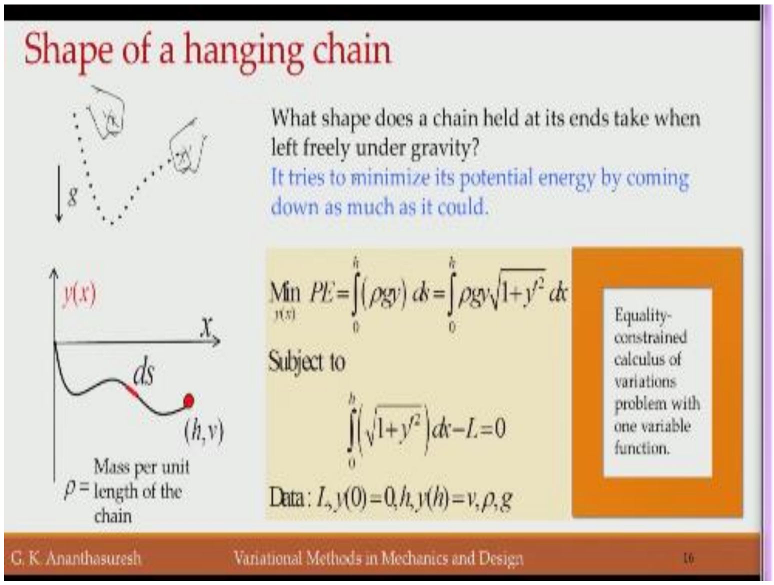

This problem is kind of geometry and mechanics it is a combination of the two this is if I take a chain between two points let us say here the hand should move a little bit there so you are holding the chain between two points it takes a particular shape and the shape how do you find make that the unknown so the shape of the chain is unknown and what do we do here we minimize the potential energy.

So PE here the potential energy it is the energy of the system by virtue of its position in this case there is gravity field so the petitioner G basically comes from you have dash energy and that is ρ GØ $\sqrt{1+y'^2}$ DX be that $\sqrt{1+y'^2}$ comes because we are starting with d/s which is DX square plus dy square and the square root take DX out then you get 1+dy/ dx ^{2} which will denote as Y prime okay and here also we have a constraint that length of this chain is fixed to L.

And that is what we need to do that again we will do the solutions later on first recognized how a functional looks like but there is an objective function or a constraint.

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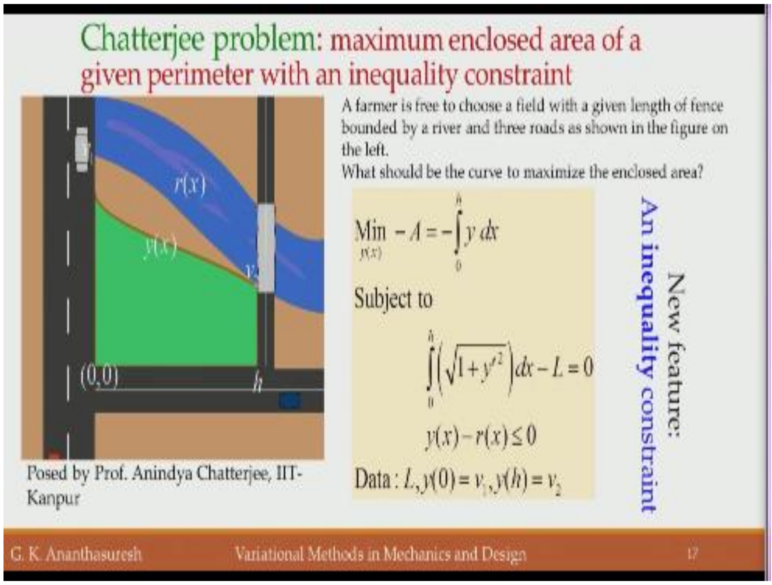

Here is a problem it is a geometry problem which I call Chatterjee problem because Posen chatterjee who is writing on board now he had discussed this and in a way posed this problem and here that a former has to take some land enclosed by three roads though here is road one.
Let me go back we change the color let us say okay this pan here is road 1 and Road 2 and road 3, 3 roads are there so here is one it is 2, 3okay.

The former can choose the land has shown here and this is a curve that we have Y of X that you can choose but there is a river so you cannot have the land that is going to penetrate into the river he has to choose it below the river that is a constraint which is N equality constraint like it is shown here you have to change the color of the ink let us choose farmers we will do green color so that is constrained says that Y of X has to be below RFX, RFX defines the curve of the river okay.

And what does the former want to do he wants to maximize the area enclosed that is what is seen here that has to be maximized to put minimize minus A or convention is to write all problems as minimization problems so you want to maximize it we say minimize minus of that area with respect to Y of X that is what he has to choose to put a fence there and you want to enclose maximum area and the length of the fence.

Because the budget constraints the former fixes it should be F okay, now how do you pose how do you solve the problem because you have a functional here and there is a constraint which is also a function and there is a function constraint on why that should be less than or equal to R of X which is the river curve this is how we formulate the problem based on what we have later on we will discuss how to solve these things.

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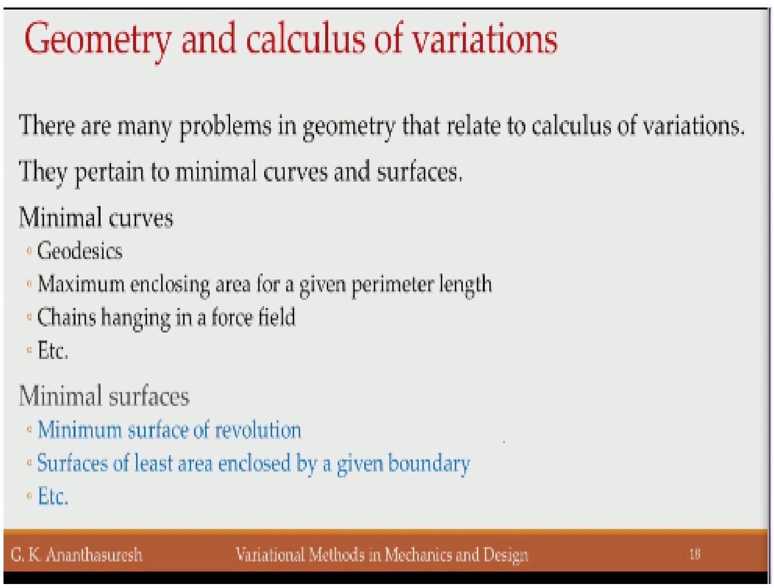

So now let us we discussed a few problems involving curves not let us go to surfaces.

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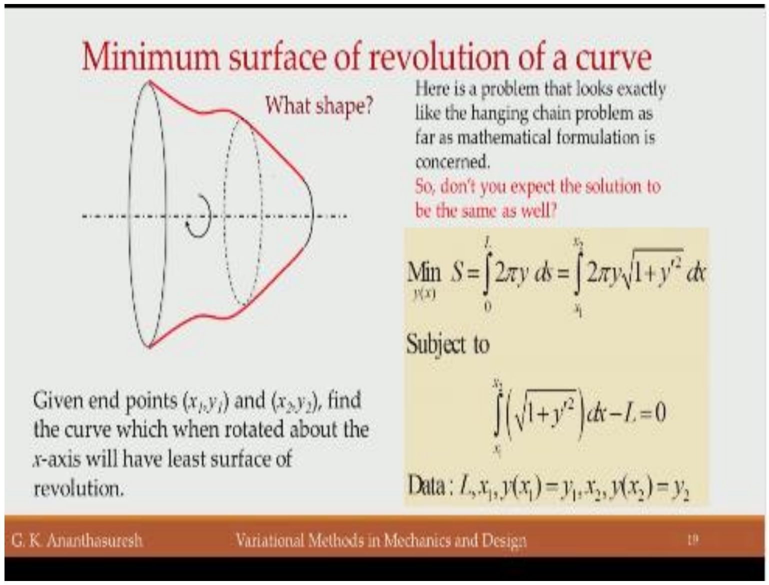

So this is a problem similar to what Newton had considered to get the surface of revolution for a body to minimize the drag on it but in this case we are looking at a problem which is not to minimize a drag but to get this surface in a way that it is minimized you are given a shape that you have to find the shape which is our function this Y of X we are given a certain length which is this L we want to minimize between two points let us say point is given here and here we would like to find a curve that minimizes this surface of revolution for this okay, so for this we have this functional which is 2Π YP the surface of revolution if I take a perimeter like this and length of that if I take anywhere that is d As and y which is the height there 2 Π y X DS then I would like to extreme eyes this surface area subject.

To this constraint which is given length okay between two given points the minimum surface and for that we get a problem such as the one shown here okay here we are taking a y a curve and then we get a mineral surface with that it is not really a surface related problem it is more like a curve which is a surface of revolution is going to come out of that curve when you revolve it around this axis okay. Is a transition between curves and surfaces?

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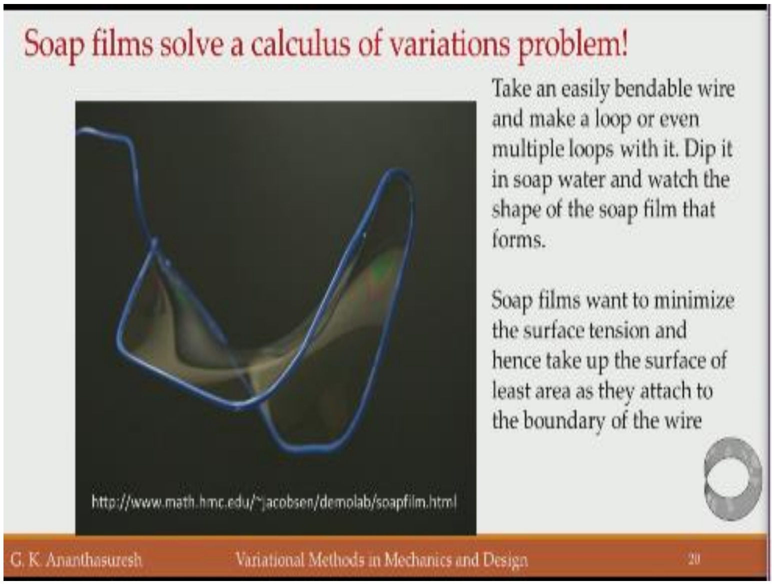

If you want to look at a truly surface problem where the surface are unknown we have to look at soap films if you take a wire and put it into any three-dimensional shape dip into soap water and take it out soap film like it is shown here we will take a strange shape strange shape is actually a minimal surface so film has to adhere to the boundaries of the wire where's opener to attach but in between it tries to minimize the surface energy and actually surface itself right it Is a very least surface area kind of surface will attach to that boundary of this.

It is a very interesting experiment to one can do it at home very easily take some soap water take a wire putting into three-dimensional shape dip it will just come so there could be a lot of interesting shapes such as the one that is shown here so you have to just take a wire bend it whichever way and dip it will you get the minimal surface so soap films do this minimal surface is very efficiently okay.

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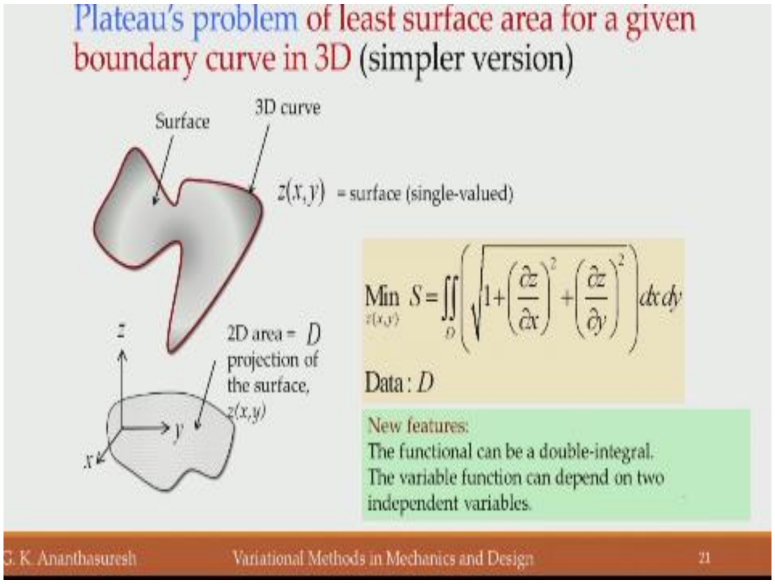

How do you post that problem here we have to minimize the surface and that is given as a functional here again an integral now if you notice we are integrating over a domain the domain is if I take a projection of the surface on two XY plane so this is our domain which is denoted as T here okay so on that we have to get a surface that area of that surface that is area of this surface is given by this expression again this will be DX^2 plus we have dy^2 and a DZ^2 .

So it is basically a little geometrically involved to get this $\sqrt{}\ 1 + \partial Z/ \partial Z^2 + \partial Y/\partial Y^2$ you are integrating there is two variables x and y over the domain that is a projection of that surface so now you realize that a functional need not be involving only one variable you can just X now here we have x and y okay so look at the functional as we go along this.

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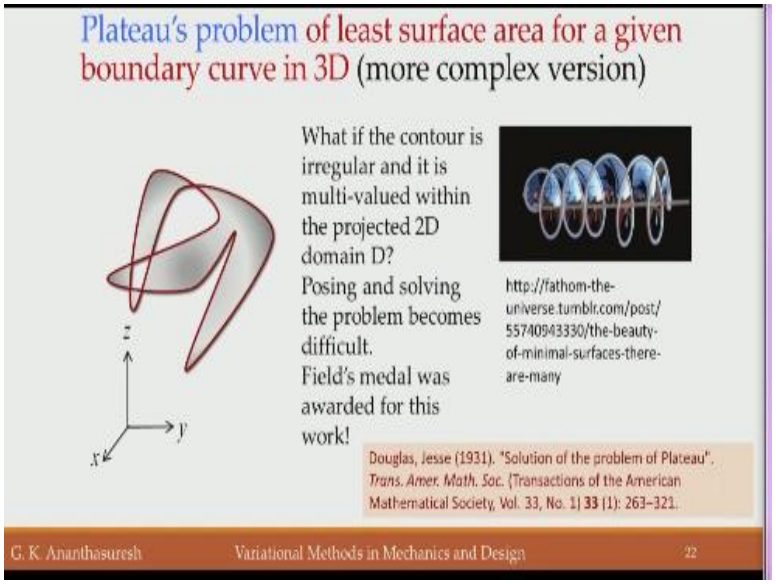

And this is actually very important problem if you take a helical helix like this a wire and dip it that also consumable surface just about anything you take complex shape like this and dip it will get you this it is a very important problem in fact one of the people who have solved it was awarded Fields Medal which is the highest price that can be one can get in mathematics like a Nobel Prize equal and that was awarded to somebody who looked at this problem for irregular shape such as this.

You take a wire and put it whatever shape you want finally if you were to solve you get a minimal surface which has lot of implications in mechanics as well as geometry is a very important problem.

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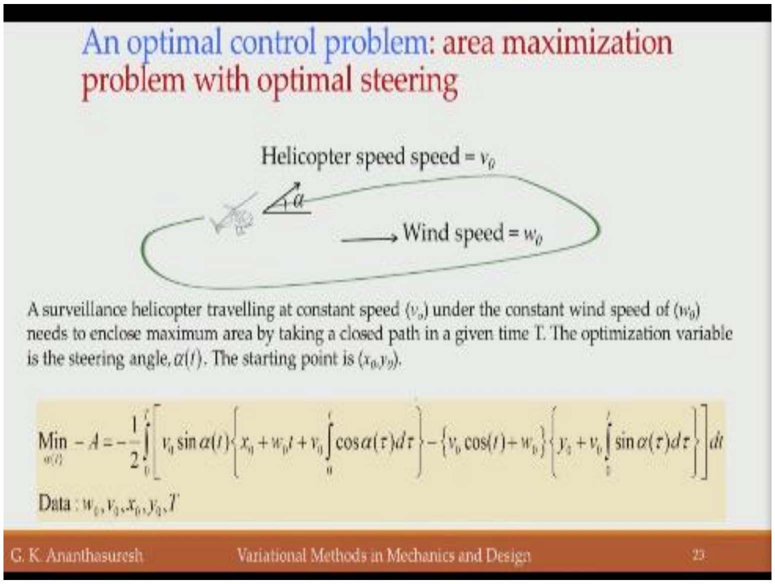

And in optimal control as well there are lot of calculation problems and one of them is given here let us say helicopter to circle around to let us say surveillance purpose in area the speed of the helicopter is let us say fixed what is the variable is the alpha the steering angle the speed is fixed so if you want to make a closed curve you have to change your steering if you are the pilot of the helicopter then you would be rotating and change direction.

For that if you want to maximize the area enclosed by this closed curve as you are circling in air that functional takes this unusual form so here the unknown is alpha as a function of T which is the steering angle this is the steering angle that becomes your variable the area again you want to maximize that area so we have minimize negative of area that area expression is like this so there is sine α T cosine α T and there is an integral within this bigger integral there is one integral here right.

Overall within that there is another integral so and then one more here so a functional can take many different forms.

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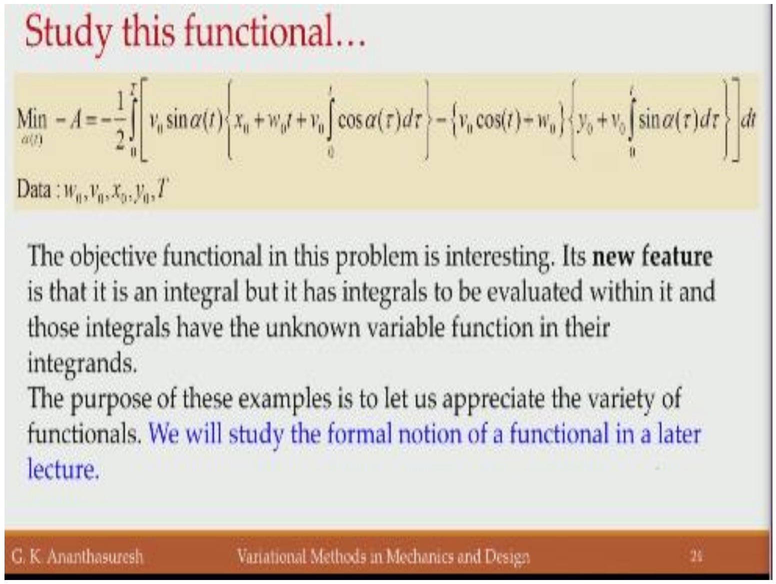

Here is a very complicated from there is integral inside an integral we saw integral 1 1variable integral two variables and then now an integral which are integral inside it so we will study the notion of a functional as we go along and take a number of problems we will take care we will end this here and continue with mechanics later.