Indian Institute of Science

Variational Methods in Mechanics and Design

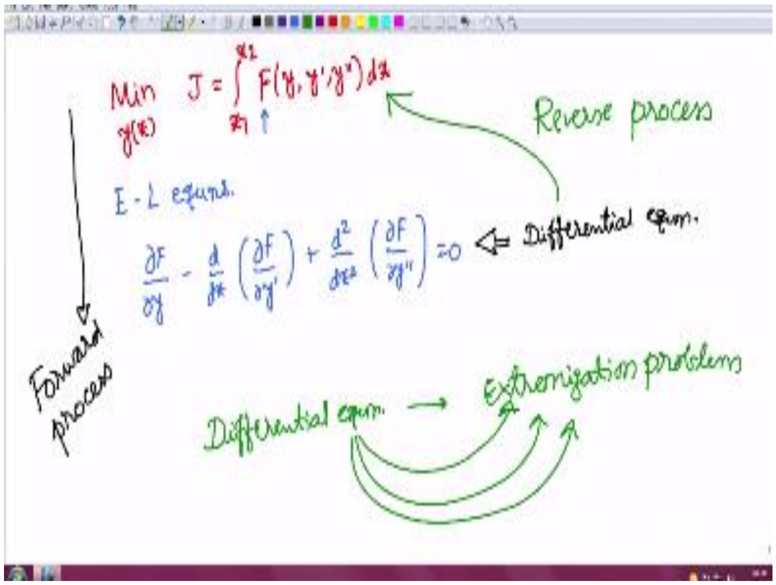
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

Hello in this last lecture of the course variation methods in mechanics and design we will discuss one small topic but also an important topic of being able to formulate a differential equation that is given to you in the framework of calculus of variations that is given a differential equation we want to come up with the functional which when minimized or maximized or rather extremists would lead to the differential equation that is given as an Euler Lagrange equation.

That is we are going backwards that is we are given a differential equation and we want to find the functional that will lead to the differential equation when you minimize that functional by writing Euler-Lagrange equations we will illustrate that for a system that is conservative and non dissipative and we will also look at a system that is dissipative and discuss how calculus of variations framework would still apply to such systems which are dissipative let us take a simple example and discuss these techniques okay. So normally when you look at a Lagrange equations let us write that.

(Refer Slide Time: 01:38)

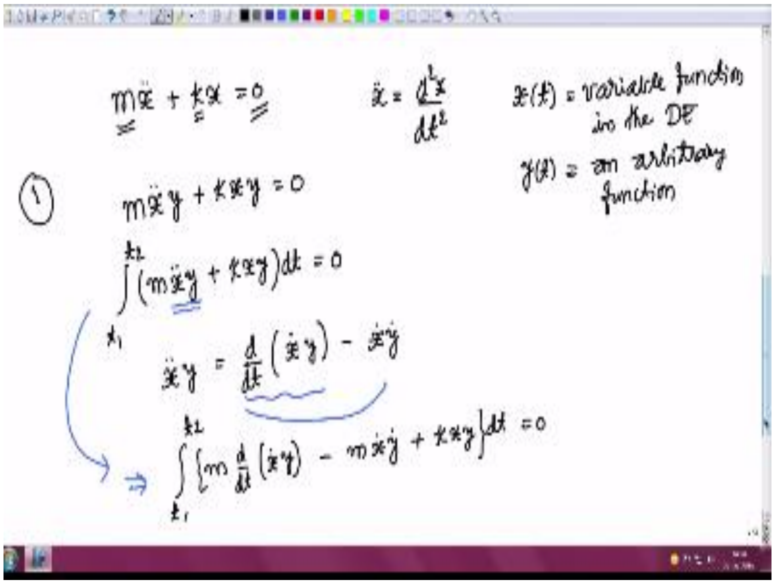


We will be let us say minimizing a functional j equal to integral let us say X 1 to X 2 integrand that depends on Y, y' and y'' and so forth and Y of X is our unknown function when you have this we have discussed much about Euler-Lagrange equations which are necessary conditions for the minimum of such a problem so that we know if noting that this is an F will have ∂ f / ∂ y - d by DX of ∂ f / ∂ y'- + d2 / dx2 ∂f /∂Y '' = 0.

Now as we have discussed in this clue in this course this is a differential equation this is a differential equation and doing this way together differential equation we can call a forward process of obtaining a differential equation from the minimization problem what we will discuss now is the backward process can we go here this is the reverse processor what we can call inverse calculus of variations can we do that we are given a differential equation.

So let us give let us take a differential equation and we want to come up with extermination problem because most of the dynamic systems which are continuous can be expressed as a differential equation and from that we want to get an extermination problem extermination problem let us discuss this process this actually can be done in three different ways there will be one way another way in at third way which will discuss with a simple example.

(Refer Slide Time: 04:17)



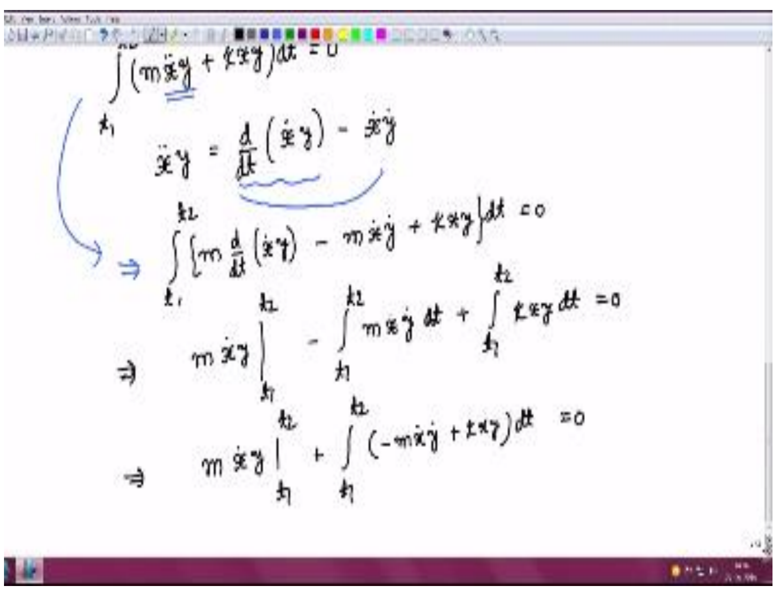
The example will take will be the familiar one let us say I take m X..+ KX = 0 it is an ordinary differential equation that describes the dynamics or rather what we call free vibrations of a spring mass system free vibration because there is no external force there is only mass and there is a spec also familiar one right so which functional been extreme wised do we get this we already know the answer but we'll pretend that we do not know and then do it with our first method of going from differential equation to a minimization problem that will be a calculus of variation problem.

What we will do is we will multiply this with a function like this let us actually not put double dots there so we will multiply with actually another variable let's call it just y MX ..+ y 2 K X Y okay, that will still be equal to 0 that will be equal to 0 when this Y is an arbitrary function that you have chosen right now what we will do is will integrate fromt1 to t2 because here then you put double dots when I put x double dot I actually mean it is d square x by d t square so t is the independent variable okay t is the independent variable.

We have dt^2 square okay so that is why t 1 to t 2 m x..+ k xy and integration variable is t so dt equal to 0 that is obvious because we have taken differential equation multiplied with arbitrary function so x of t is the variable function in the differential equation variable function in the differential equation ok y of t is an arbitrary function an arbitrary function when you multiply that this will be 0the busy which is integrating that is one method we will integrate.

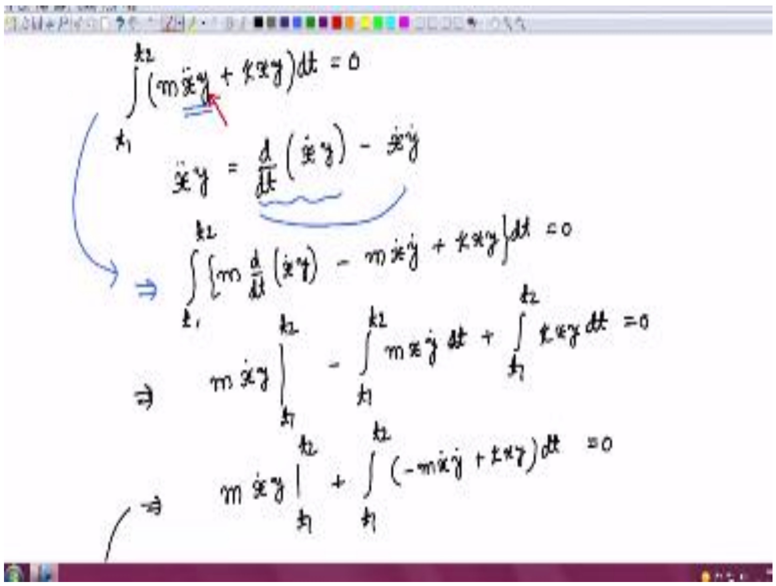
Now we write this X .. y term in this manner we consider d / dt of x .y . okay when we expand this d by DX . x . we will get x.. y in fact this is not y dot will just take it as Y and then we will have another term which is x dot y dot okay we have x double dot why we are writing this as d / dt of x . Because 1 -x . y. because when we expand this will get x.. y + x .y. which is on this side so we are going to replace this term with these two okay. So that gives us if we do it what we get is t 1to t 2 m okay d / d t of x. y - m x . y . + k x y dt = 0.

(Refer Slide Time: 08:30)



Now you see the first term there are d / d t and then integration d t that will simply be m into x . y and that is integrated so between two limits t12 ttt2 and then we have minus t1 to t2 m x . y . d t + t1 to t2 k XY t t0 right now if you look at this we would notice that we have this boundary conditions MX . yt1 t2 normally those are the ones that will be 0 when you let a give a problem and this portion we can write it together I will just write plus t1 to t2minus m x dot y dot plus KX y TT equal to 0 now let us look at this equation that we have just written and think in the reverse way that is what calculus of variations problem will give you this type of a situation okay.

(Refer Slide Time: 10:07)



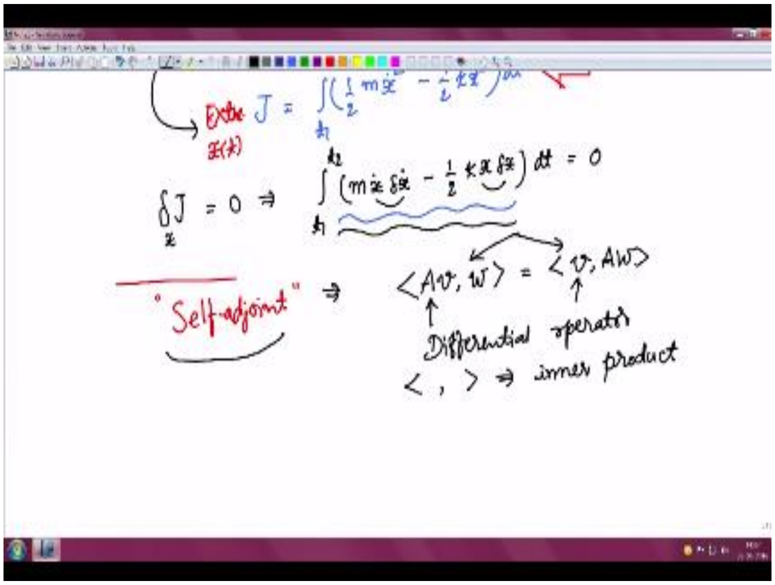
Let us imagine that why is δ X so let Y is δ X because Y is arbitrary we say why is Δ X Δ X is the variation of X of T okay then this equation that we have in that equation that we have will give m x . δ x t1 to t2 + t1 t2- M X . Δ X. instead of Y. + K X Δ X DT = 0 now this tells us from here to here if we were to consider a functional okay functional J which is t1 to t2which is half M $X.^2$ – ½ kx^2 DT okay.

Let us say we are extreme zing with respect to X of T if we do that and write Euler Lagrange equation we will get that differential equation instead of that let us actually say that we would like to take variation so let us take variation of j with respect to X okay that is equal to zero then what do we get we get T 1 T 2 half M half and x .will go so it will simply become M X .Δ X . Okay and then we will have minus half K X Δ X DT = 0 and when we do integration by parts then we will get back the boundary condition which is this okay.

What is important is that apart from this minus sign okay the extermination we have put not minimize maximization if you look at this and this we see that the differential equation that we started out will come when we take this functional and try to extremis okay so this is one way to get in fact what you see here is that there is certain symmetry in the sense that if you look at when we did our technique is simply take arbitrary function y and then multiplied it and integrate it over that differential equation we got x y integrated from t1to t2 and equate it to 0and replaced X.. y with these two terms.

Then what we got here under the integral sign is symmetry x . y. XY such symmetry is known as self ad jointers okay.

(Refer Slide Time: 13:35)

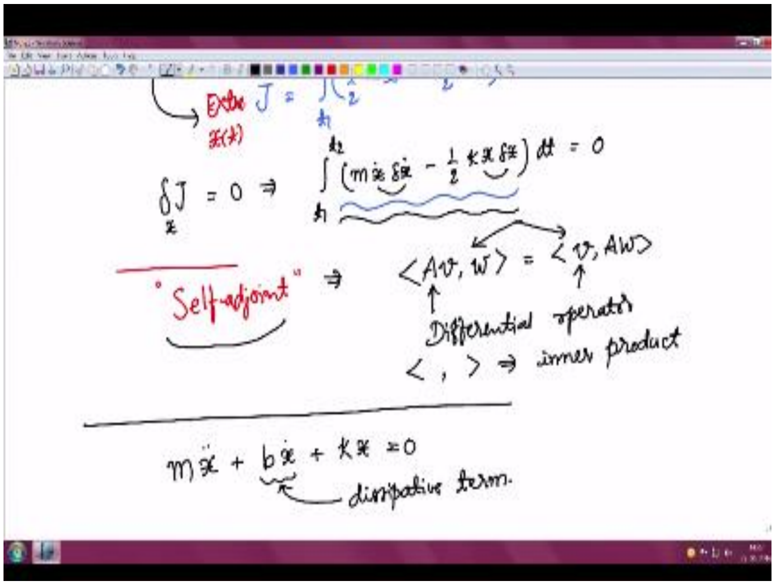


So if there is differential equation or rather the differential operator is self-ad joint by that what we mean is that normally the definition we write is we have let us say a differential operator acting on a function φ then we have another one w that is also equal to V A W where V and W are functions this is like a differential operator and these ear brackets ear brackets refer to in the product okay here is a differential operator differential operator okay.

And VLW functions this EA brackets will indicate inner product inner product as we have discussed in the very at the very beginning of this course in terms of pre Hilbert spaces in Hilbert spaces in a product is simply for integrals it is a function another function multiplied integral the entire domain that is what we have here when you look at an integral of that kind there is X dot Δ X dot X and Δ X.

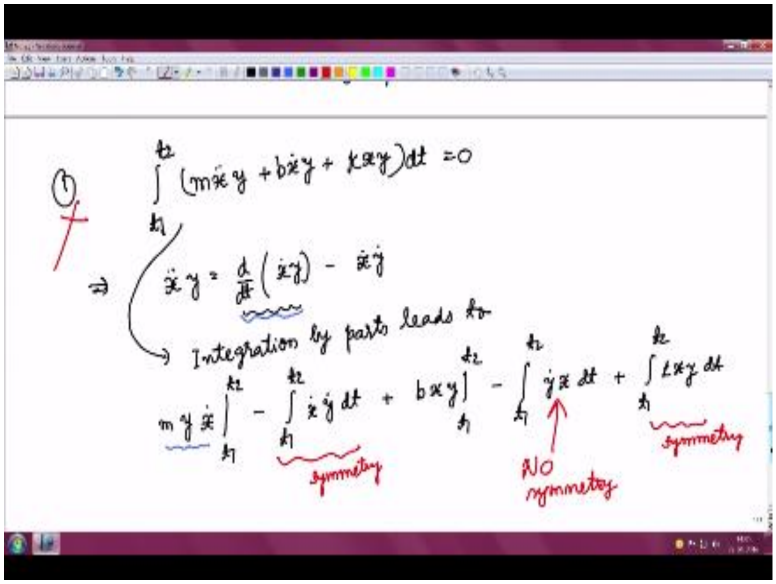
So there is this symmetry which are by interchange which is what we do over here we have simply interchanging the role of the two functions V and W if you get the same result which what we will get then this process of constructing the functionals which mean extreme eyes will give the differential equation is possible okay so many people say that self adjointness is required for doing this that is sufficient but not necessary meaning that if there is a differential operator which is not self adjoint it may still be possible to arrive at the differential equation arrive at the functional which one extreme eyes we will get the result.

(Refer Slide Time: 15:51)



So we take a differential operator that is not self ad joint will take the same example I will write MX double dot now I will take be x dot also which spoils the symmetry that we have equal to 0 this is as you know from mechanics this is a dissipative term this is the dissipative term dissipated terms will lead to differential operator that is not self ad joint that is there will be lack of symmetry so normally we say that it is not possible but I will show you two ways in which this also will lead to a way of constructing a problem which leads to a functional that is going to minimize will lead to the differential equation or usual way of doing it as we just did.

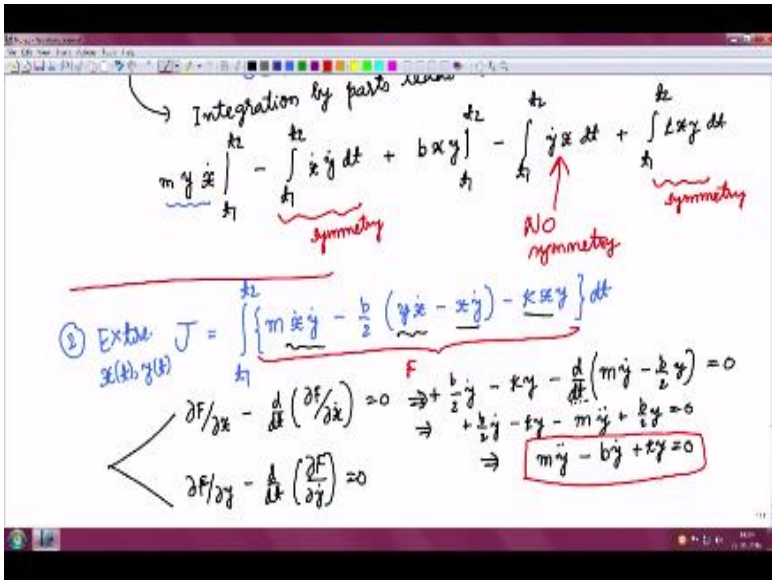
(Refer Slide Time: 16:28)



That say again the method one if you repeat for this problem you will see we want to do from t1 to t2 m x double dot y plus b x dot y plus k x yd t equal to 0 again we will replace this MX double dot y + b x start y with other terms when we do that we were able to get this x double dot why we were able to do this we said x dot y minus x dot y dot which goes under integral this is comes out of the integral is actually integration by parts so if I were to take this under integration by parts so this is integration by parts leads to so we have first okay.

Let us do first function for the first term let us take this as the first function y integral a second function that between the limits t1 t2 my derivative of first function integral of second function right that is exactly what we have because when we write this term and integrate it we will get back x dot y right and then this part is that comes from the integration way of our second term now the other one if we do let us say first function into integral a second function VX t 1 to t 2 that is fine that is a boundary condition minus t 1 to t 2 derivative of first function into integral a second function TT.

(Refer Slide Time: 19:28)



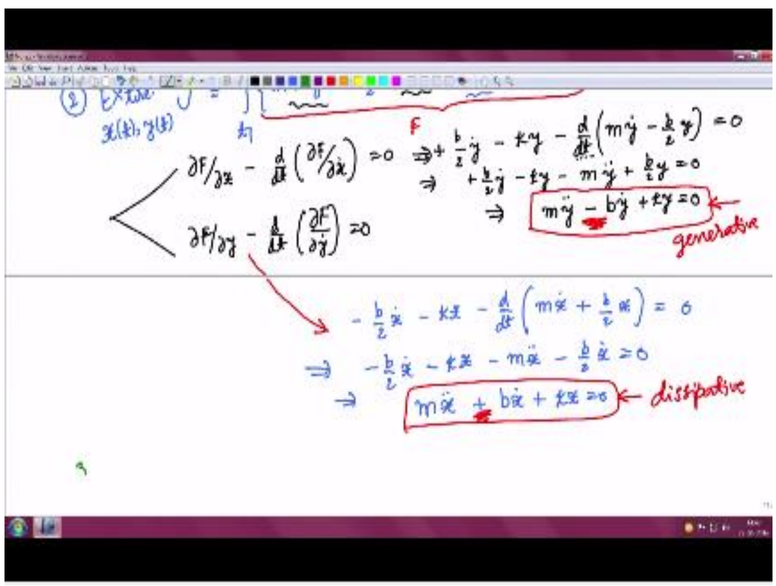
And then third one we can leave it as it is t 1 to t 2 k XY d t now apart from the boundary conditions if you see this has symmetry our self ad joint this, this has symmetry our self ad joint but this one has no symmetry so the first method does not work okay but there are two other methods that we can make it work there is a second method let us discuss that what we do here is to imagine a parallel system that adds energy our generative system because the first one is dissipative system there will be a generative system we will add both of them to get this in this case.

What we will do is we will define a functional J okay t1 to t 2 at time does not change we will have a system which is like this MX dot Y dot minus B by 2 y X dot minus X Y dot we are defining symmetrically and then minus k x y d t okay that is what we will do right so this one will say we will extremize okay with respect to two functions X of T y of T okay we have discussed calculus of variations two functions where our integrand is this that is our app so we will write two equations that we have one is ∂ f by ∂ X minus d by d t of ∂ f by ∂ X dot minus T Square by DT square ∂ f by ∂ X double dot plus this 1 W which we do not have so we do not have to worry about this in this problem that is not even there okay.

That is equal to 0 another one is ∂ f by ∂ y minus d by dt of ∂ f by ∂ y dot equal to 0 let us do this ∂ f by ∂ x varies x, x is there and there that will give us minus y dot and there is a B by 2 I will put a minus sign here that is here and then minus KY and then we have to do minus d by dt d by dt of ∂ f by ∂ X dot that is over here and here we will get ay dot and then minus B by 2 y so we got only y's in this equation that gives us minus B by 2 y dot minus KY if we do this differentiation over here we get minus m y ∂ ble dot and then we have this is with X minus of minus okay what do we get here so we had first y dot so minus of minus actually this is plus we are doing X we made a mistake the first1 ∂ f by ∂ x, x is here that is minus of minus plus this is plus sign

.

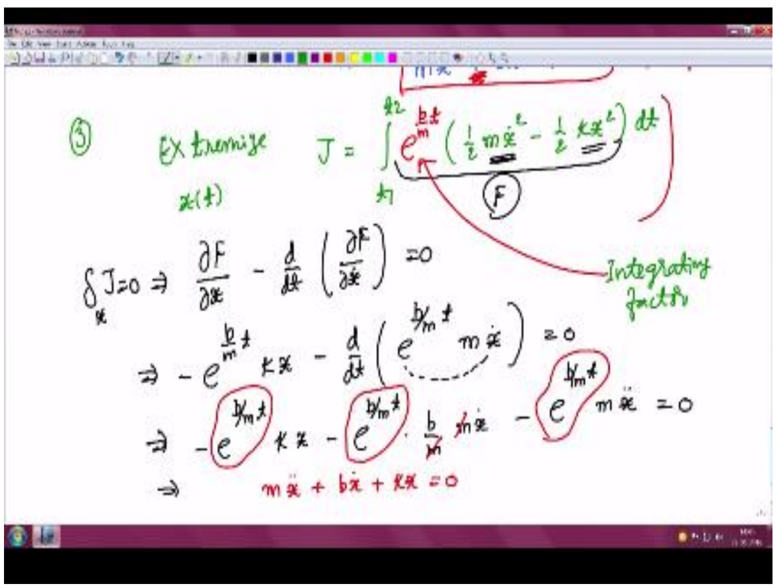
(Refer Slide Time: 23:25)



And then here minus of minus plus again be x 2 y equal to 0 what we got is basically m y double dot minus B Y dot plus k y equal to 0 this is not what we wanted see our, our equation had plus sign here for y dot right so we will find that if I do the same thing for the other equation that is for this one we will find that okay let us do for that ∂ f by ∂ Y that will have Y over here that will give me minus B by 2 X dot then minus KY and then D by DT of ∂ f by ∂ y dot that will be M X dot then minus at le plus because we are doing ∂ f by ∂ x dot which is over here minus of minus that will be plus B by 2 X equal to 0.

That gives us minus B by 2 X dot sorry this is not KY because we are doing the respect to X is change we are doing ∂ I / ∂ Y so Y is over here we actually get we are doing respect to Y dot earlier actually k x with respect to Y we are doing that will be k x over here ok that is k x and then minus M X double dot minus B by 2 X dot equal to zero what we get now is M X double dot plus BX dot plus KX equal to zero so we got what we wanted so even dissipative system can lead to a functional okay which is what we have written here which when we minimize we can get the differential equation that has dissipation the only thing is we got two equations we got that one and this one this is dissipative there is a parallel system which is generative.

(Refer Slide Time: 25:58)

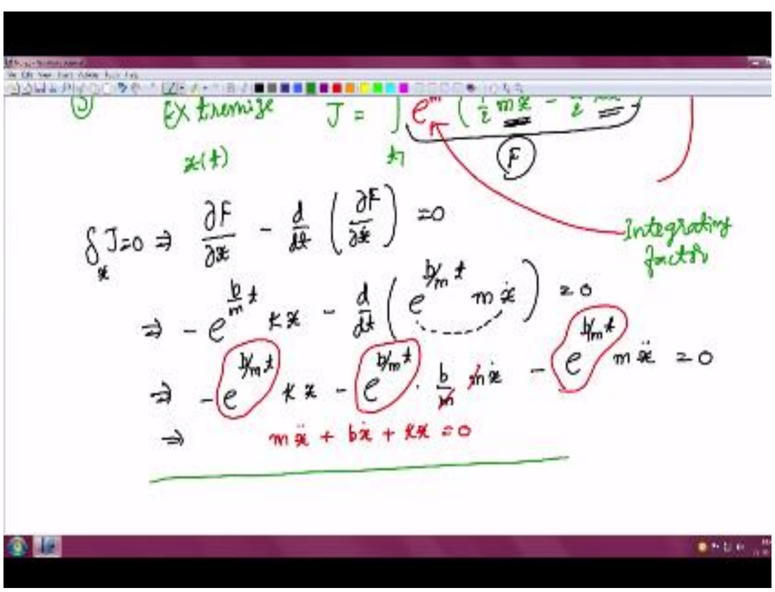


That is why here we have a minus sign that whereas here we have a plus sign so it can be done so this operator clearly not self ad joint which we already saw over here there was this term that was lacking that symmetry or non self ad joint but yet we are able to get this not only this way there is actually a third method that we can talk about third method which is with an integrating factor which is with an integrating factor meaning we will extreme eyes with respect to function unlike in the second method very how to introduce another function y of T which will be generating system here we just have this and then I put a j-integral t1to t2.

So we had half M X dot square minus half k square DT we know that that leads to MX ∂ ble dot plus KX equal to zero which is what we said first method we know how to do it now there will be integrating factor e raised to be by M let us say what happens now our integrand is this total thing okay now if we write the I Lagrange equations which is only one variable we have that is variation of j with respect to X equal to 0 as ∂ f by ∂ X minus d by dt of ∂ f by ∂ x dot equal to zero if we do this what we get now that F is this so ∂ f by ∂ X will be EPM by T X is only there that will be minus K X okay then we will have minus d by dt of ∂ f of X dot which is only this term okay.

We already took care of that we have now this term so that will give us e be by M T times M X dot equal to 0 let us expand this we have minus e raised to be by m t KS minus first if we do chain rule will have derivative of e raised to be by Mt will get be by M and then we have MX dot here then another term will get this we keep it as it is we are basically doing product rule for these two and then derivative of this which is M X ∂ ble dot equal to 0 if you see this m and m m1 m cancel and then there is this integrating factor that we had it is there in all of them that can be cancelled everything is minus sign what we get here is M X ∂ ble dot plus BX dot plus KX equal to zero.

(Refer Slide Time: 29:27)



That is what we wanted so even with a single function by having a suitable integrating factor suitable integrating factor we can obtain a functional that is extremist to get the differential equation that is has a differential operator that is not self ad joint so to say that self ad jointness is needed to write a functional from a differential equation it would be incorrect it is not necessary but sufficient that is not incorrect that is sufficient if we have self ad jointness you will get it but self ad joint is not necessary even when the operator is not self ad joint we can still get these functional which one extremized.

We get the differential equation okay so that is something to note because normally people say that itself ad joint is required it is required English language may be okay but actually not okay it is not necessary but it is sufficient that means that even with cellular joint does not exist for a differential operator we can still get the functional which one eternized we get the differential equation because three methods one works when you sell for joint because the symmetry is established the second and third method second is to have a parallel system.

That compensates for the dissipation of the first system and a third method where we have identify a suitable scaling factor with which we can achieve what we want which is to be able to write a functional which were minimized will lead to a differential equation that has dissipation and not self ad joint can still work thank you.