Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello again we looked at a dynamic problem where we wanted to maximize the stiffness of a structure by minimizing what is called dynamic compliance and we found that the adjoint equation when there is a damping term will not be the same as equilibrium equation but it will become the same when you go backward in time and that is what we were showing just before we ended the last part of the lecture now let us understand this slightly differently for that let us take a different problem.

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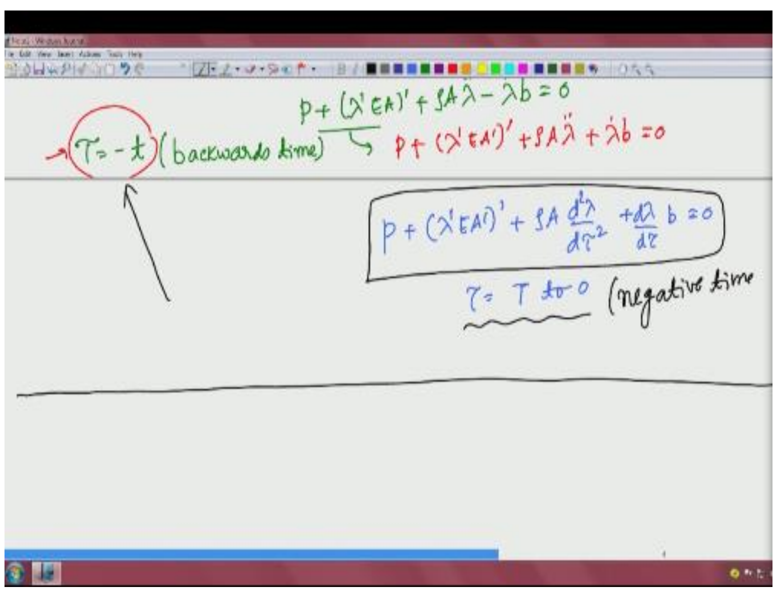

Where we look at the meaning of this negative time that we took to say that we are going backwards in time tau equal to minus T why was that necessary that is apparent from here but when we take another problem that will become more clear.

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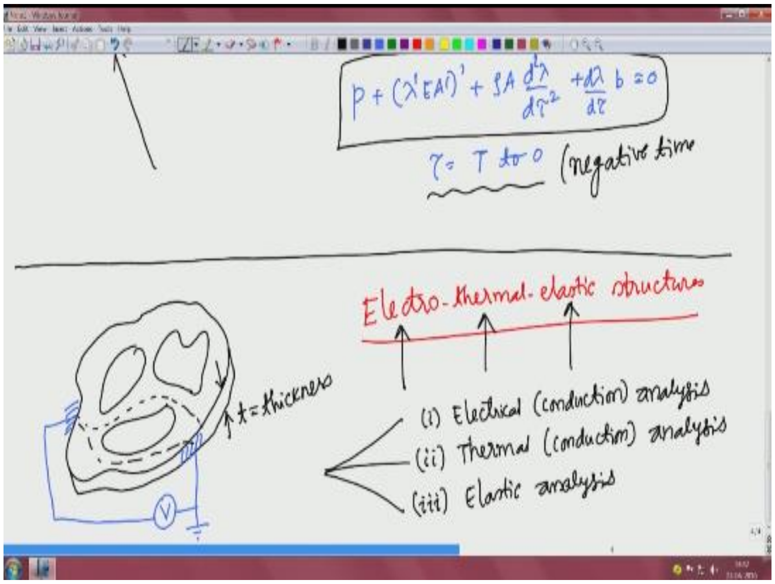

So what we do now is take up a problem let us say we take a two-dimensional sheet of some kind okay that let it have some thickness for now let us take this as uniform thickness T that is thickness and we apply some electric potential difference between two points okay let us say somewhere here and here we apply we fix it there mechanically and also apply some voltage okay when you apply a voltage and if it is a conductor there will be current going through it when the current goes it will develop heating what we call Ohmic heating a Joule heating.

Because of the heat the structure will lx experience thermal loads and start deforming so we would like to minimize that deformation okay this is very common in electro thermal actuator electro thermally activated structures what we are taking now or electro thermal elastic structures okay we would like to make this structure stiff under such conditions especially this becomes important if there are some holes in the structure let us say there are some holes in the structure when apply potential difference like that current will flow in some fashion from there to here okay.

Now when that happens these things are going to get hot but they are going to deform and we want to minimize the deformation let us say actually it will be interesting to maximize deformation but we will take a simpler problem of minimizing deformation so here we have three different energy domains electro, thermal and elastic therefore our problem statement is going to be little complicated because it involves three analysis in a sequence we have to do first we have to do electrical analysis electrical it is actually electrical conduction because when you apply voltage through a conductor the current is going to conduct electrical analysis.

And then we have to do thermal analysis it is also thermal conduction in fact the equation for both of them is identical with different properties one we have electrical properties other will have thermal properties and then we have to do normal elastic analysis which is what we have considered several times this course.

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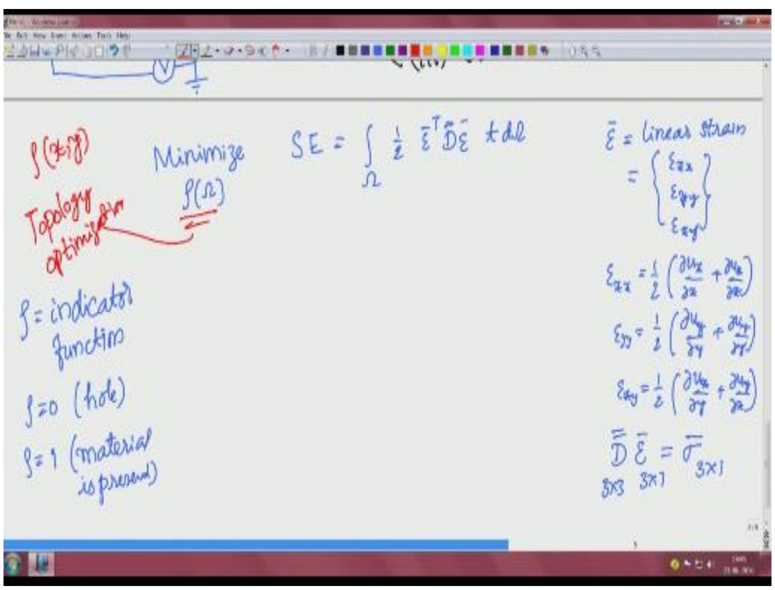

Okay now how do I write this problem I would want to minimize strain energy let us say because that will make me a stiff structure stiffer structure so this is over a domain let us call this two-dimensional domain as some Ω our area whatever Ω and the strain energy is going to be half ε^{-1}

DU ε and there will be T the two-dimensional area okay what are these things ε here this is not U should be ε also what is ε is linear strain that we consider here linear strain which will have in two dimensions ε_{xx} ε_{yy} ε_{xy} be ε_{xx} ε_{yy} those two are normal strains and then εxy the shear strain where does that come from basically ε_{xx} is I can just right now we ∂Ux/ ∂x + ∂u x identical I am just saying half and this because that is how we generally right because that applies to both ε normal strains and shear strength ∂u / ∂y + ∂u y / ∂y essentially ∂u /∂u by ∂xxy when you write you will understand why I am like this will become do you x / ∂y + ∂u / ∂x reverse okay.

So what are Ux and Uy they are the displacements of every point in x and y direction 2d problem there is some displacement in x Direction Ux some different Dy Direction you why and these things vary from point to point okay that is how we have strain energy and what is D here the D the stress strain matrix if I multiply D with ε I will get σ also will have three components just like strain has so this is 3 / 3 this is our stress strain measures or what we call constitutive matrix stress-strain relationship.

What are we changing here we will change a fictitious term defined over the domain at every point we have the indicator ρ we can call it is not a mass density even the way are you using that is an indicator function what does it indicate it indicates whether the material is present or not if ρ= 0 it is a hole there ρ= 1 material is present material is present essentially by doing this what we is we are solving what is known as topology optimization problem okay. So we have a function okay here ρis a function of xy x and y we have the domain such as the one we showed.

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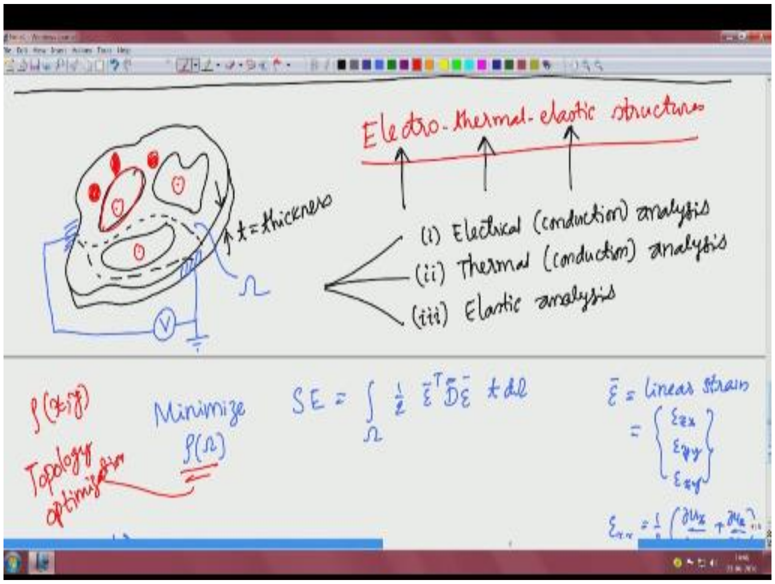

If I have at this point ρ equal to 0 I get a hole there okay if another point I have $\rho = 1$ I will have material there okay so that way if I keep on doing this I will get the holes and I will get solid things okay this is let us say this is all material all this is holes if I take thing I get a whole we go here I get material if I get a key thing I will get a whole if I go there I will get a whole that is how the holes get defined once we find this ρ x y indicator distributor distribute distribution of the material function okay indicator function defines the material distribution all right.

So these are all the things we have to define now we are fine we defined objective functional design variable and now we have to write the constraints subject to whether the constraints equal equation there is ε here so that has you x y that should be governed by some equation which we will write here in the so called weak form that is I would write this as ε^T Dε V indicate that it is a weak form meaning that instead of Ux Uy here I would take UV Vx and Vy where V is the variation of U or is a week variable corresponding to you okay.

We get this and then the T of this and there is no mechanical load in this problem but there is thermal load so that will write it as ε thermal and suppose D ε V T Ω that should be equal to 0 that is for any arbitrary V this equation should hold good now what is ε TH.

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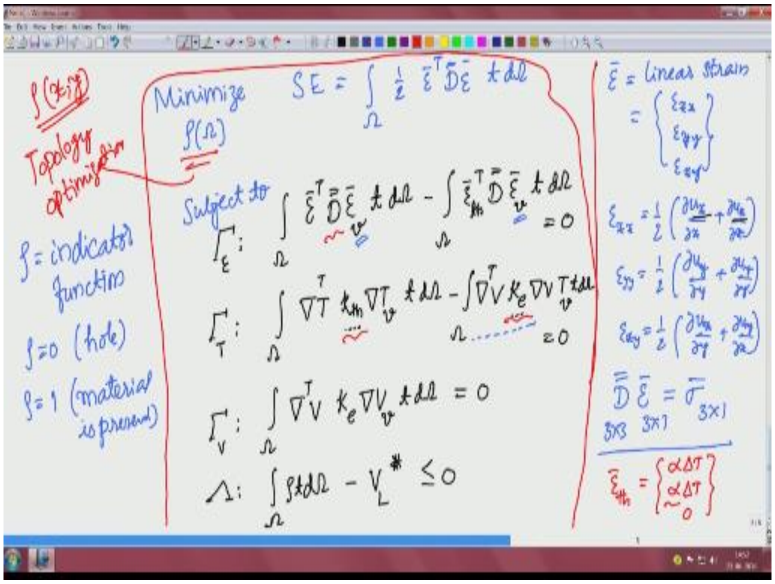

That is the thermal load okay let us let us write it over here in a different color ε thermal loading it is basically something that happens when you have thermal expansion okay that is strain thermal strain so you have that as ∝ into Δ T ∝ into ΔT and then zero only normal strain will be there but not shear strain.

Now we have Δ T come in here that is another state variable for that we need to have an equation that equation is for the heat conduction so that can be written as follows so again that will be over the domain so we will write it as del t where t is temperature okay that is the temperature rise we will call it transpose K^{th} ∇T V I call it over T Ω right so they can t be here is a weak variable corresponding to the temperature T okay so we will have that so but temperature actually comes from the heat generated.

So we have to write that also here that we will say that is also integral the entire domain that will be ∇^T T k_e that is here it was thermal conductivity will be electrical conductivity into ∇ this actually not T which should be V is the Joule heating ∇ V and then we will have this T V here and then the thickness this is equal that is a big form of the electrical thermal conduction equation where this term that we have this is the heat generation term okay it is like $v^2 r$ heating by r became electric conductivity upstairs in the numerator rather than denominator then we are left with we have T and now we have V what governs V.

So we have another equation so let me write it in like we have been doing so we have that will be the same equation V now instead of k_e TH will have k_e and then ∇ V VT this and there is no sourcing term that will be simply equal to zero so these are the three equations correspond to that since these are functional type consider integral there will be a $\lambda\varepsilon$ enthalpy I sorry γ ε gamma T and then there is γ V Lagrange multipliers constants and then we have the volume constraint that will be over this not a because we have ρ if I take that ρ and integrate over then type ρ t entire thing that should be less than equal to V* the volume star so I will just put VL because V we have used for a potential electric potential so I will say VL here that is filled less than equal to 0 here we will have the λ corresponding multiplier for this okay now let us take a look at this problem let me encircle the problem completely to see the overall structure okay what is not clear is varies row here we do not see ρ but actually ρ will be in the material properties in this D and in this $\propto$ that we have and then k_{th} and k_e all these places will have ρ how does it come.

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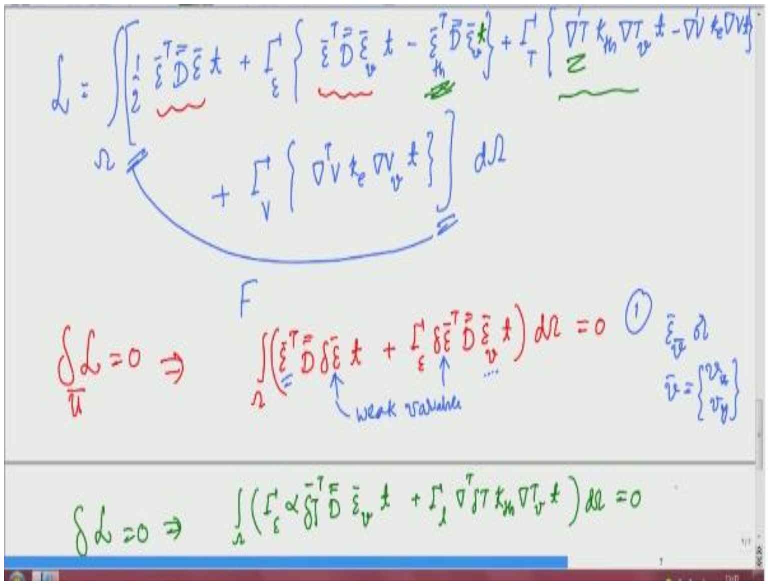

Because of the way we define the indicator function ρ if I take the stress strain matrix I will take the base material matrix as d 0 x ρ, so ρ we do not really make it 0 we take it something like ρ min okay some small value so that we do not run into singularity problems while solving we take that ρ when it is 0 it is almost like a very weak material or very flexible material is there when it is 0 equal to 1 you have the base material.

This we do for all the properties that is α also will have α 0 into P and so forth k_{th} will also be K_{th} 0 x P and so forth for k_e also okay that is only one there we will put k e ρk0 equal to k_e so we will have that so as P changes for point-to-point the material distribution automatically gets effect in terms of properties, so when we have this as usual where to write the Lagrangian it will be a long Lagrangian because you have three governing equations one resource constraint volume constraint and the objective function.

So if we write that so first one is half ε^{-1} D ε T that is a strain energy and then we have this γ ε multiplying the governing equation for the mechanical problem so we will write that ε^{-1} d ε v + t - ε^{-1} this is thermal d ε v then we have γ thermal that will be x ∇ T ∇^{-1} T times K_{th} into ∇ T V into T and then we have ∇^{-1} V k u k_e ∇ V that also will have T plus one more.

So all this is going to be the integrand okay and then we have ∇ V γ V that will be similar that is ∇^{-1} V k_e ∇ V, V T all of these things d Ω we have put t everywhere so that okay the D Ω this whatever is inside of this square brackets from here to here the whole thing is our integrand f okay once we have this we can write the Lagrangian for each of them so let me do it in RGB scheme.

Let us I will do red color I will do variation of this with respect to u which is actually ε I can say u ε that is equal to 0 when I say u have to put a u bar because that is a vector okay so we have u bar here that is a vector so we have we take that we look at wherever ε is that has you so we will just write it that way okay so that will give us this is like a quadratic term in ε then you take variation what I get Ω I leave it in the weak form.

So I will get ε^{-1} D ∇ ε okay and then T and then here also there is ε so that will give us plus γ ε ∇ε v d ε VT and where else I guess that is all ε even then there is ε thermal that will wall temperature so there is no u r ε so these things are there this over the domain be equal to 0 okay that is the adjoint equation if you take with respect to ρ you would have got the design equation.

But my intention to show you that going backwards so in order to do this problem forward where to first to electrical problem and then thermal problem and then elastic problem because if you look at the problem if I cannot do the this problem first because it involves temperature over there then I go to temperature problem I cannot do this because it involves voltage so I to first do this and then do this and then do this that is a sequence okay.

Now what you will see for adjoint equations is that it is reverse go to the elastic first then thermal then electrical okay, now if you already see this we can see that we can already do this first not a problem so we can do this to find what we have to find our ε V or our V vector the V vector is VX and VY after you find u_x u_y then you would know this and what we want to find is this thing at epsilon.

Then these are the weak variables for it okay these are the weak variables or variation for ε be solutions we can already solve it we do not need to the joint equation for thermal and electrical field okay now similarly if I now go or let me choose green color Lagrangian with respect to T equal to zero okay we need to look at this where the temperature is temperature is in this and over here and that is a two places so this will be over that will be α ε thermal has α ρT right.

The temperature rise that we have so that will have when I do this γ ε α the TR ρ temperature rise is taken care and we put like a identity thing there because we need to retain that that array feature over there into εv and well at that time the t will be here anyway so that we have forgotten there so that he will be there and it is there in the other term also that will be γ T and then ∇^{-1} ρ T okay.

That is what we will get okay when we say here this will also be I should write it as better I will write it as δT so ρT that is variation of T δT into K_{th} δT VT all of this d Ω equal to 0.
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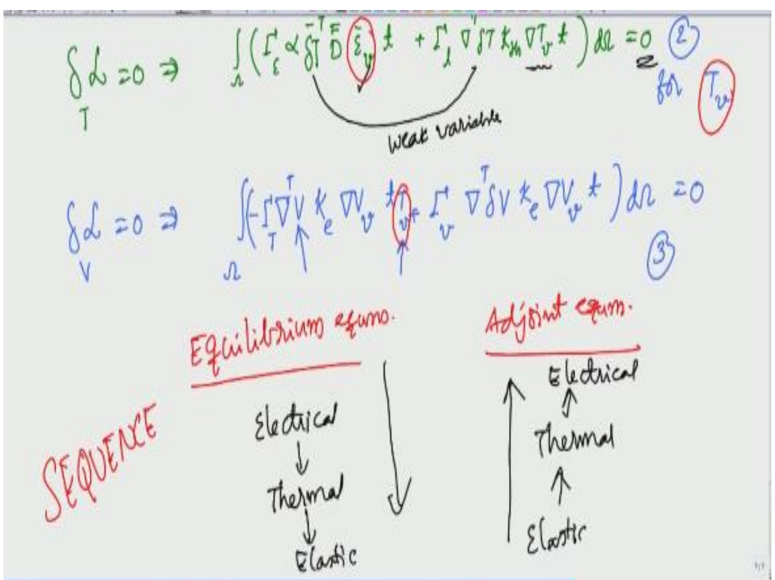

Now if you see this does not involve anything to do with a joint variable of electrical thing after solving this I can happily solve this equation for this T_V because after you find ε then we would know we would know the temp at ε and temperature know that ε V is also known and this is the one that we want to find ρT_V over here and over there so that is known because these are the so when I say we know this what I actually mean is that T is known but not ρT so this becomes the week variable here okay.

These are the weak variable so make that weak variable this has to hold good for all of those arbitrary week variable you can solve for T_V and third one that we have let us go to blue now Lagrangian with respect to V equal to 0 they look at all the terms that have V voltage so we have V over here over here twice and then over there so we will get two terms right so the first one so we have to scroll down the screen let us remember this is γ t x minus ∇^{-1} ∇ v k_e ∇ v that will be a quadratic term.

So there will be two times this will be there okay that will be over this will have a minus got that is minus their ∇ V^{-1} k_e e ∇ V okay T and of course we have that γ V and this is a minus sign let me put that properly minus sine T that is the first term that we got where v is the second term okay that is γ V and then ∇T ∇V and then we'll have a rest so plus γ V sorry this is actually not ∇ V this will be γ T naught γV γV and this will be ∇^{-1} ∇ ρ V K_e.

So here it is k_e where this came from k_{th} thermal conductivity so let us look at the term we were writing or so that is actually oh that both is actually k_e so ∇V and then there will be so that is that is k_e here okay and this is also k_e then we will have ∇ V VT all of this integrated equal to 0 so now when you want to solve it we know V already so we know V and we would have temperature coming in somewhere okay.

That the in order to solve for this we should have the temperature this γ T was there so here okay we maybe we missed a term where we are writing this there was this T_V that we missed okay, so this should all be x T_V and that will continue to be there over here so this will have a T_V after we know that we can solve this in the third we cannot solve it first because we have this T_V term there the T_V first has to be solved you not solve T_V if we see this equation we would have this εB.

You so first you have to solve for that and do this now if you see the equilibrium equation was electrical thermal and elastic adjoint analysis is reverse okay so let me just write that the sequence of solving the problem equilibrium equations and then let us say adjoint equations the sequence that is what we want to do here we have to do first electrical then thermal then elastic okay whereas here first you have to do elastic adjoint one then thermal then electrical okay.

So this is what happens when we can say dynamic problem we have to go from 0 to t the final time when we do equilibrium equation analysis but we will do adjoint then you have to go from final time to backwards in time 20 that is what we see when you have sequential a couple problems such as the one electrode thermally activated elastic structures or electro thermal elastic structures.

This goes in one sequence this goes in another sequence okay so when we do a giant analysis we have to keep track of how to do it and that will be anyway evident from the equations that we write again you remember that adjoint equations always are the same form as equilibrium equations only the only thing is that sometimes you may have to reverse the order like in this problem electro thermal elastic but also a dynamic problem that has a damping term then you need a minus sign in that joint equation to get rid other on a sign.

You reverse the time and you get it the same as equal to equation so you can use the same framework meaning the same finite element routine to solve this also accept that you go from instead of going from zero to final time you start from final time and come back to zero as if you are coming negative which is what we see in the sequence of equilibrium equations and adjoint equations being different, okay thank you.