Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

So we used minimum potentials of principle for getting the Eigen value problem for buckling and then we got the Rayleigh quotient also and noted the property of Rayleigh quotient Rayleigh quotient has a minimum locally whenever the w the function that we take becomes the buckling mode shape okay so by minimizing Rayleigh quotient for column buckling we get the fundamental buckling mode shape buckling value buckling load okay that is what we are taking now.

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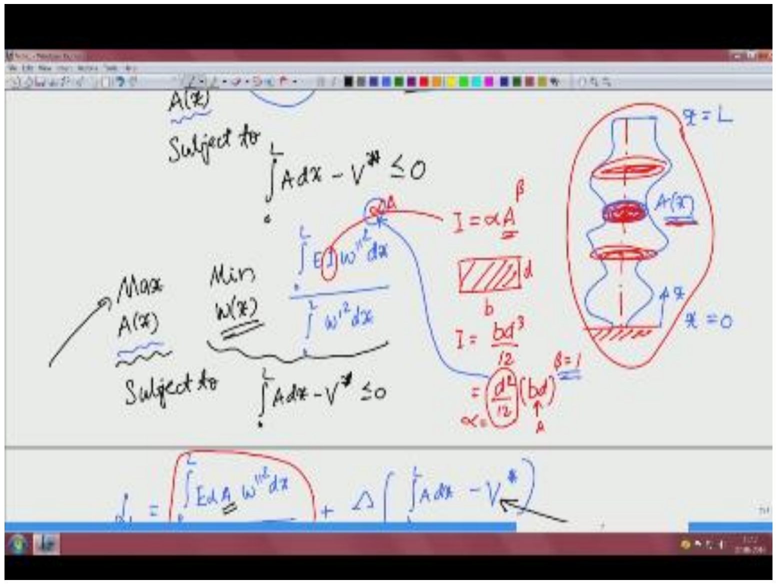

We want to design the strongest column forgiven amount of material okay strongest call of a deformed a material that is I would say subject to subject to the volume constraint a DX 0 to L minus V Star less than or equal like we had done for the bars and gives before we are doing for the column what do I mean by that we have a column I will only draw the neutral axis now okay let us say fixed free column boundary can do not matter now instead of taking uniform cross section we will take a variable cross section variable cross section okay.

Everywhere the cross section this A of X is a function of X again our X goes here in this direction X equal to 0 and X equal to L here we want to find this function A of X that is why that is became our design variable okay but how do I get P critical that is where we use a Rayleigh quotient so what we do is we rewrite this problem we rewrite this problem maximize a of X okay p critical we know that if I minimize Rayleigh quotient with respect to W X that is what is the Rayleigh quotient we recall from the last lecture this was the Rayleigh quotient for column buckling okay.

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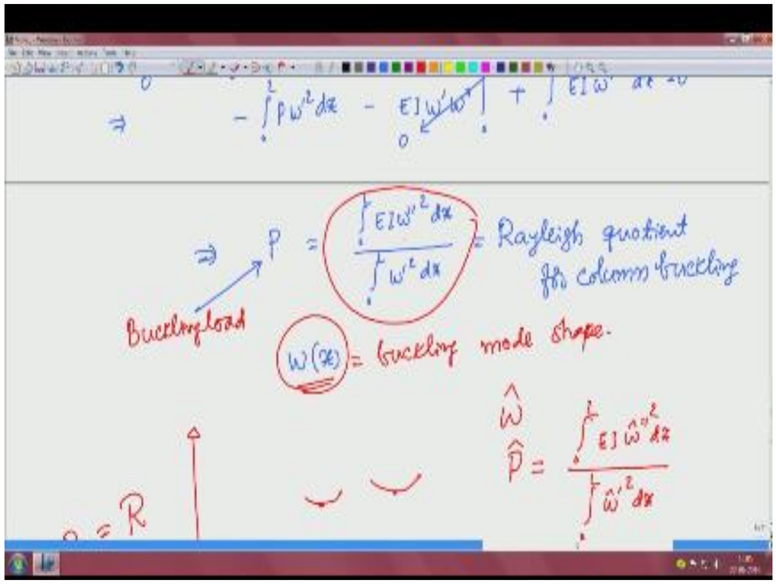

The W so we minimize p with respect to W then we get the buckling load fundamental buckling load so I can say that Rayleigh quotient I will write 0 to L e I w double ‘ square DX divided by 0 to L w double ‘ W ‘ square DX okay varies A here we have already discussed that is actually in I we had noted that I can be written as α some number a raise to β okay so it depends on the cross section shape if it is a rectangular cross section we had already discussed that if I have a rectangular cross section with Breath b and depth D this kind of a cross section then I as we know is B D cube by 12 so since BD is area I will take d square out and then call this B D which is area okay.

This is area so my data here is equal to one and this becomes α if I assume the depth of the beam okay that is where the area of recession would come so we will replace that in this case with α into a because wait I equal to β is equal to1 and what is α, α is d square by12 okay and then we have the volume constraint subject to subject to volume constraint 0 2 L A DX minus v star that is not equal to 0 okay.

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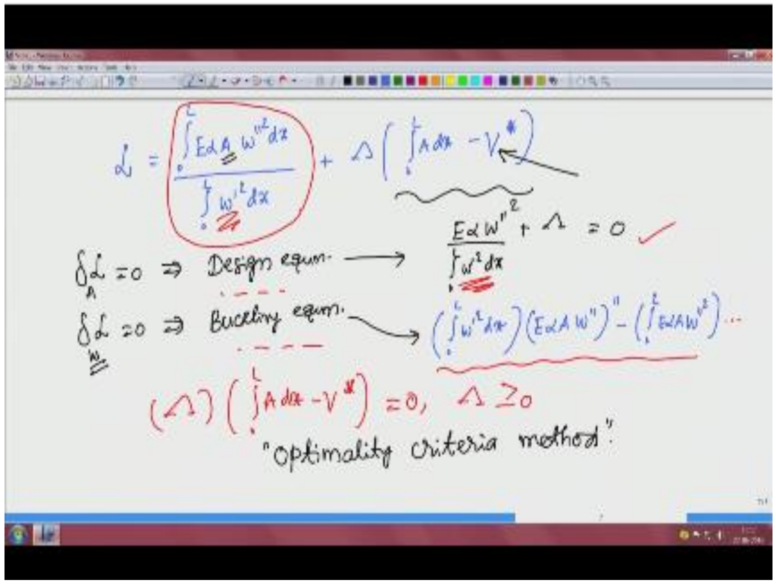

So we have the problem statement now okay the first thing we do is to write the Lagrangian for this problem like Lagrangian we have d I w, w ' square DX okay let me change the writing as I we will write it as E α A that is EI w, w ' square DX 0to L / 0 to L w ' square DX that is the objective function plus will have λ x 0 to L ADX minus v star okay when you have this we have to take variation next step we are taking of Lagrangian with respect to a that gives us design equation to design the area of cross section likewise when you take this with respect to W you get in this case the Eigen value problem itself because a P critical.

When we solve the W if you take the minimization you get the buckling equation buckling governing equation because we do not write the governing equation separate one we pause held it into the objective function like that minimization that will give you the P critical minimum value of that corresponding w is going to be our buckling mode shape so we get buckling governing equation okay and then we have the volume console less than or equal to complementarily condition and all of that okay.

So let us first do this part variation of length respect to A so A here is in the numerator only so you would do that denominator just stays the way it is okay so that gives us you know 0 to L e α

w, w ‘ square we can all other do that just directly right Euler -Lagrange equation for it okay because this is just a constant in the denominator let it be there we have the rest okay so if we if we do this we get e α A is there so W double ‘ square minus λ that comes from here except this will be divided by 0 to L w ‘ square DX that this actually plus okay.

That will be equal to 0 that is what we get when it a variation respect to a so the buckling equation if you take then we have two great with respect to W here that is there w ‘ W double ‘ we have to use this quotient rule of differentiation when you do this, this is a functional that what we have here is ratio of two integrals is also a functional we can take that okay so first we write 0 to L w ‘ square DX over here into the variation of the first part that will be e α w double ‘ e α A also will be there p α a w double ‘ since double ‘ will be double ‘ also right d square by DX square that we have for the numerator then minus this will be there 0 to L e α A okay.

W double ‘ square here okay and then Dominator we do not have a okay so we have to make one little change okay that is actually important we missed it so when we derived this Rayleigh quotient over here we assumed uniform cross section if we do not assume uniform cross section right here when you do all this if potential energy that is correct if we write this as I and you know α we are at α a raise to better whatever when we are taking these derivatives we assume that especially here that I will come we would not have gotten just this here.

This I also have gotten double ‘ okay this also does got double ‘ because it is inside right that will change a few things over here when we when you write everything all that would have changed a double ‘ and so forth so we have to redo the Rayleigh quotient there in order to get it right otherwise what is happening for us here is that we are seeing that there is no A here in the denominator okay and this other things are missing so to redo the Rayleigh quotient when we do that we put the problem like this then we get the design equation if you do it properly and then buckling equation here and then we have the complementarity condition.

That λ times this constraint a DX minus v star equal to 0 and then λ greater than equal to 0 okay this one will have to redo it and then I suggest that you do all this after reading the Rayleigh quotient here assuming that I second moment of area actually varies with X okay then we will get this Rayleigh quotient properly and then, then we can do what we intend to do if you just do it like this there will be a problem because we assume that area of cross section is constant and this but now we are talking about that being variable.

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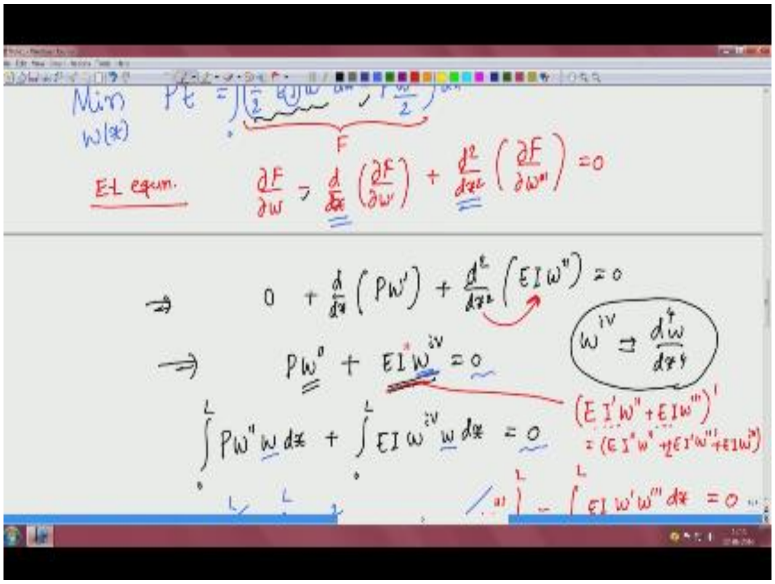

So we have to read arrive that equation that is easy enough you just that in these steps that we took in these steps that we took we should not forget that over here I is varying so there will be two terms coming so over here we will have e AI ‘ W double ‘ okay plus there will be two terms right so when you do this two derivatives we have to do over there so II I ‘ w, w ‘ and then e I w triple ‘ and one more ‘ will be there because this is d square by DX square so that will again give us e I, I double ‘ every double ‘ okay.

And then e AI ‘ triple ‘ this will have now we have done it for this when I do it for this another one of this will come that will become too and then I will have e I w fourth ‘ that is what we get over here for the buckling problem okay and after that when we do this by multiplying by w each one of them we have to reduce all that so now the Rayleigh quotient actually becomes different from what we have because we are do integration by parts for this and this and this we have done only for this now because that was there but these two terms were not there because we assume that I is constant so I ‘ and I double ‘ are not there okay.

So now when you put them back and do this integration by parts you will get a Rayleigh quotient then that will have dependence on I as well in evaluating the Rayleigh quotient then you will get the correct value over here four columns that have variable cross section where this is more here less here and then becomes more or less and so forth as it is changing we get for that okay that is what we can do when you solve this problem using what we call the optimality criteria method optimality criteria method we can get solution for this optimal column or strongest column for given amount of material v star okay.

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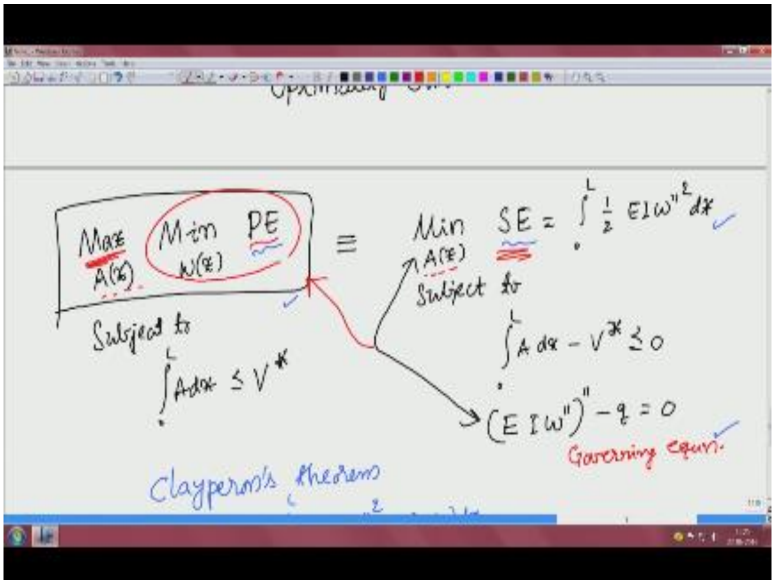

That is how this works I remember that we have here a max min problem okay that is we are designing that is by varying a of X and we are minimizing to get the governing equation that can be done for a general elastic structure also not just for column buckling problem any structure we can do this minimization of potential energy with respect to let us a state variable for abeam column whatever minimize the potential energy that happens to be equal to negative of the strain energy okay at equilibrium so we can say we can maximize negative energy with respect to a of X this problem max min problem is equivalent to minimizing strain energy with respect to K of X subject to okay.

This also needs to have the volume constraint that does not go away subject to volume concern 0 to L A dx less than or equal to v star okay this is subject to the same thing 0 to L A DX minus v star as we normally write less than or equal to 0 but here we have to write if it is a beam problem we have to write because the strain energy for abeam if I take 0 to l half e I w double ‘ square DX so at put a governing equation e I w double ‘ double ‘ there is a loading q to 0 okay so instead of posing the problem with the objective function and the constraint which is the governing equation.

This constraint is a governing equation okay instead of doing that we can put it as one maximum problem in order to see where it comes from we have to discuss what is called clarions theorem clarions theorem which is quite simple that is if I were to take the potential energy okay remember that here this is to energy here it is potential energy how are we doing this instead of putting the objective function and the governing equation we can get away with one maximum problem like we are doing for the column now column again

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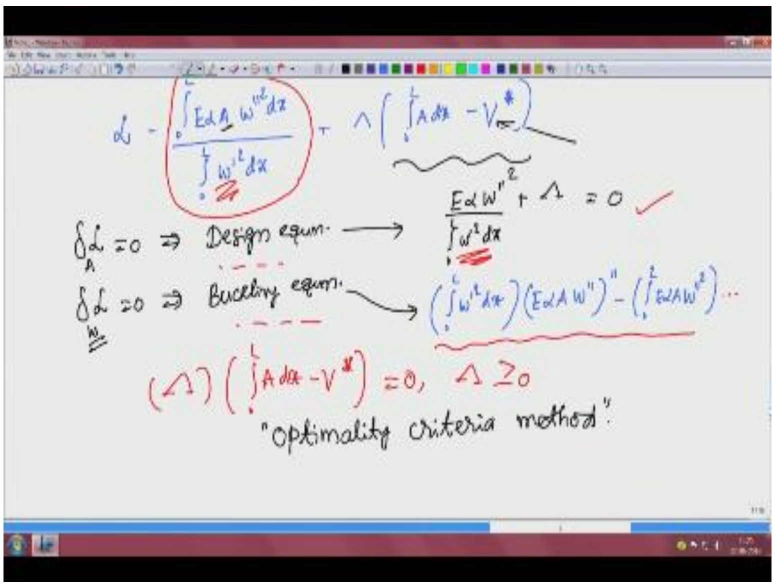

We are we discussing this because we post this as a max min problem okay with a Rayleigh quotient that made an in general for structural optimization okay so Clarence theorem where we have potential energy let us take it for a beam 0 to L half eI w double m square minus there is a loading q minus w DX this is a work potential negative of the bottom external forces now when we take the variation of potential energy with respect to W that gives us the equilibrium equation right our variation if I do so we write 0 to Le-I-e-I-o right I properly e I w double ‘ Δ double ‘ okay minus q Δ q Δ w DX equal to 0.

This we said is a principle of virtual work okay so now if you look at this problem here this Δ W is our variation is arbitrary okay that is like virtual displacement okay this is the statement of principle of virtual work this portion is internal virtual work this is internal virtual work this is external virtual work okay so again what is Ivy w that is internal virtual work evw is external virtual work both being equal is the principle of virtual work so I VW is equal to e VW which we had discussed earlier is the principle of virtual work are as we call the weak form of the governing equation that is what this is also called the weak form.

Weak form and principle of virtual work p VW and principle of virtual work okay not that Δ V is arbitrary when it is arbitrary let us say at equilibrium this must be true let us take it as W itself okay when we do that what this gives us is 0 to L e I w double ' Δ W ' is www ' so that becomes square DX is equal to 0to L q into Δ w TX okay now if you see what is potential energy here so we can take this as put half and half I will take half of this half of this to the other side okay and bring this over here okay only half goes there so half will be left behind here 0 to l e I w double ' square by 2 half is left behind other half I will take it there that becomes 0 to L e I w double ' square by 2 DX okay.

Half I have taken the right hand side half hour behind DX and this one I am bringing back here so that becomes minus 0 to l q w DX okay now if you look back and see what this is what we have here is potential energy that is our definition of attention is the strain energy this is actually strain energy and this is included in the negative sign is what potential these two added up becomes our potential energy what is on the right-hand side minus strain energy okay so panache energy is equal to minus strain energy okay is the Clarence theorem at equilibrium not everywhere at equilibrium okay.

And because of that when we want to minimize strain energy we are maximizing potential energy with respect afx minimized energy with respect to a of X maximizing potential energy with respect to FX and minimize the potential energy gives us the governing equation so this max-min formulation helps us solve problems and that is how we can do the strongest column problem by doing it as this maximum problem over here okay so this can be numerically done using the so called optimality criteria method alright so we will look at a program in the next lecture as to how the strongest column thing works thank you.