Indian Institute of Science

Variational Methods in Mechanics and Design

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Hello we were discussing Eigen value problems in the last lecture and that was in the framework of calculus of variations okay so calculus variations allows us to deal with the free vibration problems which lead to what we call I Eigen value problems continuous so we have Eigen values and corresponding Eigen functions we discussed that in the context of a bar but we can apply it for beams cades membranes actually any structure we discussed the buckling case we will discuss that today in detail.

So that we can also design a strongest column where we minimize or maximize not minimize maximize the buckling load all using calculus of variations let us look at some of the basic equations that we call Eigen value problems okay we discussed the case of a string first and the string problem.

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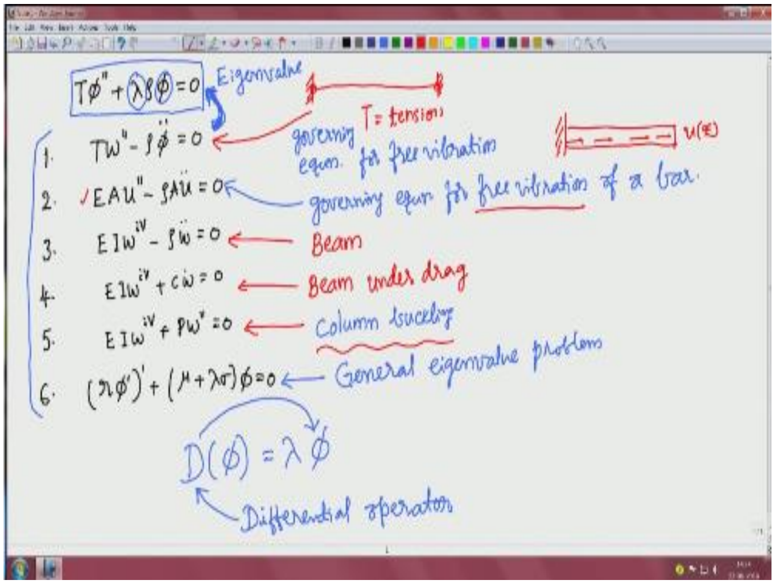

So if I were to draw a string which is at both ends it can be pinned also there is tension T in it that is what we consider and that gave us the Eigen value that that governing equation for that was this okay that is the governing equation this is the governing equation governing equation for free vibration that is when we Gateaux Eigen value problem for elastic structures and that gave us this problem which we call Eigen value problem okay from the governing equation we saw by using the separation of variables technique how we can get this Eigen value problem and λ is our Eigen value and Ø is our Eigen function.

Okay there are many solutions infinite solutions for that hat is what we discussed last time using framework of calculus of variations similar thing can be done as we have written equations here this one is a governing equation for a an axially deforming bar governing equation for again free vibration free vibration of a bar so a bar looks like this basically you have written it for uniform cross section that is why EA or outside of the way would have written EAU´ and whole prime.

Okay we have a bar with let us say fixed and free where it is deformation is this direction that is u(x) and that is the free vibration again for it if there is a bar and you pull it and it is going to vibrate there are different natural frequencies again infernal natural frequencies that come about in the form Eigen value problem if you use this governing equation just like we had done for a string okay.

Now this one over here as you may recognize is for a beam transverse vibrations of a beam free vibrations and this one here where there is only w. no w.. it is for a beam that is under some drag so this is beam under drag that is I have a musical instrument wire fix there and then you drag it in water and at that time what will be is natural frequencies that is what this equation is the one that we will discuss today is for the column but a column buckling that also this is the governing equation we are writing only governing equations here all these are governing equation just like the first one gave us the Eigen value problem okay.

This first one gave us the Eigen value problem we can get Eigen value problems for all others okay this is a column buckling and here is one that is very general you know this does not we can get all others from this in a way at least the second order things there is a general Eigen value problem general Eigen value problem okay basically I can only problems are characterized by a differential operator so there will be some differential operator acting on a function Ø gives you back that function something like this is a differential operators.

Differential operator differential operator okay that transforms the Ø function back to λ times all of them have this characteristic okay you know to understand this and then discuss in detail the column buckling problem not just buckling analysis but also design okay what we will do is we will get a little bit more understanding of the Eigen value problem.

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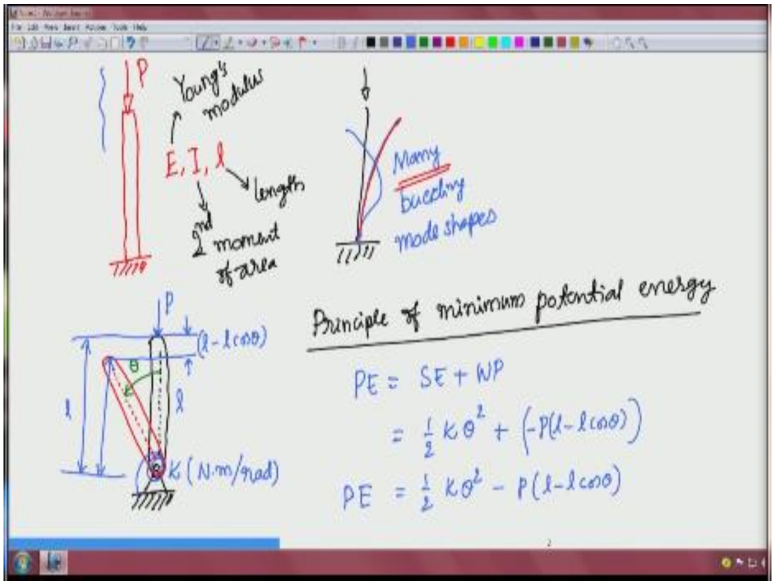

And buckling at the same time by taking a simple example the example that we take is that let us imagine that there is a column okay and there is a compressive load P on it so this column has eggs modulus E second moment of area I and length l the usual things so this is young's modulus and this is second moment of area and this is length if you know these three can you get at what load P the column B buckle buckling means that if the column is taken like this under the axial load at some point it will not be able to stay vertical and just we will buck like that are like this many other ways when we say it is an Eigen value problem.

There are many buckling we can call them mode shapes okay we are mostly interested in the one that is fundamental buckling that is the one that I am going to draw in over in red this is a fun fundamental mode and they will be higher modes a second more third more and so forth there will be many are infinite buckling mode shapes to understand this problem this buckling problem let us take a simpler system let us take actually a rigid body with a let me joint here which is attached to some frame let us put a Tensional spring of spring constant k whose unit is going to be Newton meter per Radian radiant because we have been joined it can actually turn let us say it this thing has turned like that okay.

Let us draw a reference line here and a reference line there let us call this angle from here to here let us call this θ okay that it is free to rotate we have indicated an angle now let us say that there is a force their compressive force just like what we have over here AP okay let us call the length of the bar l and there is spring constant torsion spring concert rotation spring constant k which is Newton meter per Radian as it turns there is going to be a torque generated here okay let us take this problem and then say what happens when we start with θ equal to zero and apply load P and then see what happens so we will use energy method or our principle of minimum potential energy.

Let us apply the principle of minimum potential energy let us apply principle on a potential energy okay when we do that we first need to write potential energy which is strain energy plus what potential strain energy is there in the form of the spring that is ½ k θ^2 okay k θ is torque times the angle θ averaged or this thing will be this like translational spring if it displaces by Δ and spring constant is K ½ k Δ^2 is the strain energy here ½ k θ^2 + work potential is negative or the work done by external force so here we have P which is downwards and the p when it acts here there will be a difference between these two points okay.

That is a distance we need to take original one that is from here to here is l and here to here is this is l and this is θ that will become l cosine θ so this is going to be l - l cos θ okay so we write that as l – l cos θ in the same direction so that will be positive and it is a negative or the work done so we will have negative so overall what I get for potential energy is 1/2 k θ - P x l minus n % θ.

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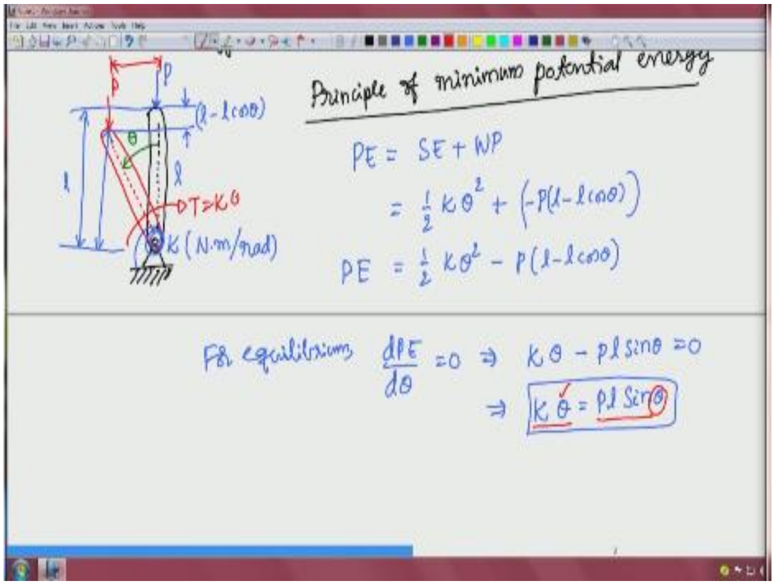

We know that in order for equilibrium okay for equilibrium for static equilibrium we must have dBE / d θ is our variable here that is equal to 0 that will give us k x θ ½ k θ^2 will give k θ when you take different derivative and this one there is a minus there is a minus cos θ derivative cosine θ is again-overall it will be minus force and the derivative minus will give us p l sine θ that is equal to 0 okay so what we get is k θ = Pl sine θ that is our static equilibrium equation okay.

If you look back at this problem what we have here we see that the moment balance for this will exactly give that if P has moved here so when it deforms the P is over there right so that moment is P xl sine θ l sine θ is the distance between these two that is l sine θ okay so Pl sine θ will be opposed by the torque that comes here when it is rotated there the torque you going to be k θ and this torque so balance we get the equilibrium equation look at this equation and this has the nature of an Eigen value problem.

They a differential operator except that here it is not a differential operator we are taking θ applying doing some function getting back data times a constant okay and you can see this problem also has many solutions.

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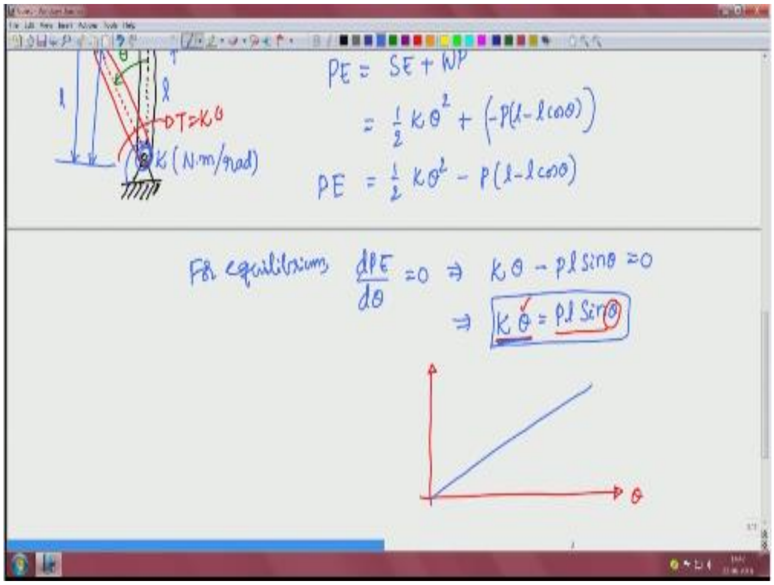

And that depends on the value of k and pl okay so if I were to plot this to show θ then k θ let me do this in blue color okay k θ that will be something like this okay.

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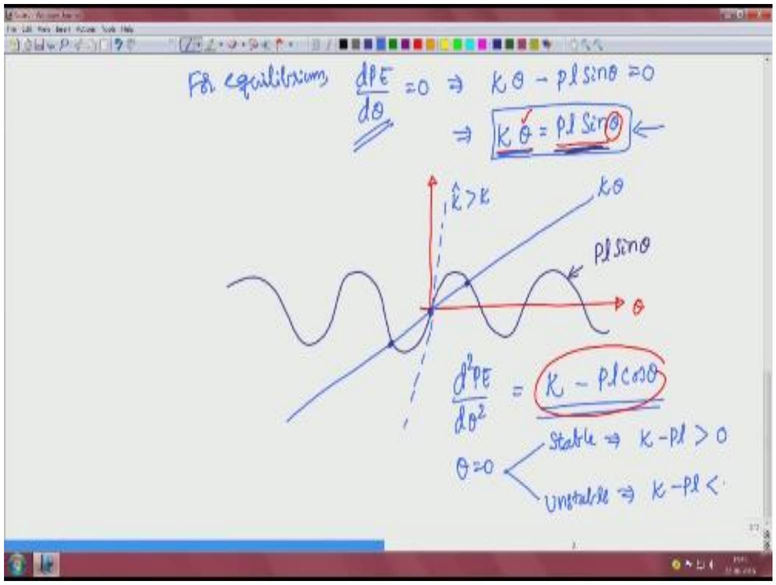

I can do both sides if I need to write this is k θ now let us also do same green color p l sine θ this it is a sin θ regular sine θ over this color okay and the other side also okay now we are all they cross do you see this point and this point and this point we have three solutions here but were k θ and this other one which is this which is T L sine θ okay now we see that depending on the value of K number of solution changes if k kappa were to be very high and the curve were to be like this okay this is some k hat which is very large larger than k we have 0 is the only solution that is a solution of this equation.

As Kappa decreases RP increases likewise so more and more solutions come right when you had more and more solutions whether it will be stable or unstable equilibrium will depend on the derivative of this again so we will write d^2 PE/ d θ^2 one more derivative will give us k minus P L cosine θ this quantity so what we find here is that let us say θ equal to zero is always a solution whether it is a stable solution or not depends on the value of this second derivative stable means that this thing this k- Pl cos θ that I cos θ that is zero if you have that equal to zero cosine 0 is 1 so K - Pl should be greater than 0 okay it will be unstable equilibrium if k- Pl is less than zero.

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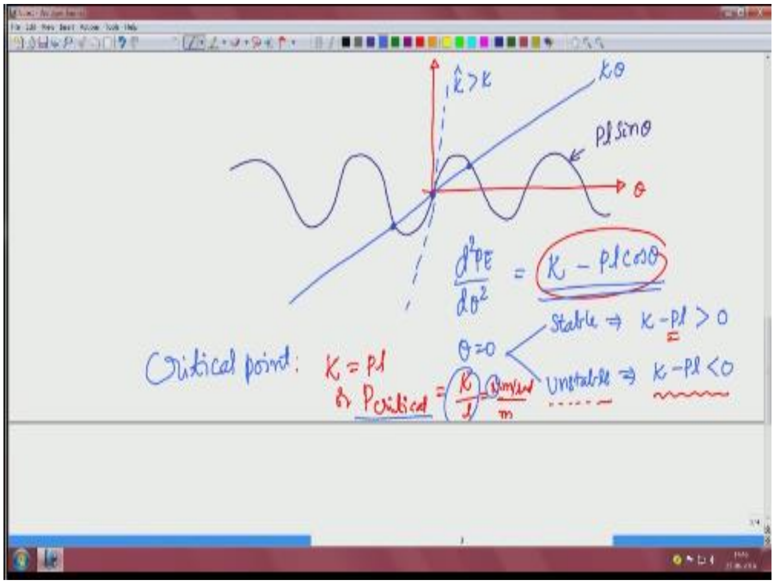

What will be the in between value or the critical point critical point where it changes from stable to unstable get to begin with when have very small p obviously k- Pl will be greater than zero as I increase p that is when it has a chance of becoming negative and hence and stable okay as a change there will be a critical value which will be k = Pl or T critical field critical in this case is k/ okay this kappa again Newton meter per Radian and meter so you get the units of Newton anyway radiance does not have unit.

So we get the force P critical force we get that okay and of course more and more solution is coming that is a nature of this Eigen value problems okay but we got a critical you k/ l of this problem all right.

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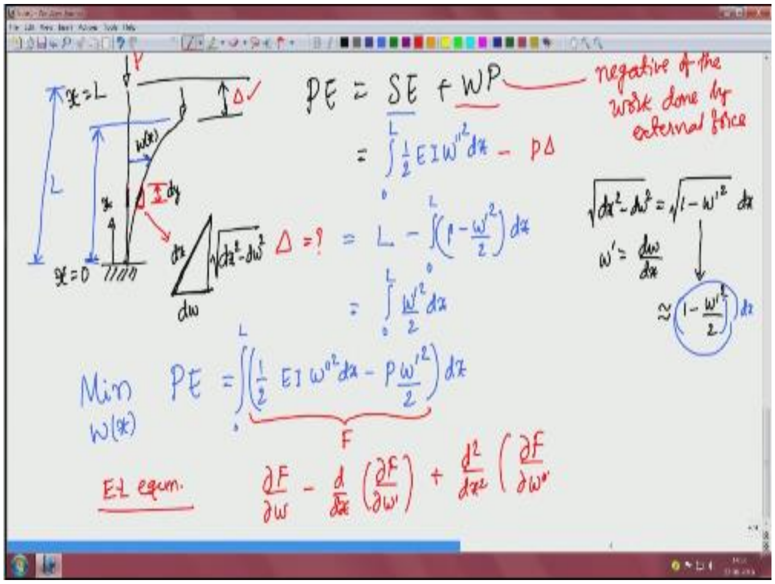

So what we do here for a column is a similar analysis where there are multiple solution that come about and a column as the one that we have we started out today so this particular column as this load increases the load is that much and that merge and that much at one critical value okay at one critical value this makes this to just buckle like that it just buckles under that load okay we can use calculus of variations to do what we did with minimum potential energy principle we will do that they are also accept that it will not be in one variable like we had this data here we will do that for the continuous function for the column okay.

So let us take that example see if I have a column I will not show two lines I will just show this way there is a load on this okay if it were to buck up let us say it is fixed and free boundary conditions so it has moved down something like that okay we want to know how much this is because we need that to write potential energy we have strain energy plus work potential okay strain energy for that we need to know when this is going I will call it WX here we are taking X like this normally we take horizontal / columns we take the vertical x is equal to 0 here x is equal to l here okay.

Length of the column so now strain energy I will write it in blue I will integrate 0 to l ½ EIW w $W^{\prime 2}$ dx that we know already if there is transverse displacement w (x) what is the strain energy I will write the work potential okay which again has in it strain energy sorry this is work potential that we have this is the negative of the work done by external force work done by external force external force in this example this problem is P the P there okay so that since it is negative I put a minus sign and then p times let us call this Δ .

I am okay first of all I do not need to put integral I do not know yet it will integral will come p times Δ P times Δ negative of that is work potential so what is Δ okay this Δ is if I take a little segment there okay little segment and that one which was here will be dx that will be dx inclined what we have is dw what we want is this that if I integrate for all segments I get this Δ okay let me blow it up let me draw larger one like that okay this is dx which was what was here which has come here okay that is dx what is this that is the incremental transverse displacement w so Pythagoras tells us that this one if I call that dy or something okay what we want that height right that is dx^2 - dw^2 okay.

Let me just write $\sqrt{}^{2}$ dx ^{2} -dw^2 I can write this as 1 - $W^{\prime 2}$ take the dx out where $W^{\prime}$ in our notation is dw/dx you know that this^2 $\sqrt{1}$ -$W^{\prime 2}$ can be approached summated as 1 -$W^{\prime 2}$ by 2 approximation when $W^{\prime}$ is small which is what it is it just buckles we are not worried about post buckling analysis just when it buckles small displacement that what we get now what is this Δ so what we guarded dy is the length from here from here till here what do we get this particular thing that we have posed dx will also be there right so dx if integrate this I get that but I have subtract from L so this Δ that we want is the length the total length from here till here that a total length from that we are subtract this 0 to l 1 - $W^{\prime 2}$ /2 dx okay.

If I do that I get l and there is this one that will also be elven integrate what minus of minus I get 0 to l $w^{\prime 2}$/2 dx okay with that I can write my potential energy strain energy which I have half e I w double $^{\prime 2}$ dx integral 0 to L now I also have the same integral minus P times Δ that also comes under the integral sign I get this okay dx that is a potential energy using principle of minimum potential energy minimize that with respect to this WX and when you write the Euler-Lagrange equations for it okay.

So I will write Euler-Lagrange equation for it and for that this becomes our integrand f so there will be $\partial f / \partial w - d / d x (\partial f / \partial w' + d^{2)} / dx^{2} (\partial f / \partial w'')= 0$ and there will be boundary conditions okay.

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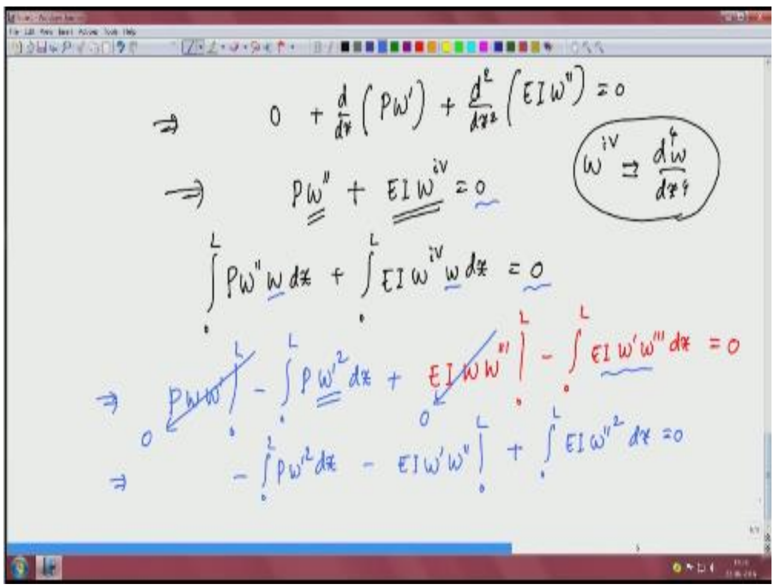

That is what we have we write it for this the first one anyway that is 0 because there is no w here w ´is there so minus W ´is there that becomes minus of minus plus okay $\partial f / \partial w'$ d / dx (∂ / w ´0 will become p that is y minus of minus I already took care of that minus and this minus becomes plus pX w ´2 + w'^{2} they canceled so that is what I get and then plus d^2 / dx into EWI´´ that is what I get when I take derivative of that respectively ´ ´equal to 0 so what I got now is P W you p is constant I take another derivative become w´ ´plus I get plus EI if I take E and I are constant they do not vary with x content area of cross section same material throughout the beam throughout the column I will get this to be fourth derivative equal to 0.

So when I put W^{IV} I mean d this is equal to D to the 4 w / dx to the 4 that is our notation okay so we have this now this is the governing equation for the column when there is a P symmetry had gotten for a simple bar such as this we had gotten this equation right k θ equal to P L sine θ and what we have here is similar okay pw ´´plus EIW4 derivative 0 right now this equation we can get the Rayleigh quotient for it from which we can get the buckling value and Eigen value problem because I can value problem if you look at it if I take w ´´okay.

That is what I have taken here a differential operator is being acted on it w two derivatives you are taking you are getting back w ´´that is the nature of the Eigen value problem this is an Eigen value problem so you can write the Rayleigh quotient for it okay so for that what do we do from our previous lecture so we multiply w ´´with w integrate from 0 to l dx similarly we do for the other one EI W fourth derivative w dx that will equal to 0 all we have done is x w you x w and integrated that is all if it is zero here it will be 0 there okay now if you do for this one the first term we get the first function in macula second function first function integral a second function 0 to l minus derivative of first function w integral second function that will become^2 dx okay and now let us do for the other one in red color.

This one second term so that also first function which is w integral second function that willbe come triple ´ 0 to L minus integral of derivative of first function integral of second function dx that is equal to 0 one more time we are only do anyway boundary conditions this would go to 0 we have a column either w´ will be 0 okay and what we are left with is 0 to L P $W^{\prime 2}$ dx and then likewise wr or w triple ´will be 0 here the boundary conditions that you get they will also be gone we are left with this here we will do integration by parts one more time we keep on doing until we get like a like a squire term quotient okay .

So if I do this integration by parts minus again first function I will take that integral a second function 0 to L minus of minus plus 0 to L derivative of first function integral a second function that gives us square dx equal to 0 and once again this also boundary conditions will make it good zero.

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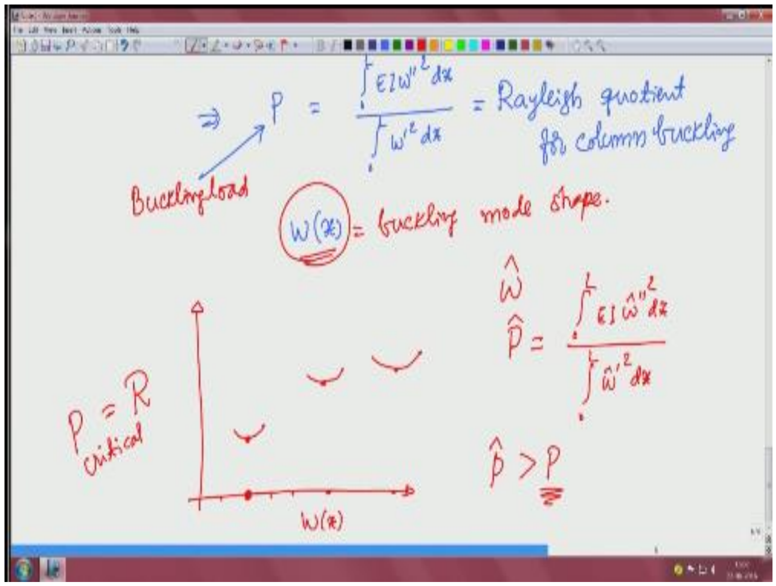

Now we get the P from the above we can write this as 0 to l EL w ´ ´2 dx divided by 0 to l w ´2 dx this is a Rayleigh quotient for column buckling so Rayleigh quotient has both numerator and denominator positive so it always positive so buckling load is passed you cannot be negative it is compressive already we have taken then it is just positive real a coefficient for column buckling okay why is this important as we said if you take any W here this becomes the buckling load.

And this is buckling mode shape we noted one minimum characterization theorem pertaining to Eigen value problems in the corner Rayleigh quotient Rayleigh quotient has this amazing property that it becomes a minimum whenever w becomes a buckling mode shape you can take any function okay let us say let us just imagine a you know w axis okay when I say W access that these are functions you know if I take different points will have different functions they are all functions you see if you take that wherever it the Rayleigh quotient for column buckling or any problem when ever this becomes a buckling mode shape or a mode shape then Rayleigh quotient will have minimum there okay.

Let us say another one over here will have minimum again there another one over there will have minimum again there so everything is a minimum so all these points Rayleigh quotient locally becomes minimum that was a minimum characterization theorem that we had another thing because the minimum is that if I take aw w hat then I write this Rayleigh quotient for this p hat okay by evaluating E I w hat$''^{2}$ dx divided by 0 to l w hat$'^{2}$ dx you get a p-hat value this P hat you will find is larger than this P critical that we have the real buckling load in other words whatever value you take you will get a value that is larger than critical load if you minimize Rayleigh quotient okay.

The earlier question that we have there or in this particular problem it is P critical if you minimize this Rayleigh quotient for buckling you get the buckling about using this what we do is we will solve a problem.

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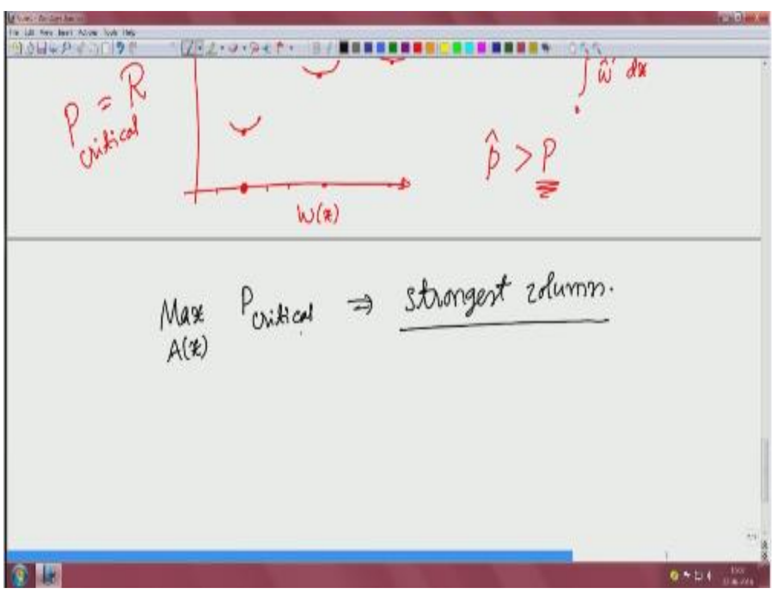

Where we want to maximize the P critical that will be the strongest column we want to maximum P critical by varying let us say area of cross section of a column okay this will give us the strongest column if I strongest meaning it is not going to bucko buckling has this phrase that we buckle under pressure right that is you are becoming unstable so here when you say strongest column that will have maximum P critical that is what we will discuss in the next part of the talk thank you.