Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello again so Euler entered the field or enters the scene in 18th century.

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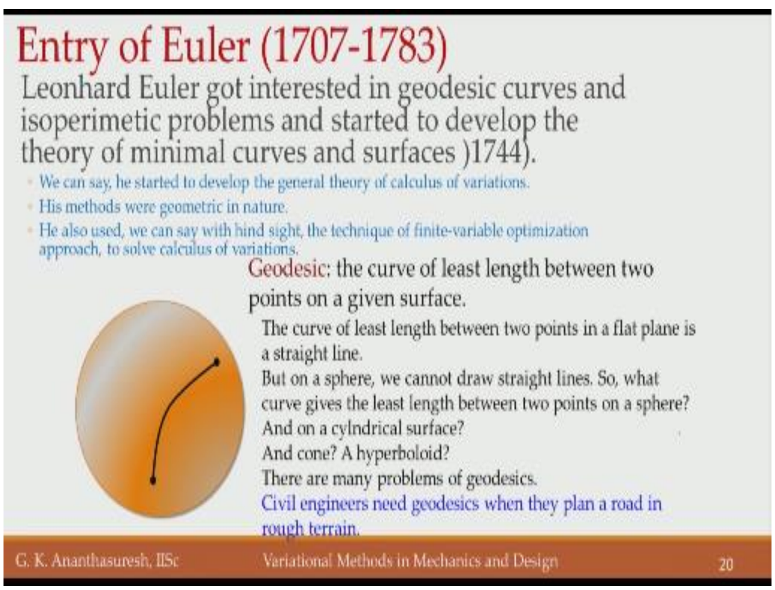

Then he actually contributed a lot to calculus of variations so Euler got interested in what are called now geodesic curves okay. Geodesic curves are the least distance curves on any given surface in a plane if we are given two points A and B the least distance curve is straight line because straight line is the least distance toward go from point A to point B.

On the other hand if you are on the surface of a sphere such as the one shown here you have a point over here and a point there then straight line cannot be drawn on a sphere right. So it has to be a curve that can be drawn on the sphere it turns out that the curve that will give you the least distance on a sphere will be arc of a great circle that will be passing through the two given points A and B okay.

Such things are called geodesic curve, geodesic curves are the curves that give you the least distance path any given surface whatever surface you take you know can be a highly on surface or a hyperboloid or any surface you take if you are given two points if you find a path that takes the least distance to go from point A to point B that is a geodesic okay.

The geodesic is something that civil engineers use even it today if you want to layer road between two cities or two places they have to look at the ups and downs of the terrain and find the road that will be the least distance following all the ups and downs okay. So the geodesic so I will have got interested in these and started solving those problems and others and there are general theory for calculus of variations okay.

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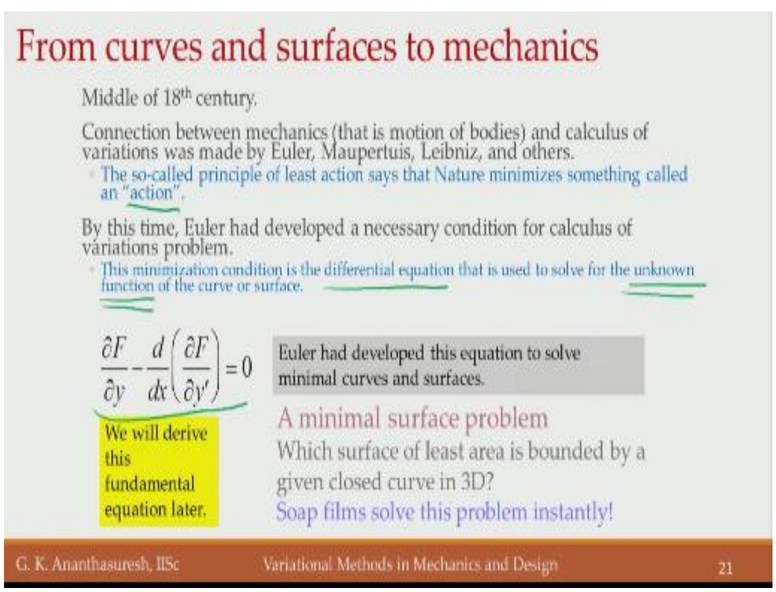

Basically these curves and surfaces have lot of relation with the mechanics also in the middle of 18^{th} century lot of other people some of the names are given here including Euler, Maupertuis, Leibniz and we already found in the context of back to one problem and Lagrange all these people had started looking at geometry of curves and surfaces and applied that to the mechanics to develop some integrals something is called an action integral and other ways of looking at mechanics.

And they try to look at minimization most minimization in all of these problems which when you try to minimize and find a condition that lead to a differential equation okay to solve for the unknown function. So unknown function is the cornerstone of calculus of variations right so that has the unknown function unity when you have known function the equation that enables you solve that is a differential equation.

The differential equation comes out as a necessary condition or a condition for the minimum of a functional or an integral kind of thing that we have seen in the last slide okay. So this is known as Euler-Lagrange equation okay, he developed this equation for the geodesics and other things Euler okay, we will understand what FE is and what this actually means later on for now let us recognize that there is something called Euler equation.

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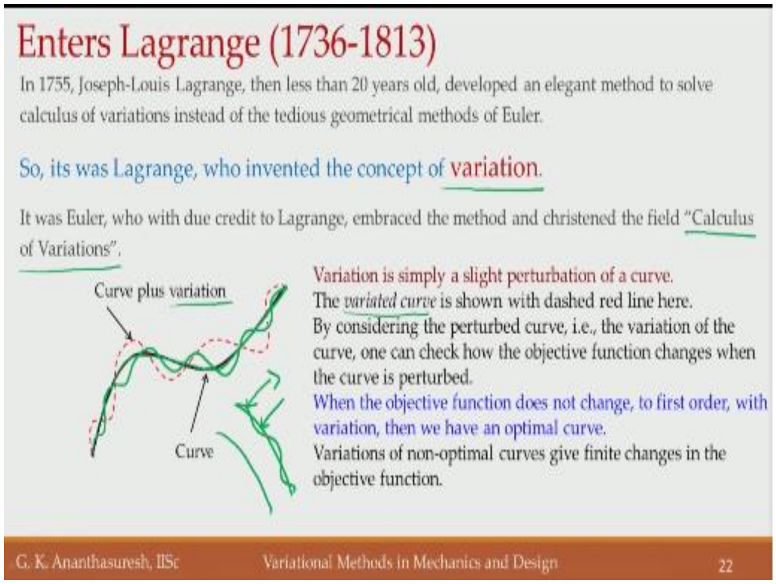

But also the equation is known as Euler Lagrange equation because Lagrange who came after Euler like where their contemporaries for some time in he when he was 20 years old when Euler came to know that this young man Lagrange had solved the problem that he had solved and derived the same equation but in a different way.

And Euler thought that Lagrange's method was more elegant and in fact Lagrange had used the concept of what we today call variation and Euler liked it so much for the elegance of the solution and he said do not take my method take Lagrange method instead which is what we adopt today, but equation that we had put in the previous slide is still known as Euler-Lagrange equation okay.

It can solve, you know surface problem minimum curve problems and many things in mechanics and other fields and what are the things with regard surfaces is the soap films which we will see in the next lecture how soap films take the minimum surface given the amount of soap film that is available to them and basically one can make the surface energy okay.

But also it turns out to be minimum surface problem and we look at all of those things in the next lecture okay. So Lagrange introduced this concept of variation to this field of calculus of variations. Now is the time for us to understand what variation means okay. Let us say we have a curve such as the one that is shown here in solid another curve which is shown in dashed red line which is a perturbation of the original curve.

Here for clarity I have shown the perturbations a lot that is if there is this solid curve right, let us say it is perturbed like you imagine a string somebody has perturbed it starts vibrating right, then it becomes something like this and that is the variation. So you have a curve and you have perturbed it a little bit so it deviates from the curve in some fashion deviation can be very small something like this also right.

So that is the variated curve or a curve plus variation okay, when you do that this gives rise to the notion of local minimum when do you say a particular curve let us say I have a curve like that when do I say this curve minimizes something I say it when if I consider small perturbations around this curve such as those if this perturbed curve is substituted into that function or objective that we have, then the amount of the measure of the value of that functional array integral will be higher than what it is for the original curve.

So for all the perturbed curves the value of the functional should be here than it is for the original curve that should be greater than the value that is there okay. In which case we call that curve a local minimum okay, so that is why variation concept which is a perturbation which is a very important concept in calculus of variations it was introduced by Lagrange okay.

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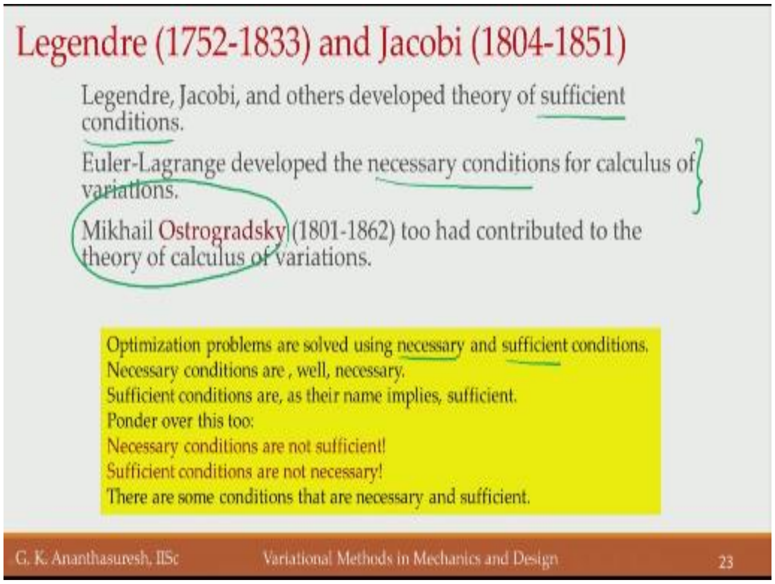

Later on after this Euler-Lagrange looked at the necessary condition Jacobi and Legendre and others consider the sufficient condition also okay. And so while Euler-Lagrange equations give you the necessary conditions the Jacobi condition and Legendre condition they give you sufficient conditions for this.

In between another person a Russian mathematician we have Ostrogradsky also contribute calculate variations if you look at some of the Russian books now they actually call them Euler-Lagrange and Ostrogradsky equations. So optimization problems always have necessary and sufficient conditions it is important to understand what they mean in fact they do not mean anything more than what they say necessary conditions are necessary that is absolutely needed and sufficient conditions are sufficient.

Basically the two words say what they mean necessary conditions may not be sufficient or not sufficient and subject conditions are not necessary. Sometimes you can have some conditions which are necessary and sufficient okay, these are important words, but I do not have to worry about them because they basically say what they mean necessary means is necessary, sufficient means is sufficient okay.

So Euler-Lagrange had dealt with the necessary conditions and other people had dealt with the sufficient conditions that is another development in the field of calculus of variations okay.

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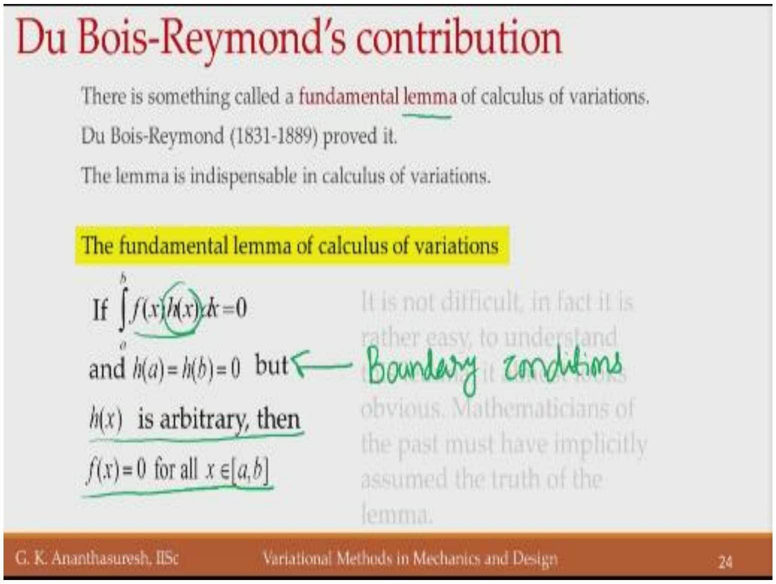

And much, much later so there is first person named Du Bois-Reymond he a fundamental contribution which is called the fundamental lemma of calculus of variations which many people had assumed as we will see later on but he had proved it rigorously and that is the fundamental lemma. So lemma is something that is slightly less in status as compared to your theorem and that is what is Du Bois-Reymond had proved okay.

This fundamental lemma of calculation is actually very simple let us say that you have a function f(x) if the integral of f(x) from A to B x H(x) so the integrand now is f(x) and H(x)dx=0 and this HX is arbitrary like our variation it can be any perturbation okay. So when H(a) H(b)=0 they are the boundary conditions okay these are the boundary conditions these are called n conditions or boundary conditions okay.

If this is equal to 0 for any H, so H is arbitrary then what a fundamental lemma says is that f(x)=0 for the entire interval A to B including the ends okay, that is a fundamental lemma which is intuitively very clear if you are integrating a function with an arbitrary function for any arbitrary function the integral has to be 0 if f(x) is zero identically for all values A to B then this is the fundamental lemma which is intuitively clear, but he had proved it rigorously.

W will use this fundamental lemma when we derive the Euler-Lagrange equations later in the course okay.

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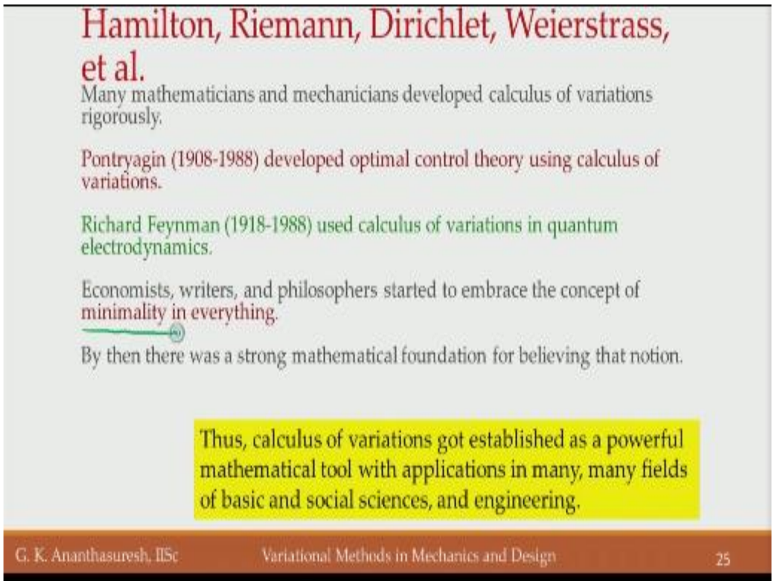

Much later when it comes to mechanics or what we call variational methods Hamilton, Reimann, Dirichlet, Weierstrass a number of other researchers had applied calculus variations to mechanics and derived wonderful energy methods or variational methods for mechanics and at the same time Pontryagin much, much later in 20th century he developed the optimal control theory that is also related to the calculus of variations and Richard Feynman the noble laureate are famous for his physics lectures.

He used calculus variations in electrodynamics before that Schrodinger had used calculus of variations in deriving what we now call Schrodinger's equation and economist, writers, philosophers a lot of other people also use the concept of minimality even if it is not calculus variation there is a notion of minimality in all of these things everybody uses some way of minimizing like we said in the last lecture.

In life we try to optimize things are minimized so the calculus variation if you know you will have a powerful tool which is a foundation for many mathematical physics concepts okay.

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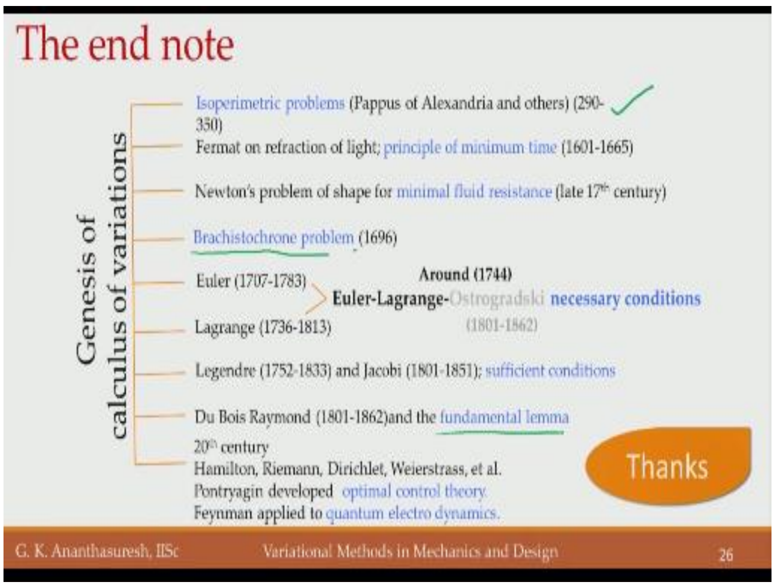

Just to summarize what we have discussed today if you look at historically from ancient time BC there was a isoparametric problem and then the Fermat conjecture about refraction of light that is light rays taking minimum distance path and then the classical problem of brachistochrone which is a very, very important one that was in the at the end of the 17^{th} century and then Euler and Lagrange derived and Ostrogradski also contributed to necessary conditions and Legendre Jacobi and others did the sufficient conditions.

And a fundamental lemma was proved by Do Bois Raymond in the 19^{th} century and 20^{th} century used calculus of variations in mechanics heavily by all the people that listed optimal control theory has it and quantum electrodynamics and showing this equation all of these use calculus of variations. So it is a very important tool today we discussed the history of calculus variations and in the next lecture we will discuss some of the problems in geometry as well as mechanics, thank you.