Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello we have been discussing calculus of variations and its applications in mechanics as well as structural design or optimal structural design. And we are looking at new answers of calculus of variations where we consider general boundary conditions that led us to this way straws Edmund Connor conditions where we have functions that are non smooth.

And then we also looked at invariance and know this theorem which talks about transformation of the variables in a calculation problem and how if there is a parameter there is something that is constant that in the case of mechanics gave us the conservation of energy, conservation of angular momentum these two things we did. Similarly you can also do conservation of linear momentum is a parameter in this theorem that α is a translation as opposed to being a rotation that is what we did.

Today we are going to look at calculus of variations a slight variation theme to apply to what are called eigenvalue problems sometimes about free vibration. So principle of monopoly energy is only for statics. Now we are going to move to free vibrations and we also discussed Hamilton's principle that is for dynamics in general okay.

So we start with again a very simple example and that is always one dimensional and in this case a one-dimensional example is a string a musical instrument the vibrate it produces pleasant sound that is to do with the frequencies and mode shapes of the string we look at that in the framework of calculus of variations today.

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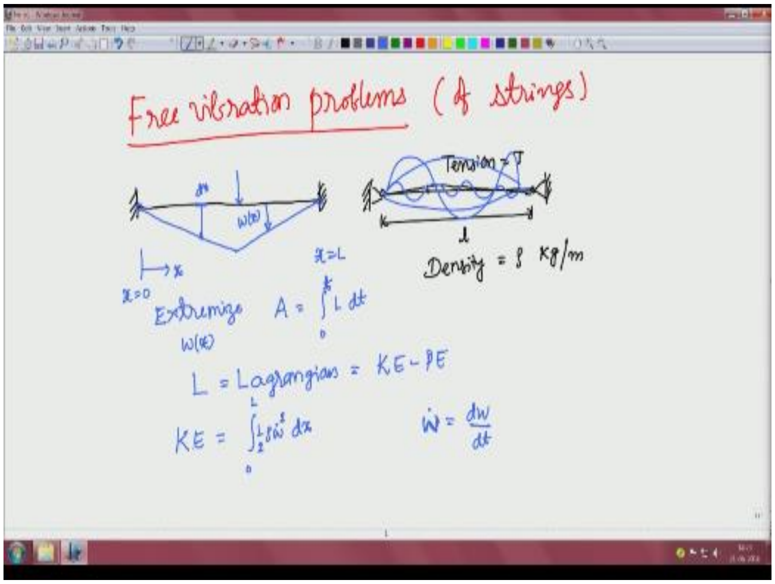

So let us consider a string, so what we consider here today is free vibration problems in the framework of and specifically of strings that are taught meaning that intention using the framework of calculus of variations. So let us take a string that is kept taught between two points okay it can be a joint it also does not matter for a string, the string does not have any bending strength in the sense that if I apply a force here a string is going to deform like this it does not bend like a beam does it just goes like that okay.

Now so that is why we can we can show whichever way you want between two points we have a string okay, and there is tension in it there is a T tension equal to t let us take this length to be L a string that is length L tension T and the material property that we would need here is density mass density you note by P okay.

So now we will actually define this ρ as you know kilogram per meter cross section unit cross section, so we take it density like that way linear density kg/m if you have such a thing we want to write the governing equation for it when we pluck it. Let us say somebody comes and plucks it you know something like moving it over there then for the moment it will become like this right you become like that and then it starts vibrating in some bit hit me vibrate like this or like that many which ways you have a string we are part having it basically okay.

So how do we write equation of motion we use calculus of variations which is to say that we extremize this is optimization but we do not commit to minimization or maximization that is what Hamilton's principle tells us extremize what do we extremize the deformation wherever it goes let us call this w(x) as usual x is measured along the axis here x= 0 here x=L okay, extremize with respect to w(x) that is a function as usual in the calculus of variations.

What do we do we have this action integral which is integral of Lagrangian with respect to sometime t Ldt and what is L that is our lagrangian which is kinetic energy minus potential energy okay. What is kinetic energy for a vibrating string, so we take a little piece wherever that is vibrating it is a DX okay.

Whenever there is a DX we are to integrate from 0 to L the kinetic edge of little thing as it you know vibrates by the W there okay. So when that we have that then we have write it as half into mass per unit length okay, that times DX ρ DX will give me the mass now to put MV^2 half MV^2 now I have half M how to put V^2 that I will put as w.^{2} because w is the transverse displacement when I say w. what I mean is dw/dt okay this is not ω this is w. So that is dw/dt that is the kinetic energy okay.

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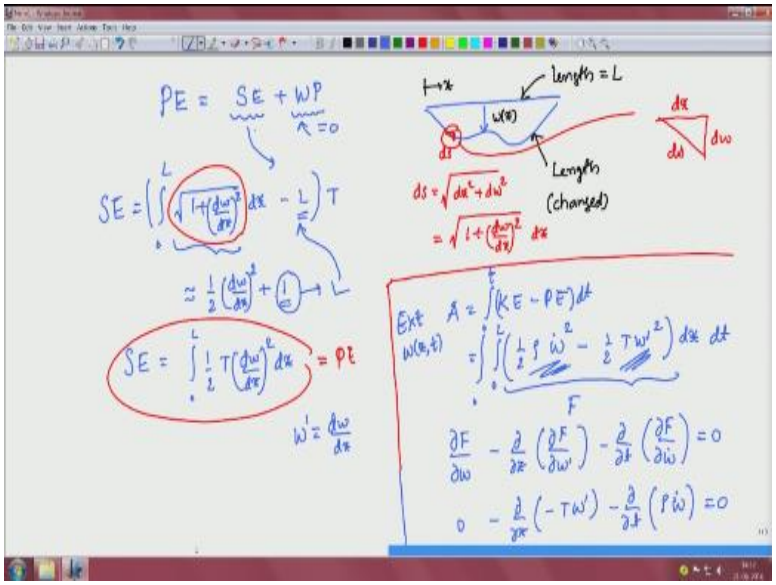

Now let us also write the potential energy potential energy that has two component strain energy plus work potential this is the internal energy stored there are an elastic body string is an elastic body and then external force since we are talking about three vibrations this is not that, that is equal to 0 that is a negative of the work dimensional force that is equal to 0. For the internal force we have the tension T and how much it is stretching is what we need to put.

So when we have a string and when it stretches okay, let us say I have this string when it stretches in some manner that is our w(x) that is w x everywhere again X goes from 0 to L. So what is this length we know the length of this equal to L before stretching what is the length of this curve okay, this is the changed length the difference of these two times T will be the strength energy our internal energy stored in the string.

To get this change length let us take a little element somewhere as you really do that is ds the d/s is Pythagoras theorem square root of dx^2+dw^2 okay, that is this portion that is dx this is dw okay, I mean that is too small to see let us blow it up see if I take this as the ds this way this is dx that is dw that is ds okay. So now this can be by taking dx outside I can write this as $1+(dw/dx)^2$.

So I can write my strain energy here as 0 to l the little thing how much work how much is stretching that I will write that is I will do square root of $1+(dw/dx)^2dx-L$ and this whole thing I want to multiply by my tension T in the string okay, that would be the internal work or our strain energy okay.

Now we are going to approximate this because w is going to be small the small vibrations score square root of $1+(dw/dx)^2$ for small sloops or small vibrations we can approximate this as ½ $(dw/dx)^2$ okay square root of 1+x for small values of x = 1 plus in this case the 1 will become dx and it will cancel L, so when I say, I am approximating it this is 1+ times dx I am approximating only this portion square root of $1+x^2$=R1 square root of 1+ x can be approximated as ½ (x)+1 okay.

Now dx is there when you do this integration that will lead to L okay when integrated that will lead to L that will cancel this L so our strain energy turns out to be 0 to L half then we have the T there $T(dw/dx)^2dx$ okay. Again when I integrate 1 that leads to EL that L cancels this - M they cancel each other. So how the strain energy and potential energy again let us recall the potential strain energy a kinetic energy that we have written is over here.

And now we also have strain energy which in this case is also equal to potential energy okay. Now we can write our Hamilton's principle okay, so what we have is that action integral is kinetic energy - potential energy which is equal strain energy here we write kinetic energy zero to L ½ P $w.^2$ – ½ tw'^2.

So w' is dw/dx this whole thing over DX is what has action integral okay. Now once we have this we write our Euler-Lagrange equation, so our integrand this thing is our integrand f, so what we write is ∂ f/∂ x minus sorry not x this will be 0 w∂ f/w is our variable right. Then we are extremizing this okay, with respect to w(x)∂ f/∂ w minus since we have two variables okay one more thing this is just L we have to do this from zero to action integral is 0 to t sometime t Lagrangian dt.

So we have dt also we have to invent variables X and T so we get ∂ f/∂ wand then ∂ /∂ x(∂ f/Iw')-∂ /∂ t(ρf/∂ w.)=0 okay, we have a function. Now w is a function of just dodged X it will be X and T that is what I mean we say vibration things are varying with time and space okay. So I think let us go back and change if I wrote w(x) here this is X and T this function of X and T both are variable okay, that is equation.

So let us write it now so what we have do that is anyway 0-f is there so ∂ /∂ x(∂ f/2w') there is already minus here, so that - T the two and w'^2 in here they canceled. So I get w'-∂ /∂ T of again half and two in $\omega.^2$ will go and we ρ the .ω. that you become Pw.=0 or what we get is here minus of minus plus we get tw'' okay.

And over here, now okay I think the sign we forgot about one thing we had the strain energy that we have written w and the times the change in length that is our strain energy. So we have potential energy there okay seems fine.

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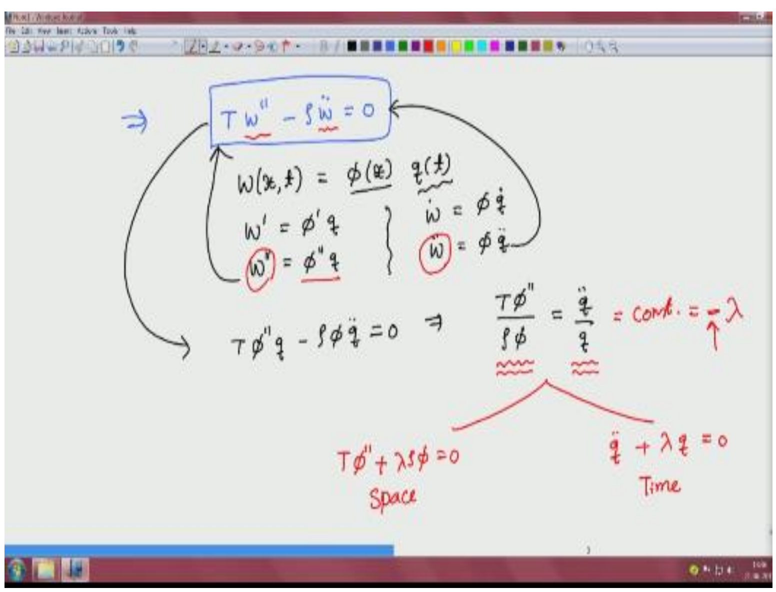

Let us see okay dw'' and then we have – Pw.=0 okay that is what we get tw' Pw dot equal to 0 okay. So this is our governing equation okay now in order to solve this we assume separation of variables meaning we will assume that wx,t has separate space part that is X part and then the time part okay if you do this w' will become free prime Q because only this has X this does not. Similarly w'' will be f'''Q.

Similarly w dot will be ϕq. because only q is a function of time simply ∂ ble u ∂ ble dot will be ϕq.. okay if you substitute these results into it what we get this tc''q-P pq..=0 okay are substituted for w'' this one over here and for w.. for this one over there okay. So this gives us this result so we get T ϕthe ϕ''/Pv=q..x q okay. We basically from here we arrange them.

So that things that depend on time are taken to the right hand side that is Q and Q.. are the ones that depend on time things that depend on space fee ∂ ble primary or on the left side and given constant tension in the string T and the mass density linear mass density that is mass per unit length of the string is P there on this side.

So if you look at this particular thing varies with X this particular thing varies with time right, but both are equal what does means is that this must be equal to some constant because we have two things left hand side is dependent on space right hand side depend on time and they are equal that means that whatever quantity that is equal to cannot vary with space that is X cannot with time that is T.

So that is what we have this one for convenience we call it -λ even if you take +λ nothing happens you can still arrive at the same conclusions but making it minus helps us because that λ turns out to be what we call natural frequency of the system okay, that is free vibration will be frequency that is what we get rather square of that. So what we have here when I assume –λ I get two equations here from this one is tc'' that is -λρϕ when it comes to the other side become λρv=0 and here q..+λq=0.

Again note is the minus sign okay, we get two equations this is all for space this is for time because Q we assumed in the separation of variables that this part depends on time this part depends on space okay.

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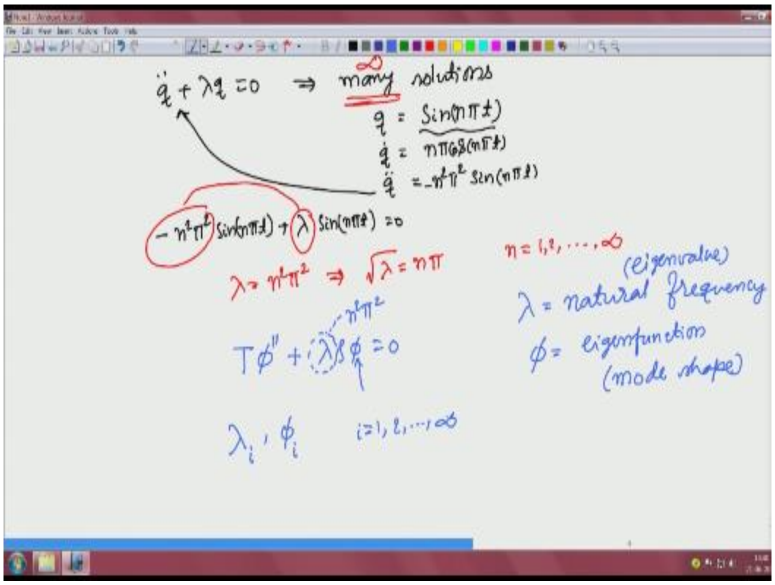

So if you look at this equation that let me write again $q..+\lambda q=0$ there are many solutions for it so both sine and cosine depending on the initial conditions that you have about Q you have many solutions right. For example, if I say Q equal to sin nϕ okay if I say sin nϕt okay, then Q. is nϕ pi cosine nϕt and then $Q..=n^2\ \phi^2$ back to sign with a minus sin nϕt okay I have substitute this back herein there substitute there $Q..-n^2\phi^2$ sin nϕt+λ what is Q sin n ϕt sin n ϕ t that is equal to zero.

Then you find that this $\lambda=n^2\phi^2$ that is when it will be 0 so that is that $\lambda=n^2\phi^2$ or you could have the square root of λ is n ϕr square root of λ is nϕ where n can be one two three of the infinity. So there are when I say many solution there are infinite solutions I can assume n = 1 to infinity I have that many solutions. Now I take in this λ I can put back into this equation and solve for ϕ okay.

Now that we know many λ are there so I take $t\phi''+\lambda\rho\phi\lambda\rho\phi=0$ where we know that this $\lambda=n^2\phi^2$ for each value of n I get to solve this equation for ϕ. So λ is our natural frequency and ϕ is our eigen function or we can call this eigenvalue and this is eigen function any word that has eigen as prefix has to be written together eigenvalue is not two words it is single word eigen function, eigen mode whatever you say is a single word with what follows I can function are what we call in vibrations mode shape natural mode shape a normal mode shape okay.

We have we get all of those here, so this ϕ just like λ, now we have λ I wear this I goes from 1 to infinity λ I and ϕ I there are many, many one of them infinite ones okay, and they all have special properties.

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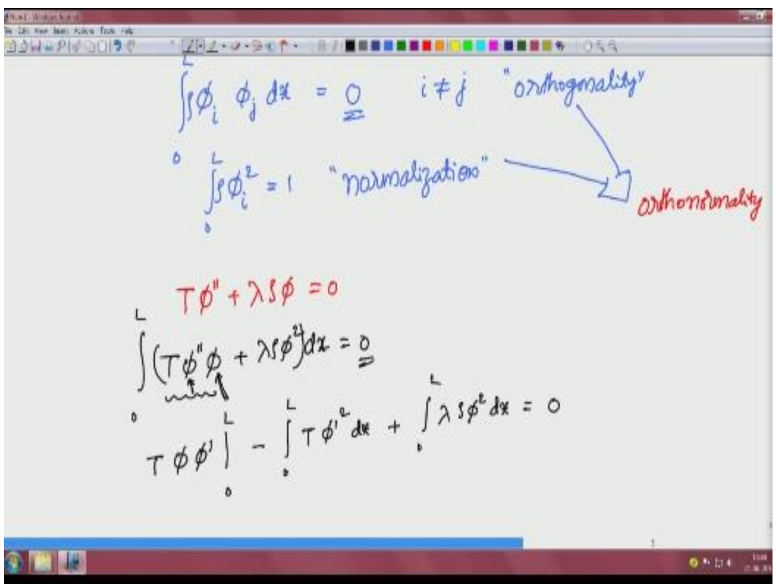

What are the properties that any two of these let us say I take ϕI +ϕJ these are orthogonal so you know that when there are vectors know like in vector calculus vectors let us say my finger is one vector, my pen is another vector, if they are perpendicular like this we can say that there are orthogonal or perpendicular to each other okay.

When two things are perpendicular whatever one has this has this does not have whatever this has this does not have like x-axis, y-axis do not have anything in common except origin of course okay. So similarly for functions when we can talk about two functions being orthogonal what does that mean here this ϕi's are functions of X okay ϕi, ϕj we have their functions of X when I take two functions ϕi, ϕj two eigen functions I would say that they are perpendicular to each other if this product is 0 for i not equal to j okay.

Perpendicular if i = j then I can have a normalization as we call it, so if I say ϕi^2 I will also multiply a ρ here okay, that is a convention something is 0 does not matter if you multiply I do this 0 to L to be equal to 1 this is the normalization condition because mode shapes eigen functions can be of any amplitude what decides the shape itself this is normalization this is orthogonality when put together it becomes orthonormal.

So these two things are satisfied for these ϕ functions we call something orthonormality okay. And you can also prove that these λ that we got they need to be they will be not need to be there will be only non negative okay, that is there will be either zero or positive they can never be negative. In order to see that okay.

Let us take remember this equation tϕ''+λρϕ many, many times we will use this. So we have tc''λρv=0 we will use this to prove that λ have to be non-negative okay, that we can see by taking this multiplying by ϕ I will take pw'ϕ and λρvrmp multiplying again and take TX integrated for zero to L. So basically I have taken that equation x ϕ and then integrated over X from 0 to L okay, that still be zero nothing changes. Now here in calculus of variations integration by parts place a big, big role.

So we will do integration by parts, so we will do the first function as that one integral a second function that is that one that will be that are the two ends then minus derivative of the first function is the first function that will become 3 prime integral second function that is also ϕ'. So ϕ'^2dx+ we have this other part 0 to l ϕPv^2dx=0 okay.

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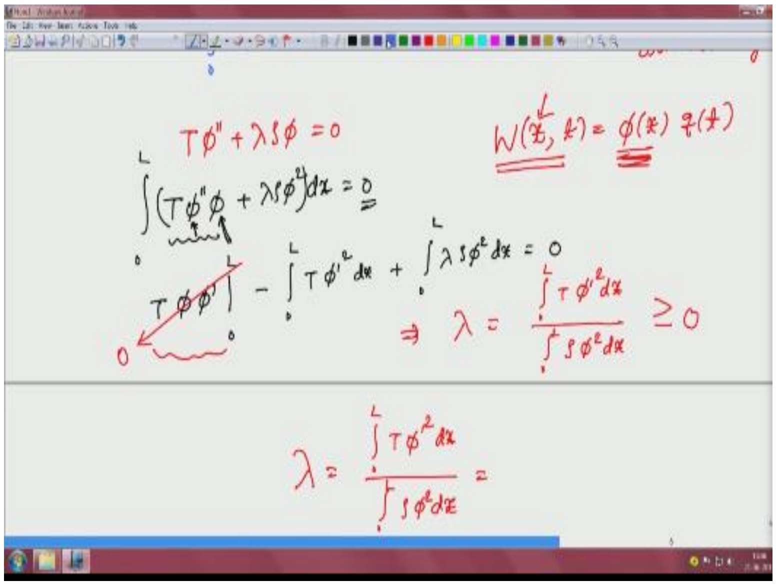

Now if you look at this quantity since ϕ which is our spatial function by the string vibration w again remember that W which is a function of X and T we have taken it as P(x) Q(t) our separation of variables technique. So whatever boundary condition is that with respect to X P will satisfy whatever etc with respect to time Q will satisfy our interest here is ϕ.

So when W is 0 at the either end ϕ will be 0, so at 0 and Lx=0 at X = L this actually goes to 0. So we are left with this okay. Now here we get something very important λ is equal to we can write integral 0 to L TV'^{2}dx/0 to LPv^{2}dx okay, you see this λ. Now has no way of becoming negative because tension T is positive see if the tension T is negative that is the slackness there is no tension you cannot vibrate a string so tension is by definition positive mass density cannot be negative.

So that is also positive and then we have ϕ'^2 in the numerator and ϕ^2 in the denominator. So this tells us that this λ is greater than or equal to 0 any λ I just took arbitrary if I put ϕi λi it will be true ok so this λ that we got as a ratio this has a name.

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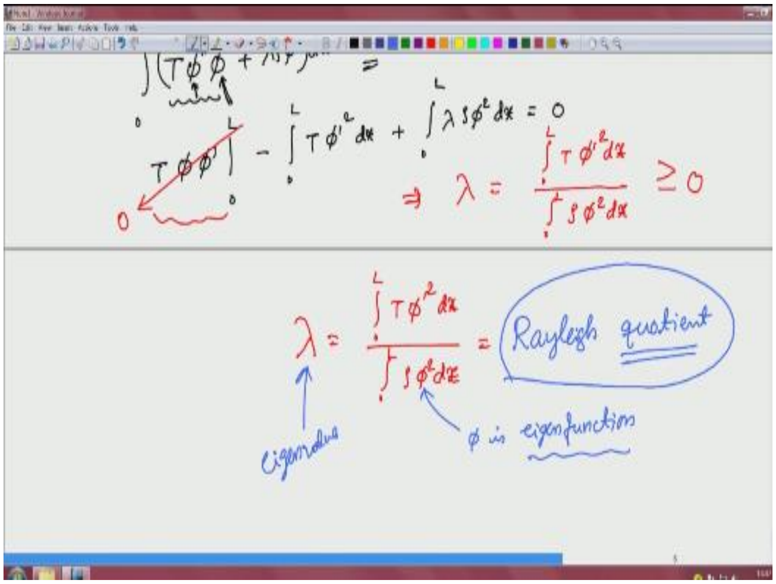

And that name is Rayleigh quotient, zero to LPϕ^{2}dx this has a name that goes by the name Raleigh quotient, quotient is like a ratio it gives you the eigenvalue okay and ϕ is of course what we said ϕ is eigen function I can get λ as a quotient right if it turns out that if you take ϕ anything it not be correct eigen function you take any function that is also the boundary conditions when you do it you get a λ value.

And that λ value is always larger than the real eigenvalue okay, we will come to that minimum characterization or the eigenvalue problem. So there is a gain another optimization coming up okay. We will stop here and then continue but know that we have this Rayleigh quotient which is very important both in mechanics as well as structure optimization as we will discuss thank you.