Indian Institute of Science

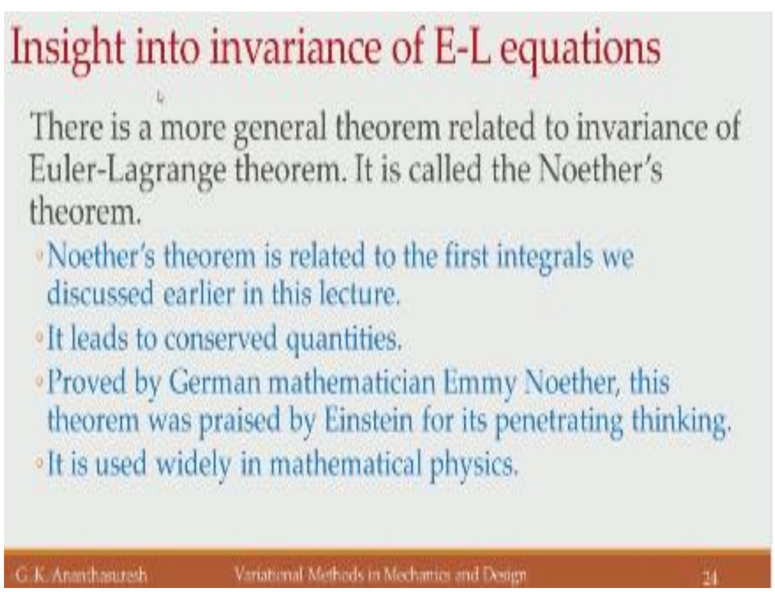
Variational Methods in Mechanics and Design

Prof. G.K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

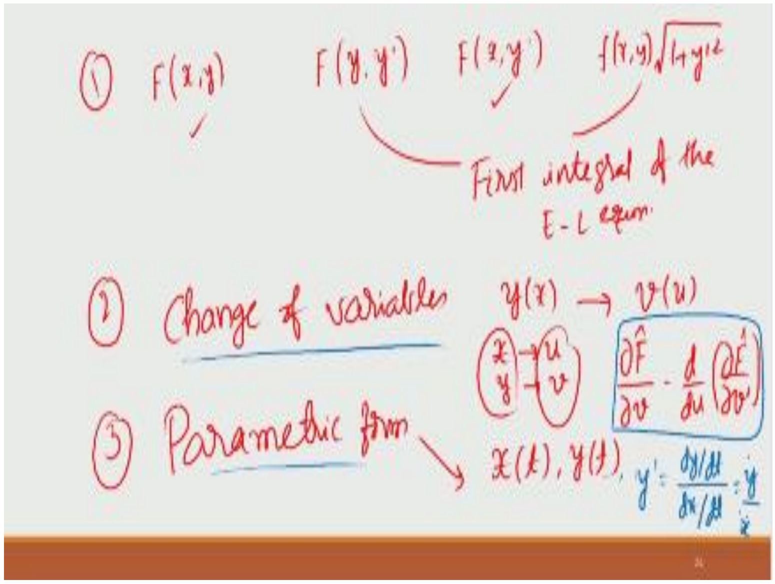
Hello today in the last lecture we discussed a way to solve Euler Lagrange equations they are formidable even for the blackest token problem the very basic problem calculus of variations if you do not know more you can formulate differential equation but you cannot solve, so people have developed techniques to solve when the integrand of the functional which is an integral if that has some special forms there are ways to solve we discuss those special forms, one of them is called the first integral of Euler Lagrange equations then we discussed two more things and.

(Refer Slide Time: 01:02)



So first let me write what we just discussed that we have some special forms, let me get the pen.

(Refer Slide Time: 01:08)

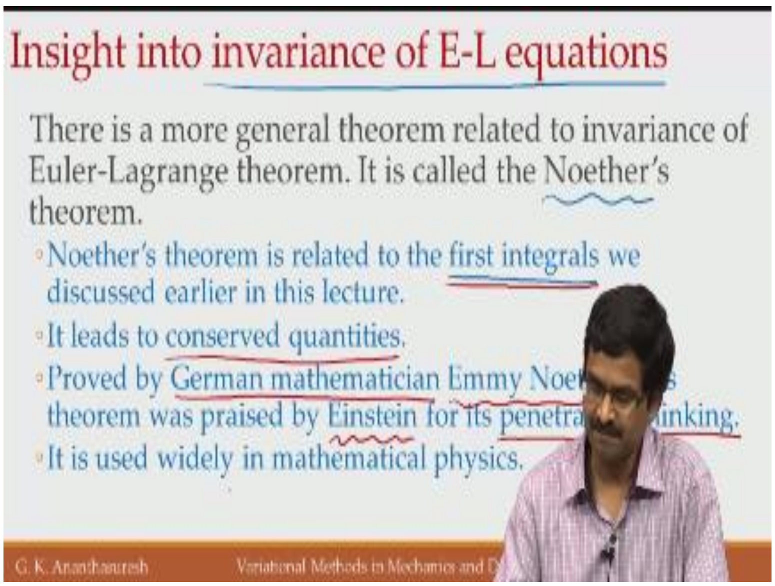


We had a special form of the integrand to be dependent only on x and y that only to be trivial and then we took integrand where it is a function of y and y' and f as a function of x and y' and then we had where integrand without the form $f(xy)\sqrt{1+y'^2}$ okay, this and this turn out to be rather straightforward and these two lead to what is called the first integral of the Euler-Lagrange equation, okay then that is the first thing we discussed then we discussed two more things one is change of variables and the other is parametric form, okay.

In change of variables x and y y changed to so you have y(x) that becomes v(u) okay, that is x takes the role of u, y takes a role of v, in parametric form we introduce another variable let us call it t, x(t) and y(t) if you notice that the form of Euler Lagrange equations remains invariant that it does not change when you change xy to uv form remains the same meaning that when you can convert to uv we would write δf it would have changed the format of change let me call it $\hat{F}$ or something $\delta f/\delta u - d/du(\delta f/\delta v)$ sorry, $\delta f/\delta v$ it will be not u in that that would have been where $\delta f/\delta v$ and do d by d of $\delta \hat{f}/\delta v'$.

So the form of the Euler Lagrange equation does not change when you do this change of variable similarly parametric form also, parametric form what is y' here y' will become dy/dt what we call $\dot{y}$/dx/dt so it becomes $\dot{y}$/ $\dot{x}$ okay, so that is what we discuss now. Now when you are doing this change of variables and putting in parametric form there is a deeper result called know this integral, okay.

(Refer Slide Time: 04:12)

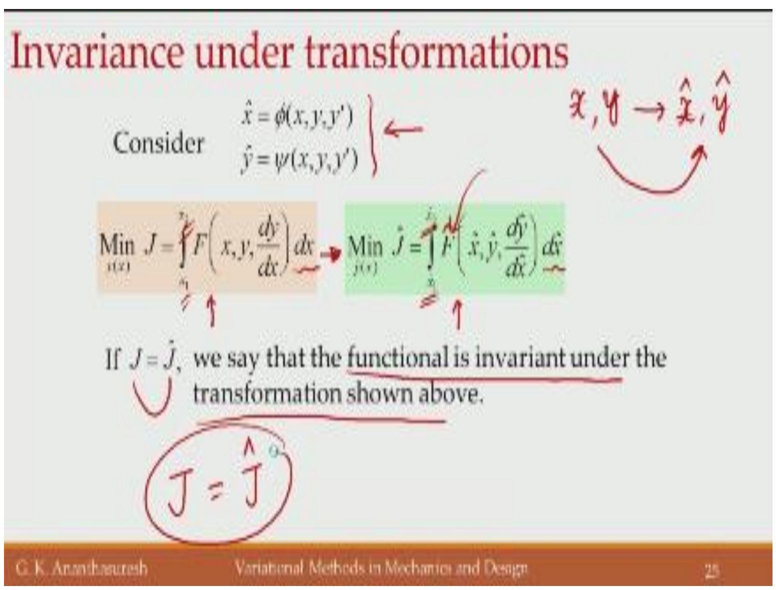


So when we have this invariance of the electrons equation that we just talked about there is a more general theorem which is named after a women mathematician by name Emmy Noether's who was a contemporary of Einstein and lot of other great people of that time okay, it is related to the first integral that we have discussed today okay, it is related to the the first integral that we have discussed and first integral we already saw when you looked at one example that it gave the conservation of energy.

We took a mass spring model with the Hamilton's principle where we extremist action integral using calculus of variations we got conservation of energy out of that in general this notice theorem also leads to conserved quantities we will do an example today is that as I said it is a named after lady mathematician who was a German Emmy Noether okay, a great mathematician and for this Noether's theorem a person of the stature of Einstein praised Emmy Noether for this theorems penetrating thinking.

So we will have to spend time on this theorem go slowly to understand what this is okay, it is widely used in mathematical physics where you have these change of variables that you do to simplify things then Emmy Noether's theorem comes to the risk you it is widely used in mathematical physics when you say you know calculus of variations first thing people ask you do Euler Lagrange equations you tell them fundamental lemma of calculation tell them next in they ask do you know this theorem, if you say you know it then they will believe that you know calculus of variations well, okay. Now let us discuss that theorem okay,

(Refer Slide Time: 06:18)



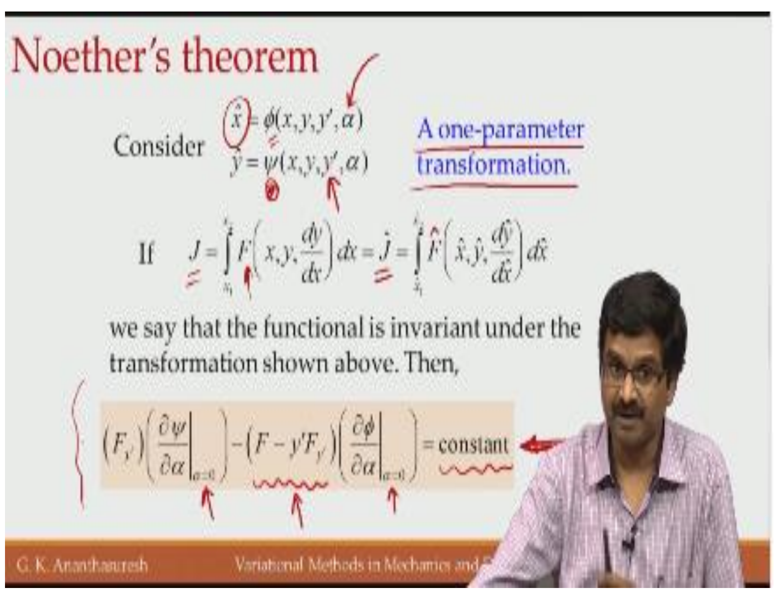
It is about invariance of Euler Lagrange equations and the transformations let us say that we have the variables x and y okay, x and y we change them to other set of variables denoted earlier we said u and v now it is $\hat{x}$ and $\hat{y}$ okay, when you are changing from xy to $\hat{x}$ and $\hat{y}$ then you have like these two expressions you know ϕ and ψ that relate $\hat{x}$ with xy and y' and $\hat{y}$ with xy and y'

that we are changing we are expressing $\hat{x}$ and $\hat{y}$ in terms of xy we are changing variables and this is what is the transformation going from xy y' to $\hat{x}$ $\hat{y}$ and you can do other way also okay.

So if you know that then the original problem which is shown here can be transformed to this problem okay, so f is a function of xy dy/dx now it becomes $\hat{x}$ $\hat{y}$ $d\hat{y}/d\hat{x}$ limits also is x1 to x2 because that is the integration variable now that is $\hat{x}$ so the limits also change okay, and in fact the expression itself will also think should also be $\hat{f}$ the integrand will have a different form it is same expression substitute everything it will have a different form okay, we say that if J =$\hat{J}$ okay, f may be different but you have different variables different limits and all the things that are changed when you transform that to that, okay.

If j=$\hat{j}$ we say that the functional is invariant under the transformation shown about that is a first invariance that is we have changed from x and y appearing is in J, J was depend on x and y you have to integrated everything it is a number and we have $\hat{j}$ which is done in terms of $\hat{x}$ $\hat{y}$ if this is true we say that the functional is invariant it is not changing its value under this transformation okay, that is what we consider first, okay.

(Refer Slide Time: 08:51)



Now what this know this theorem say says is that now consider the same $\hat{x}$ $\hat{y}$ that we discussed in the last slide, now also a parameter that is introduced then it becomes one parameter transformation okay, we have parameter α okay, so basically what is saying is that $\hat{x}$ is expressed in terms of xy $Y'\alpha$ it will also be y'' and so forth but as we have been doing we always take the simpler case that is it depends on you first derivative in second derivative third derivative then work it out we can work out a current of this theorem once you understand this so we are taking only upto y' introduce this parameter in the transformation, okay.

We will see what is this α if you wonder why are we introducing where is using it here in our example we take a case where we have a coordinate system we rotate the coordinate system but we have done mechanics or some structural design problem using one coordinate system if that corner system is rotated that could be the θ that will be the equivalent of this α so as we change the coordinate system do our dynamics change or mechanics change.

Our structural optimal profile will change it does not is what this know the theorem says or rather it says how it would change okay, what remains constant if you do that, okay. Now if you take that one parameter α here then the J or $\hat{J}$ are equal you see that is what we said then that functional is invariant under the transformation okay, then we have this F now we will have this I should say $\hat{F}$ I think I am not writing but when you write it this form of this expression depends on $\hat{x}$ $\hat{y}$ and $d\hat{y}/d\hat{x}$ will be different from this thing here, okay.

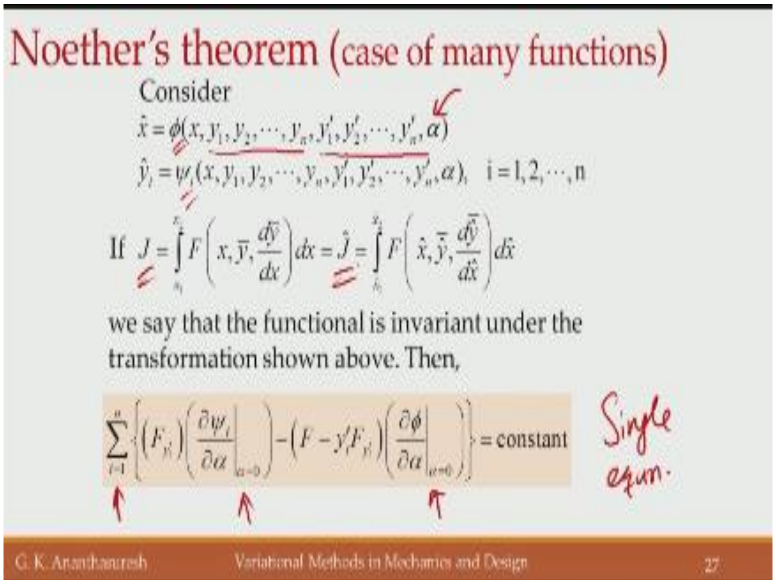
Now if it is under if this function is invariant because J and $\hat{J}$ are equal then what Noether's theorem says is this result okay, we have to look at it carefully what is it saying we have two things ϕ and ψ using both in this theorem here okay in this statement here it says that $F_{y'}$ F sub y' that is $\delta f/\delta y'$ x $\delta\psi/\delta\alpha$, α is a parameter it could be rotating coordinate system that could be θ the size and expression over here right, we are taking $\delta\psi/\delta\alpha$ and then substituting α equal to 0 α goes away, okay.

And then we have $F\text{-}y'/F_{y'}$ times $\delta\phi/\delta\alpha$ where α again is equal to 0 that is constant is like a first integral that is what we said Noether's theorem is related to first integral that we discussed in the just previous lecture, okay we also see that first integral does of here what we had taken okay, but only thing is we have some additional things you know this first term is additional and what multiplies over when to α equal to 0 that is a theorem we are not proving it here just understand.

And when we take one example you will see how it is useful especially derive some conserved quantities that are there embedded into the dynamic system that you are taking or any problem that you under consideration again we are only take in the case up to y′ one can extend it to y″ also there will be more terms coming here okay just like in the fudge integral a Lagrange equation more terms get added when you take more derivatives.

But what the theorem is saying is that when a functional is invariant under a particular transformation that is given to you then this thing is this particular thing is true okay, where this one parameter transformation, right.

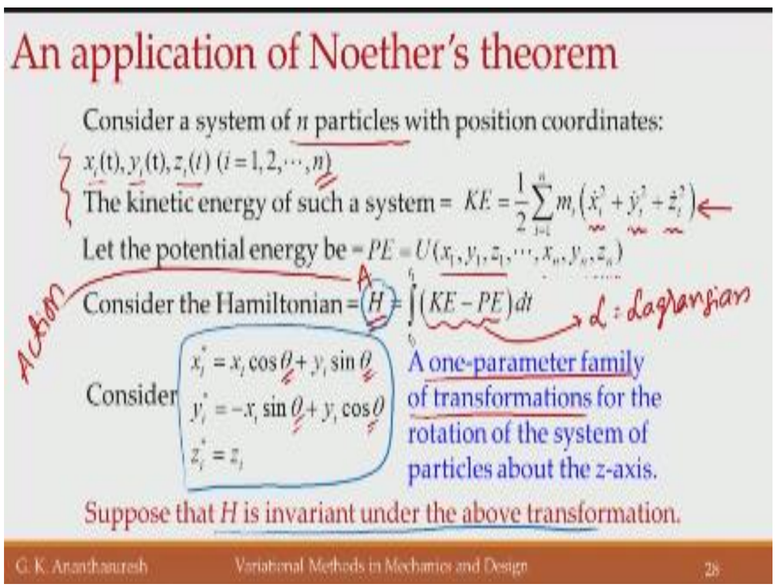
(Refer Slide Time: 13:03)



What if you have many functions that what we normally do we extend multiple derivatives we are not doing yet what if you have multiple functions y1 y2 y3 up to yn and then y1′ y2′ up to yn′ still single α one parameter then basically what changes you know this theorem is this summation so whatever we said is constant now we have to do that for all functions that are there 1 to n still a single equation, okay.

There are some overall things that coupling so you have many functions then this is true okay, that is just an extension of this theorem for multiple functions okay, again you see this has to be done α equal to 0 both ψ and ϕ. Let us I and fire define like that for each of these functions 12 3 up to n again we should see that the J should be equal to Ĵ that is a functional should be invariant under the transformation.

(Refer Slide Time: 14:13)



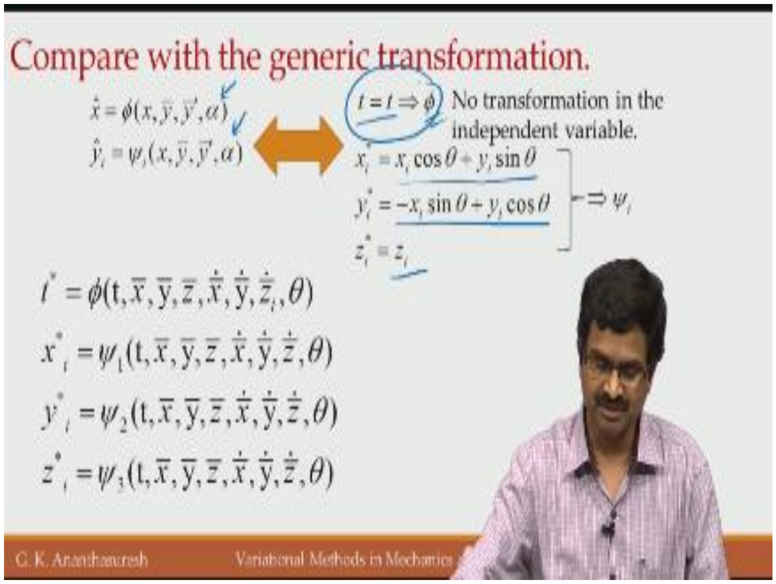
Now let us apply to a problem to understand this one parameter transformation using which we have defined in the Noether's theorem okay, consider a system with the n particles a mechanics problem n particles with position coordinates xi, yi and zi as a function of time okay, that means theyaren't particles moving in three dimensions so each particle has its position vector denoted with x let us say first particle is x1t, y1t, z1t second particle x 2t, y2t, z2t and at so forth we have n particles we have 3n functions all of which are functions of time okay, kinetic energy of such a system is easy to write each particle $1/2mv^2$ that is why we have in three dimension xi^2, yi^2, zi^2 that is a total kinetic energy.

And potential energy will be a function of x1 y1 z1 and x2 y2 z2, x3 y3 z3 up to x 1 y 1 xn yn zn, okay. Now this is what we call action or Hamiltonian okay, so sometimes it is called action integral or Hamiltonian people will use both of them this part is what we call Lagrangian that we use in Lagrangian dynamics kinetic energy – potential energy, we have kinetic energy potential is already written so we can write this action or Hamiltonian.

Now we consider this one parameter family of transformations that is why I said the α is now the θ we are rotating, okay because these things that we have is for a particular coordinate is not your chosen. What if I change the coordinates that I am rotate in the coordinate system okay, I am rotating about z axis looking at this look at this equations you can tell that Z coordinate is not changing that means that we are rotating the coordinate system about z axis okay, my index finger is z axis and x and y are here okay yeah.

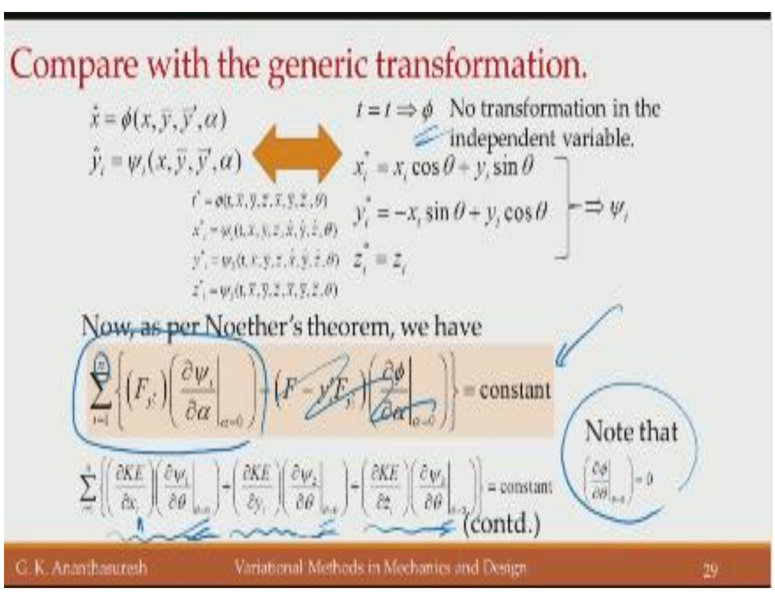
Let us see you can see that I am rotating like this about that axis right, when I do that z coordinate does not change whereas x and y would change all right. So if I take this one parameter that that we said α that α is now θ okay, if we have this you to first show that H is invariant under the bow transformation that is we have this action or H Hamiltonian that should remain the same when you change that if that is true then the others theorem will be applicable and we will see what will be that one statement that we get okay.

(Refer Slide Time: 17:28)



So we write that down so we have x̂ that α now that α is θ here okay, here xi* is this in terms of θ yi* is this is this is a transformation of chosen there is nothing there and t equal to t the time is not changing that remains is same so for the ϕ there is no transformation in the independent variable earlier just to understand what i mean by t equal tot there is no transformation let us go back and look at eight mint of Noether's theorem.

(Refer Slide Time: 18:03)



We had this $\hat{x}$ okay let us look at this thing $\hat{x}$ was a function of ϕ in this case instead of X we have time the dependent variable that t equal to t we are not transforming that whereas $\hat{y}$ equal out of that here is xi yi zi which are all functions of t, okay multiple functions so this second theorem there we considered multiple functions applicable in our example okay, instead of y1 y2 yn we have xi yi zi all of which are functions of time so instead of x we have t and then three functions, okay.

That is how we should look at so what we have here are three functions xi where i goes from 1 to infinity yi owe not infinity and + zi t and t, so t takes the role of x on this slide and these are y1 y2 y3 all of which are functions of x okay, so having understood that now we are explicitly writing t* xi* yi* zi * now we are just taking different variables like average quantities here, $\bar{x}$ bar $\bar{x}$ $\bar{z}$ that we have okay.

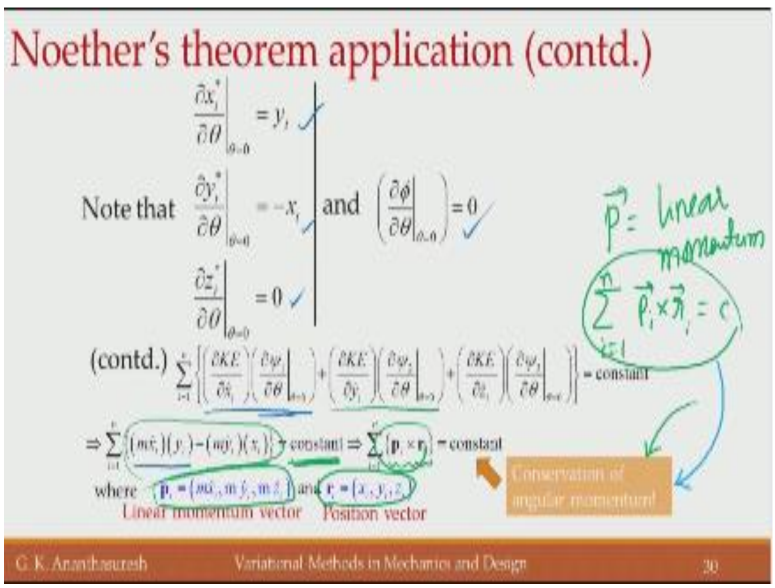
If you take this now what we find here is that in fact you should verify that with this transformation okay, so with this type of transformation we are not really going to see a change in our action integral that you have to verify no particular transformation to work it out and evaluate you know all of these things here that this H value does not change that should be invariant okay. Then notice theorem is applicable with that that is a statement when they are i equal to one to n now n is 3 for us okay.

Because we have taken these functions y1 y2 y3 we have those and when you write all this what you get Fyi ′ will become $\delta ke/\delta xi$ and then why i ˙ z i ˙ and then we have these things that should be evaluated at θ equal to 0 okay, so we are basically looking at $\delta\phi/\delta\alpha$ because ϕ does not have anything so that is just 0 that does not occur we are focusing only on this thing we have n particles this i goes from 1 to n.

Otherwise functions we have our only three that is x1 of t y1 of t z1 of t, okay when you do this so we are saying that that what I just said $\delta f/\delta\theta$ equal to 0 because t equal to t there is no transformation really there so this is 0 and that is why this term will not exist here only this let me just say that this is not really three we have particles n particles but our three functions x1 of t, y1 of t and z1 of t that is what we have and that is again split here separately, okay. With the y1

1 quantity that is x1 and y1 and z1 each of them separately that is what we have if you substitute these things using this transformation.

(Refer Slide Time: 21:45)



We also note that $\delta x^*/\delta\theta$ at θ equal to 0 is why I then you go back and look at this transformation take derivative with respect to this $\delta\psi$ that is what it is with respect to θ subsidy equal to θ equal to 0 then what you get is this result and this one then this one okay from there again this is Z is not changed so it's zero ϕ that t is not changes where this part is 0 but these two are nonzero, you substitute all of them what you get here this term I will color code becomes that because those $\delta\psi 1/\delta\theta$ here that we have comes out of yi.

And then let me use a different color to underline so this particular thing becomes that and these things are 0 because δψ3 that we have is like dψ*/δθ that is 0 that will not be there we get this equal to constant that is what now the theorem says but now if you look at it it is mxi˙x y i-my˙x xi is nothing but PxR where P is linear momentum vector we are talking about particles moving in three dimensions so linear momentum basically mass times velocity linear momentum MV.

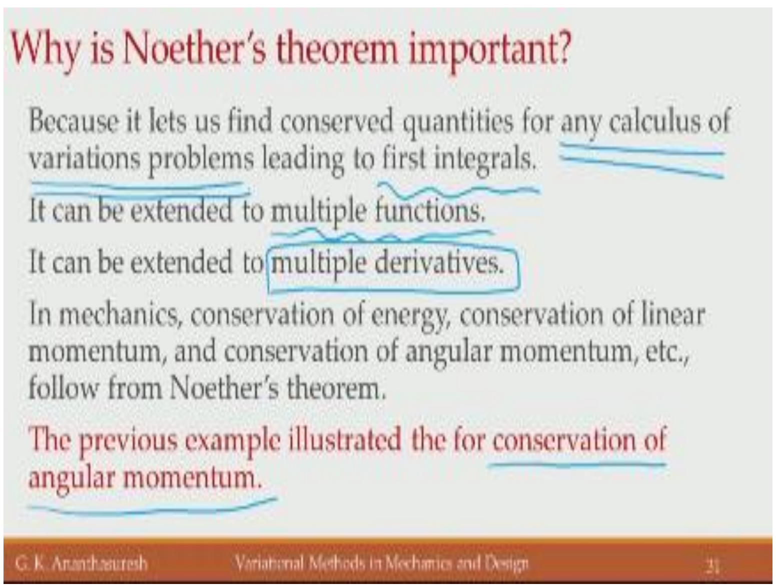
But now V is a vector because it a dimension so that is given by this thing over there it has components mxi myi˙mzi for each particle where I was from 1 to n linear momentum vector and then position vector R okay, so PxR that is what we have here that is what you get when you look at this quantity that is lying there PxR summation for all the particles equal to constant is nothing but the principle of conservation of angular momentum, okay.

So Noether's theorem gives you the conservation angular momentum when you write down the equations right, we have the Hamilton action integral that when we extremist you get the Euler Lagrange equations are what we call Lagrangian equations for dynamics of a system can be rigid body elastic system or mixed whatever, whatever system when you write a Lagrange equations there the first integrals of motion gives you conservation of energy that we saw earlier you can get conservation of linear momentum also now we got conservation of angular momentum.

Because we considered a one-parameter transformation so when you write that action integral which ever called system you take its value does not change that we are not verified but you should verify by calculating since that transformation is invariant that function is invariant under this transformation that is a more precise way of saying the functional is invariant under the one parameter transformation one parameter here is that rotation about z axis.

When you hat then we work it out the expression that is therefore Noether's theorem says something it will constant that something turns out to be summation of the angular momentum follow this momentum vector it is a P is linear momentum PxR is angular momentum for all the particles 1 to n this being constant that statement is what we have recognized as the angular momentum overall the particles that is conserved and the absence of any external movements and forces acting on it, okay.

(Refer Slide Time: 25:54)

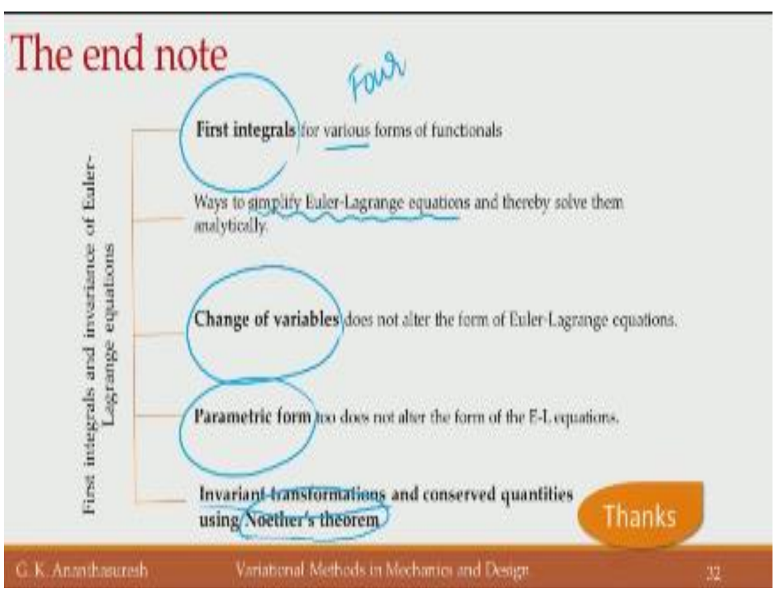


So why is no this theorem important because it lets us find conserved quantities in any calculus a variant problem lead into first integrals when something is constant you solve the problem already the first integral comes out of it okay, that is problem is partly solved especially the context dynamic systems you always have second order there will be acceleration term we have X there will be d^2x/dt^2 when you have first integral you reduce it to dx/dt you know that momentum will have only velocities angle mentor will also have velocities.

Because linear momentum vector is p1, and when it is PxR so that is basically in terms of only velocities you not find the second derivatives of the respect to time so you already solved part of the problem, okay and it can be extended to multiple functions as we saw but XYZ n particles 3n functions we consider this and it can also be extend to multiple derivates okay, so that I had said earlier but I am emphasizing again that we can work it out or state Noether's theorem for multiple derivatives are also that is what we've been doing from the beginning first we take only y′ and then take x″ x″ and so forth the same thing can be done here as well. In mechanics it leads to conservation of linear momentum conservation of angular momentum because the wave energy we have already seen okay.

Let us previous example showed us how to do this for conservation of angular momentum okay, that is why I know this theorem is important from the view point of mechanics. The viewpoint of structural optimization is hot to say it is I have not tried to apply this any time when we did this because they are the coordinate systems and things like that are not that important, but it will be worthwhile to explore that to see that this Noether's theorem is important for any calculus variation problem where you have some transformation I adjust that people have probably not applied to all problems that exist okay.

(Refer Slide Time: 28:15)



So to summarize today's lecture so we considered the first integrals of a Lagrange equations for four difference the various thing various forms we have four special forms we took and then we use them to simplify Euler Lagrange equations so that we can solve more easily than otherwise then we also discuss a change of variables and parametric form that led us to this invariant transformations and this very important theorem of Noether we are not proved this Noether's theorem but it is a very profound thing when you have multiple functions or them coming together to give one equation is important right.

So normally lagrangain equations if there are n functions y1 y2 yn you would expect to see n equations there no this leads to only one equation, because if we took the example of n particles what we concluded was that angular momentum is conserved for the whole system of n particles it is not some that is true for what only first particle second particle third particle is not like that all of them put together it is constant that is why this Noether's theorem is a profound theorem it requires us to think a little bit as to how it works and we go through the proof then it will become more consolidate in your mind okay, so that concludes today's lecture we will go back and look at a few more things in calculus of variations and application of it especially to the dynamics problems and so forth okay, thank you.