Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

We are back to calculus of variations after discussing how to formulate the mechanics problems and structural design problems using calculus of variations, we are now looking at calculus of variations again in order to see how we can solve the equations in the case of the structural design of a bar and a beam we discussed how we can solve analytically sometimes and we also saw how to solve numerically when we cannot solve analytically.

Now let us come back to calculus of variations or go back to calculus of variations to see after we write Euler-Lagrange equations how we can solve how we can solve them and there are some standard techniques to solve and one of them is to do with invariants that exists in a Lagrange equations when you change variables or when you express the variables that you have in terms of parametric form or some other variables and a way of integrating Euler -Lagrange equations a Lagrange equations as we know our differential equations.

When we say we solve differential equations we essentially mean that we are able to integrate them either analytically or numerically in this lecture we will discuss a few methods that enable us to do this analytically which is what we call integrals and invariance of Euler Lagrange equations let us get on with this topic.

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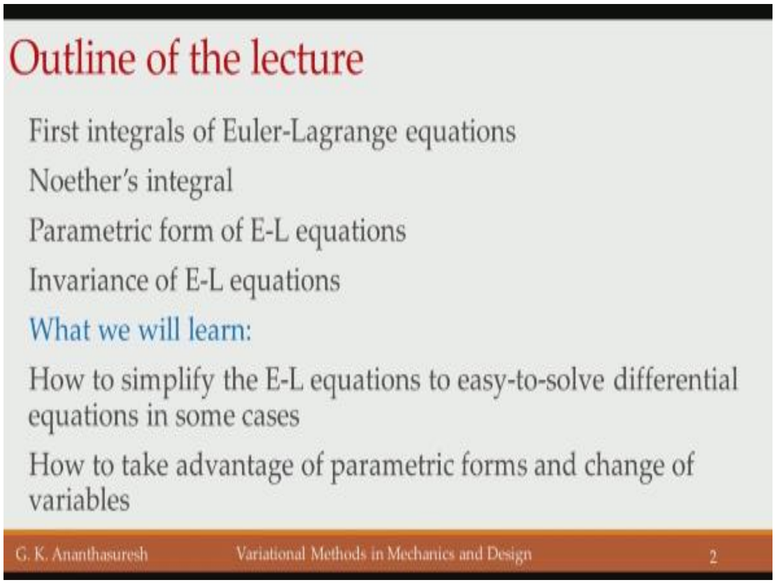

So here is the outline of the lecture first we discuss what are called the first integrals of Euler Lagrange equations they are called first integrals because when you have a Lagrange equation which are differential equations if it is a dynamic system that we have considered for which you have written a Lagrange equations usually there will be second order differential equations in terms of time now when you say first integral you will get one order reduced right that is why it is called first integral there is a interesting thing after M in others or what is called a no this integral we discussed that also and then we will see what happens to EL equations by L we mean Euler Lagrange Euler-Lagrange equations when we use parametric form when you have two variables x and y you can express both of them in terms of another variable as a T.

X(t) y(t) then what happens or Lagrange equations likewise when do Euler Lagrange equations remain invariant that is you change to parametric form or change of variables and still a Lagrange equation should be the same form they should be of the same form so what we will learn in this lecture is how to simplify the Euler- Lagrange equations so that it will be easy to solve analytically sometimes and numerically definitely easier than doing without this so called first integrals.

And we say when we also discuss how to take advantage of the parametric form and other change of variables to solve calculus of variations problems.

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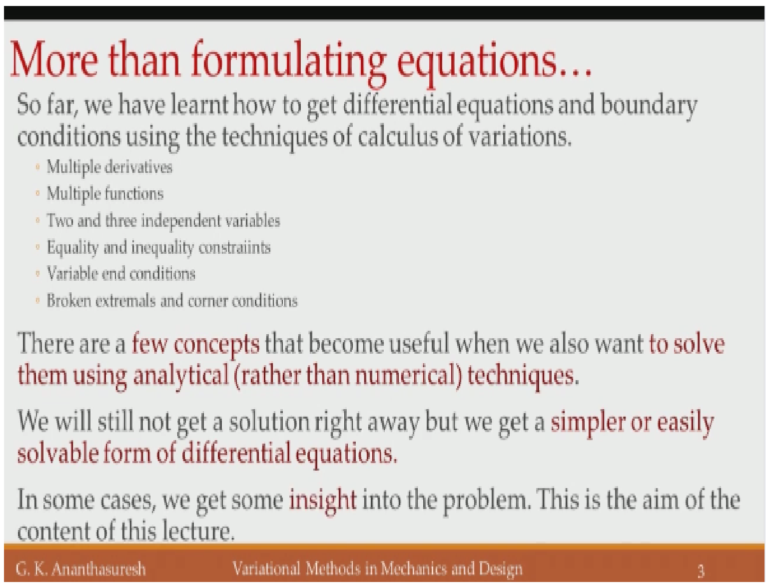

So it is more than formulating equations that is what this structure is about we know given a calculation problem we can write equations now Euler- Lagrange equations there are differential equations and then the boundary conditions also now we go a little bit beyond that and just to recap what we have discussed in calculus of variations we have discussed this multiple derivatives and multiple functions two and three independent variables equality and inequality constraints variable end conditions and at the end we also discussed broken extremal so-called via straws admin corner conditions.

We discussed all of these now we are putting them to use in mechanics and structural design but this topic is there as to how to solve this, so there are a few concepts that we need to understand to do this first integrals are in okay after doing this let us be want that we will still not get a solution but we arrived at equation that are simpler or easier to solve okay that is something that whatever we discuss in this lecture does not enable to solve any problem but it will give you a method to simplify the equations under certain conditions not always.

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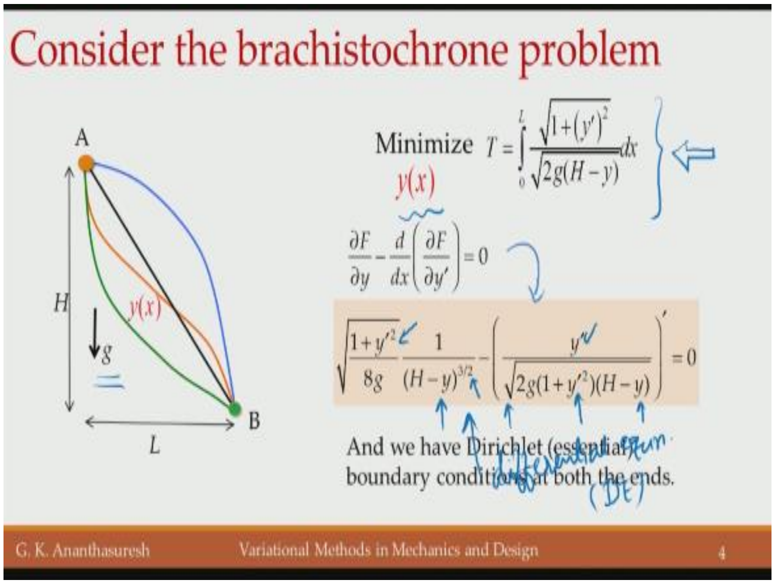

Let us consider the bracket our own problem itself to see whether we can solve the problem that is given there, okay that is our calculus of variations problem minimizing the time that this bead will take to come from point A to point B under the influence of gravity so this is the problem and Euler Lagrange equations if I write in this case one equation because we have only one function y(x) we have one differential equation in y this is differential equation sometimes we just write it as DE.

Look at this equation it has y it has y'^2 here y ' Y ' another y under a square root sin and this is 3 over 2 lots of nonlinearities, so looking at this it will be very hard for people to solve it and say that yeah this is the solution of this that is y(x) is a cycloid that is what we know is a solution for the black stone problem it is difficult to solve right even if you know the solution you have to substitute it and find out but you cannot deduce the solution from this because equation is very nonlinear okay.

That is Euler Lagrange equation that you know ∂f / ∂y minus d/dx of let me just erase what I wrote you can see the equation so that is ∂f /∂f/∂Y - d/dx of ∂f at 0 y ‘ equal to 0 gives you this equation okay.

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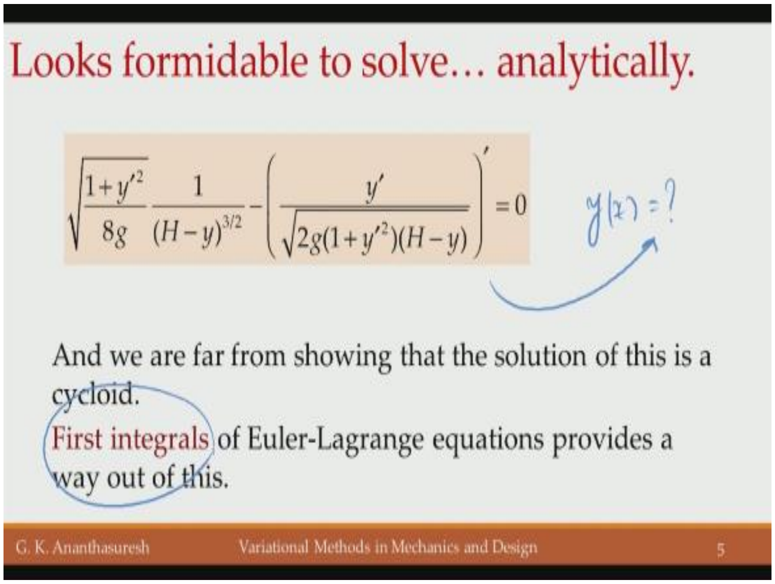

And we have boundary conditions also there if you look at this equation it is it looks formidable to solve I mean how do I solve how do I get y(x) that is our question from here from here how do you get there right that is where the so-called first integrals help you and you can do that under certain conditions.

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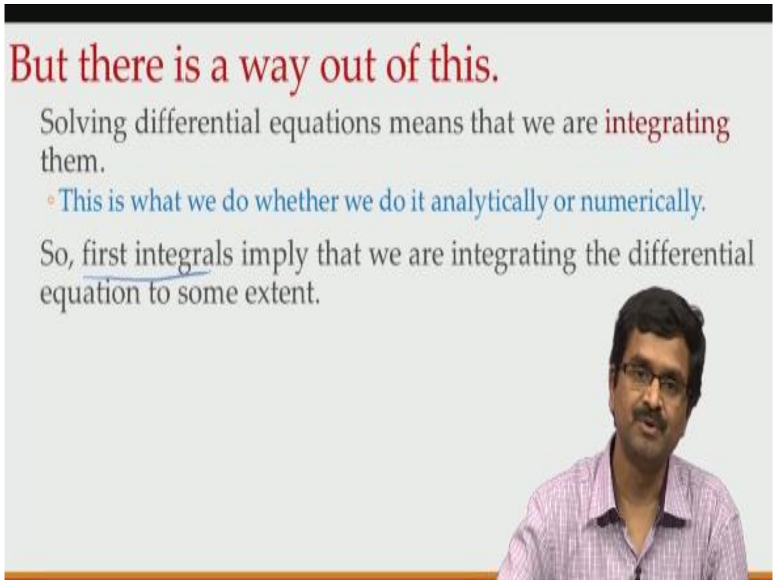

Let us look at those conditions okay so to do it analytically we need to look at a few form special forms that anyway you to do this analytically by calculating this or computing this first integrals okay.

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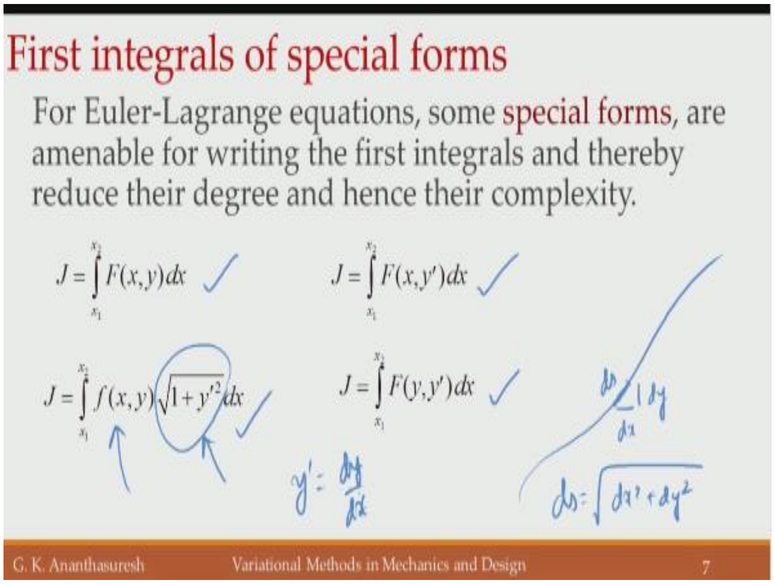

These are the special forms for which we can do it easily okay so there is this first special form where the integrand depends on x and y only there is no y ‘ nothing just x and y there is another one which depends on x and y ‘ only there is no why there is x and y ‘ there is this third one where we have y and y ‘ no x okay the fourth form which has x y and y ‘ all three ok this is no y ‘ no y no x okay those are the three things here a general form where we have a general function f(x, y) you can have any function that you like.

But y ‘ has to be in this special form which is very common in geometry as well as mechanics because as you know square root of y ‘ square basically comes from the arc length in TS Pythagoras theorem says that if I have a curve if I take this small length here okay so that small length is dx and this is dy so this is D s that basically comes to $\sqrt{dx^2 + dy^2}$ and that basically is what gets you will take dx out you get 1 plus y ‘ square again recall that y ‘ is dy/dx okay. So these are the four special forms that we are going to consider one by one.

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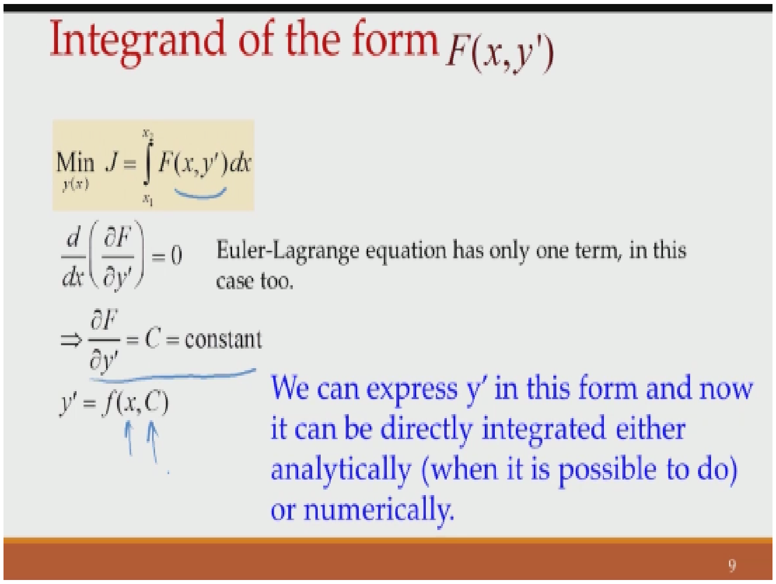

The first one is quite easy right so integrand here is only function of x and y Euler- Lagrange equations will only have one term ∂f by ∂y there is no y ' x '' nothing so you have only this if you do this just like capital f when you take derivative that with respect to y you will get another expression or equation only in terms of x and y that you can solve okay this more like an algebraic equation because there is no derivative of Y you have a solution already okay.

I will no boundary conditions also because even the simplest boundary condition involved the integrals dependence on y ' since in this case does not depend on Y ' you get a very simple solution here it maybe implicit does not matter but you got a solution okay an algebraic equation now let us take the case where there is no y but it depends on x and y ' okay in which case if you see there is no y so were at a Lagrange equations ∂f/ ∂Y is zero there is no dependence and y.

But ∂f / ∂y ' is present so d by dx of ∂f by ∂y ' equal to 0 is the Euler- Lagrange equation now we integrate on both sides you get this result that is ∂f /dou y ' equal to a constant see that you can determine from the boundary condition okay, so since F depends on x and y ‘ ∂y/ ∂y’ will still depend on Y ‘ then you simply say that y ‘ equal to some constant okay.

If that dependence and X is not there it will be constant otherwise independent x and some constant c you can hope to solve it because you have y you can integrate it on the right-hand side there is no y ‘there is only x then you can integrate and get the solution so this is one of the ways of doing it so our problem is almost solved here.

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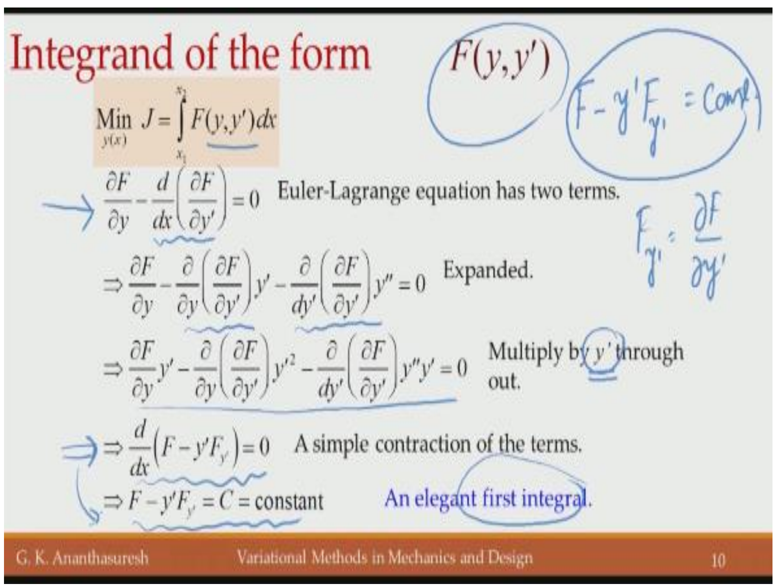

Now if you take an integrand of this form where there is y and y ‘ there is no X consider the case where is no y ‘ there is no why now the concern a case where there is no X okay then you write our Lagrange equations because both terms exist because Y exists or ∂f / ∂Y will exist and y ‘ exists so do f by ∂Y ‘ exists and you have to take this and you get that so now we are expanding the second term.

So this term is expanded using chain rule ∂y/ y’ first term remains as it is this will be ∂by douY off ∂I/ ∂y’ into y ‘ like chain rule okay and we also get the other term that dou/ ∂y ‘ because we know that this F depends on Y and Y ‘ so ∂f / ∂Y ‘ also depends on Y and Y ‘ we have done why first and then y ‘ next okay now this whole equation is x y ‘ because something interesting comes out when you do that.

So basically the previous equation is x y ‘ all through when we do that the equation that we get can be written or condensed to this form we are simply multiplying y ‘ and in that process what we have on our hands starting from a Lagrange equation turns out to be something like this okay d/dx of a quantity that is f minus y ‘times FY ‘ is equal to 0 what does it mean then it means this that is equal to constant.

And that is the elegant first integral that is what is called the first integral so when the integrand is of this form then Euler Lagrange equation have to write and then struggle to solve it you can simply write that F minus y ‘ FY ‘ is equal to constant that is the solution for the first integral again FY ‘ in our notation is $\partial f / \partial y$ ‘ okay, that is the first integral we understand that we have a differentiation at this step we are getting the solution by integrating rather straight in a straight forward manner.

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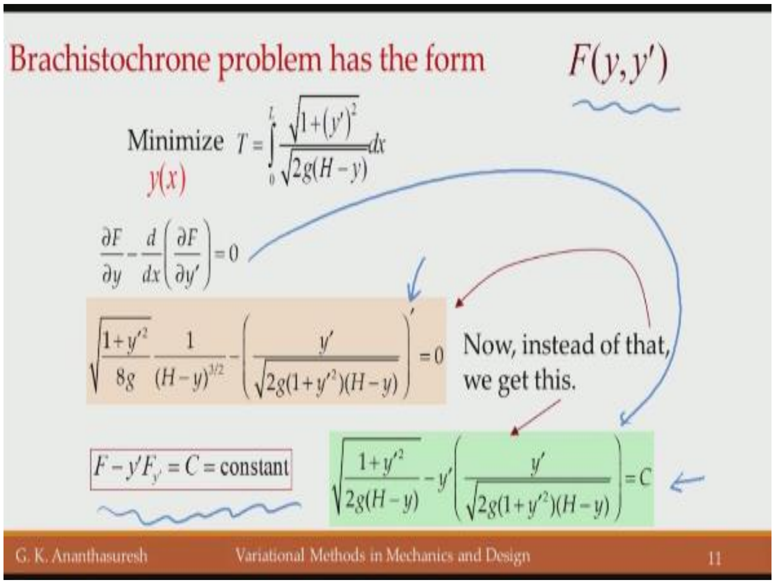

So if you look at the Brachistochrone problem that is exactly what it is depends on Y and Y ‘ so for this instead of struggling with this differential equation we can go from here to here because of this first integral so that is what it means it may still look formidable but if you know how a cycloid looks like then you have a solution here.

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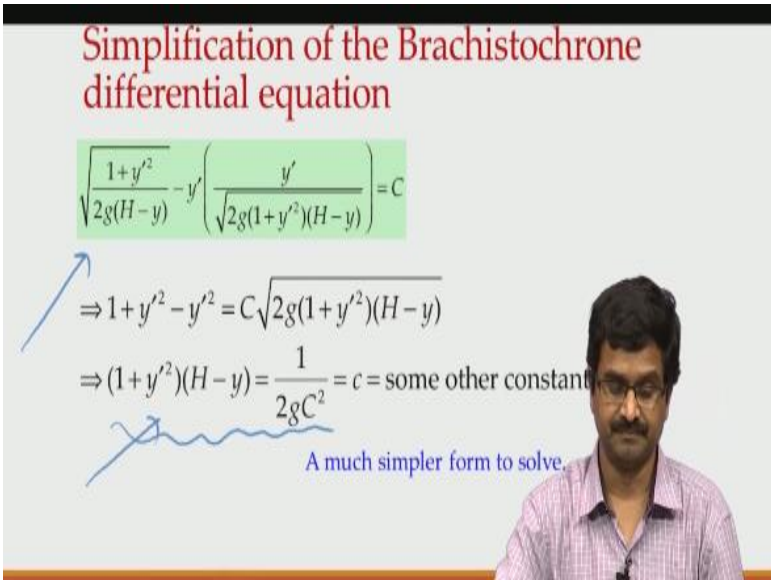

So we take that so whatever we have that F minus y ‘times FY ‘ if substitute for this problem you get a much simpler thing which looks like this ok 1 plus y ‘ square times H minus y equal to constant is much easier to solve because it involves now y and y ‘ in a rather simple fashion okay as it is a much easier to solve as compared to this okay.

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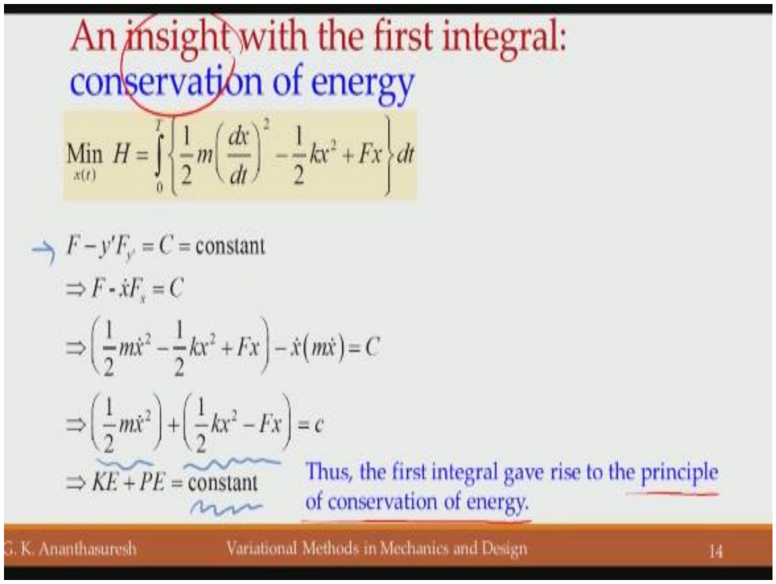

That is the first integral again first integral is what f minus y ‘ into FY ‘ equal to constant that is the first integral there is an insight that you can get from this for that let us consider the Hamilton principle for a simple spring mass system that is shown here okay so here we have Hamilton or sometimes we will call it action all right and then what we have inside is actually what we called Lagrangian and so forth and so I do not as L dt.

So we have this principle now if you want to let me erase this thing and get the pen back so this has this is of the form if you look at what's in here there is X here and there is also X dot okay just like having y and y ‘ when independent variable is X Y of X dy/dx now we have time has an independent variable that is that we have X and X dot the position and the velocity okay having noted that if you look at a problem such as this we see that we can take advantage the first integral okay.

If we do that first integral which is f – fy’ and y’ equal to constant you substitute those symbols in this problem you will see that kinetic energy plus potential energy is equal to constant okay in other words what we have is the principle of conservation of energy okay so when you take a Lagrange equations and get this first integral that leads to conservation of energy okay that is important thing or insight that comes out of calculus of variations in the first integral.

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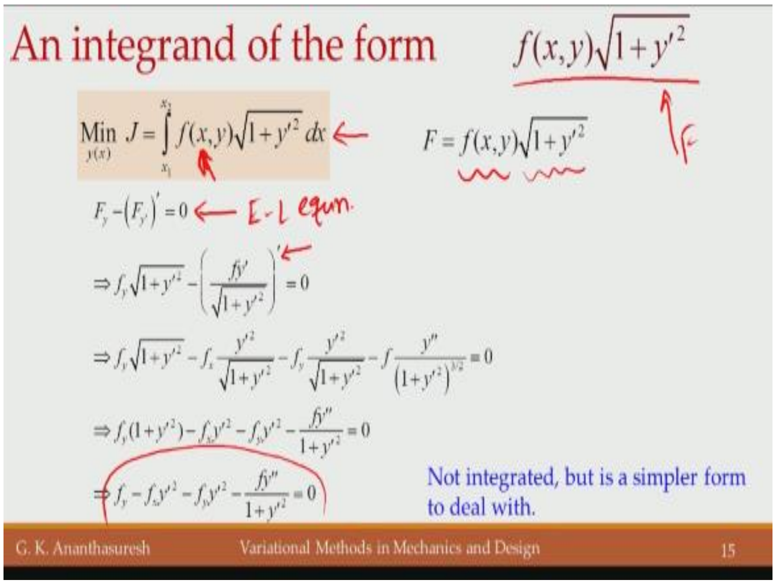

That is you have simplified the problem okay let us take the fourth form which is like this integrand that F right which has a special form which is y ‘appears only like that X Y can appear any which way then if you take this already this is a case of where it depends on Y and Y ‘it can also depend on X but slightly things change the first integral is if you look at this Euler - Lagrange equation okay will be slightly different in Euler Lagrange equation.

We written ∂f/ ∂y minus d / dx (∂y’) if you do for this problem noting this is the integrand and do a simplification not simplify actually expanding the derivative use the fact it is chain rule because it involves y and y ‘because f here contains x and y right that is what we have here all right so if we do that you get more terms and finally you get to a form that looks like that which is not that easy like the first integral.

But it is more systematic so you can easily write that without weighing about the square root being underneath and all that they are gone now you get a much simpler equation to solve these are the four special forms for which you can get this first integrals or get a solution right.

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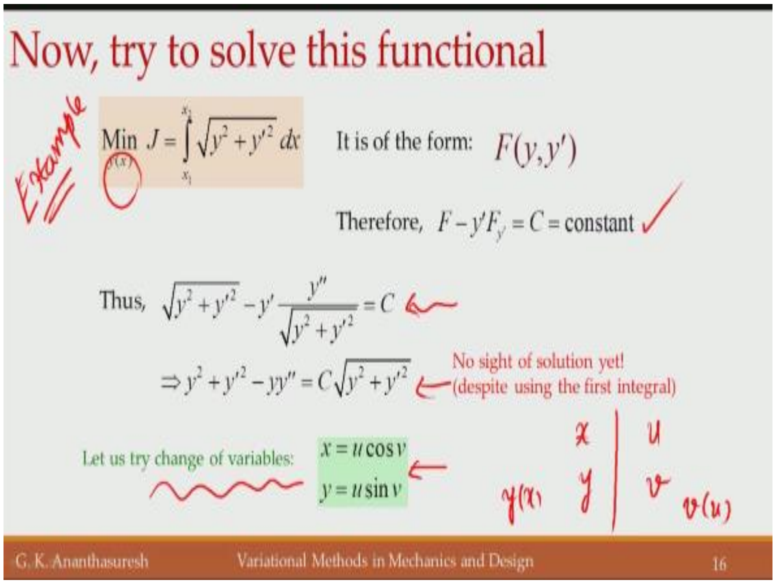

So if you try that for a given problem you may be able to solve it better let us take this example let us take this as an example where you are minimizing J which is from X 1 to X 2 square root of y square plus y ' square then the first integral is valid because it depends only y ' so we wrote f minus y ' x equal to some constant okay if we work it out the simplification if you do all this you will get to a form like that it is still not anything to write home about no sight of solution yet we do not know how to solve it just became a simpler looking equation.

But we do not know whether this has any easy solution or not so what we do is we try the change of variables okay so this is a problem where you have to find y of X so what we will do is instead of taking x and y we take u and v okay we had y of x here but we will take v of you how do you change the variables like this x equal to u cosine v y equal to u sine V.

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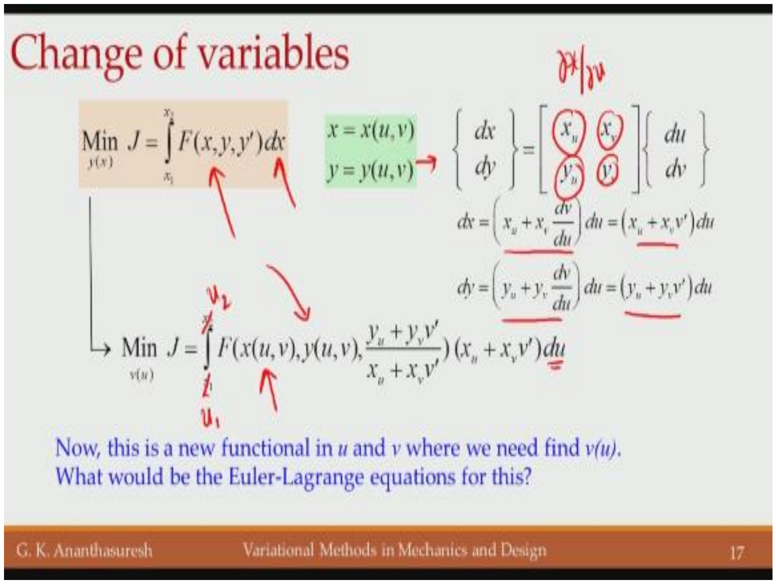

When you do that we have to go back to this equation we had in terms of x and y convert them to V and you in fact if I right d you here I should actually write U1 to U2 because you has taken the role of X now and V has taken the role of why okay so now from here we have dx here that has two cannot 2d you the way it is dx dv is given by this when I say X you hear what I mean is that ∂X/ ∂u okay.

We are changing from xy to uv but you have to make do this Jacobean with all four things which are given here very straightforward okay and d you will be replaced with dx will be replaced with D you using this all right, when you do this you notice that when there is X know depends on UV that is your thing.

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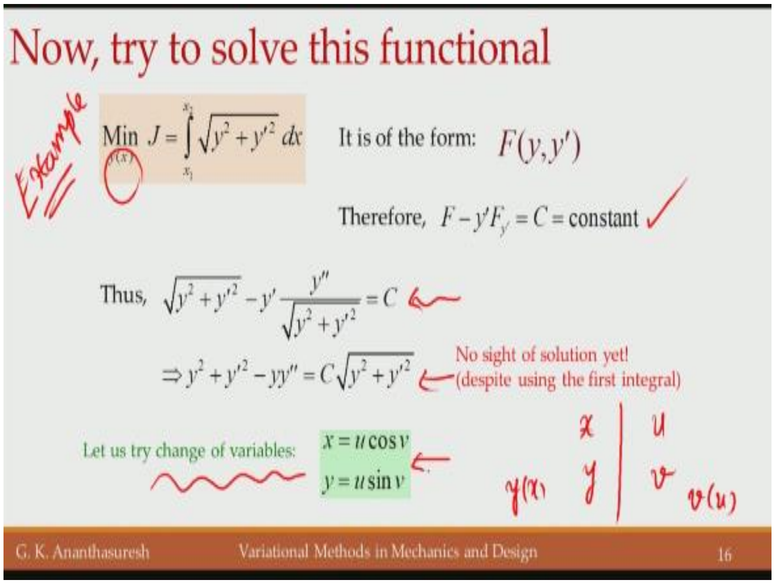

Because we assume X equal to function of U and V in this particular example we have taken X equal to u cosine v y equal to u sine V that is U and V or involved .

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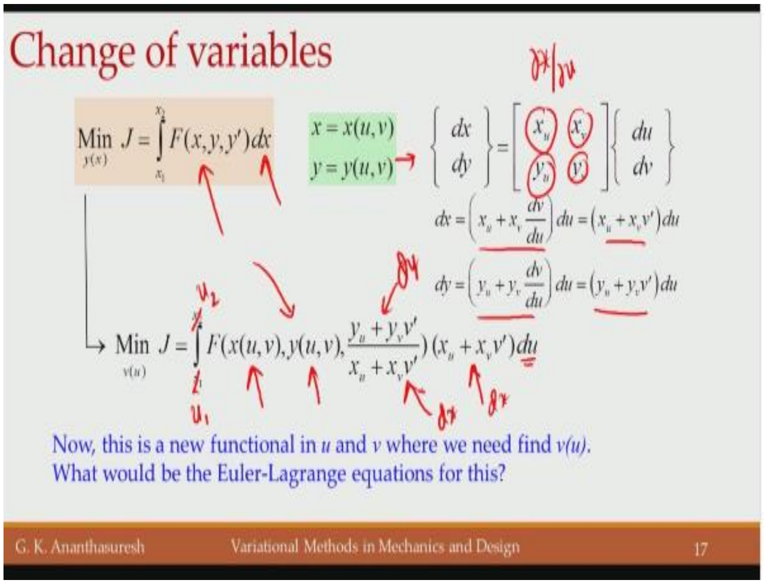

So now integrand f wherever x is there you will substitute that equivalent thing in terms of UV and same thing with y and same thing with this dx now that becomes xu + xv times V 'and here we do the same thing so we have dx and dy if this is dx and that is what is here this is dy okay and that is how the integrand is going to look like.

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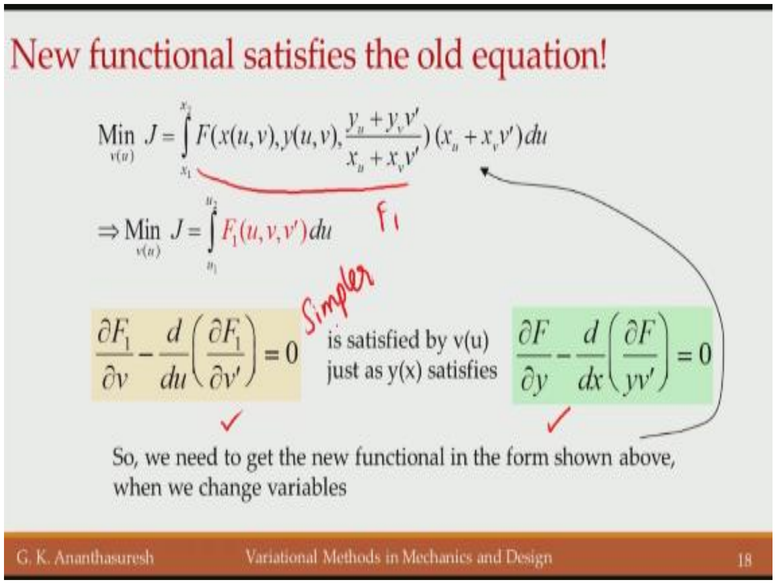

And if you were to write equations Euler Lagrange equations instead of F you have what I can call f1 to write down all the things here we get that and if you look at this now we get one equation for V in terms of you because that's independent variable now instead of y interms of X okay we get this alternate format that may be simpler why do we do change of variables only to make our life simpler okay. In this case the substitution is actually simpler.

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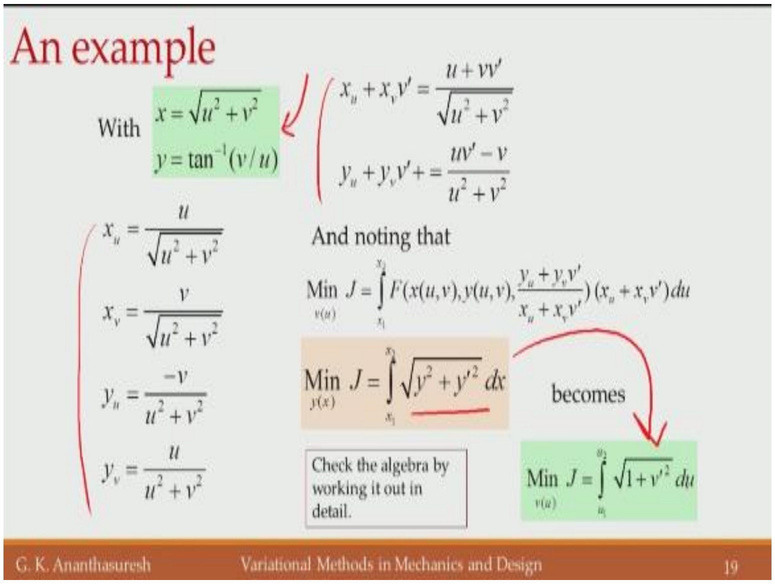

For that example that we had taken which is this example integrand is y square plus y ‘square if I choose A transformation that is this which is same as what we have now x and y we had in one way where I had u and V in terms of x and y now x + 1 2 into u and v if you substitute for all of these four things and various expression that we have this particular thing becomes that which is y and x becomes u and v okay.

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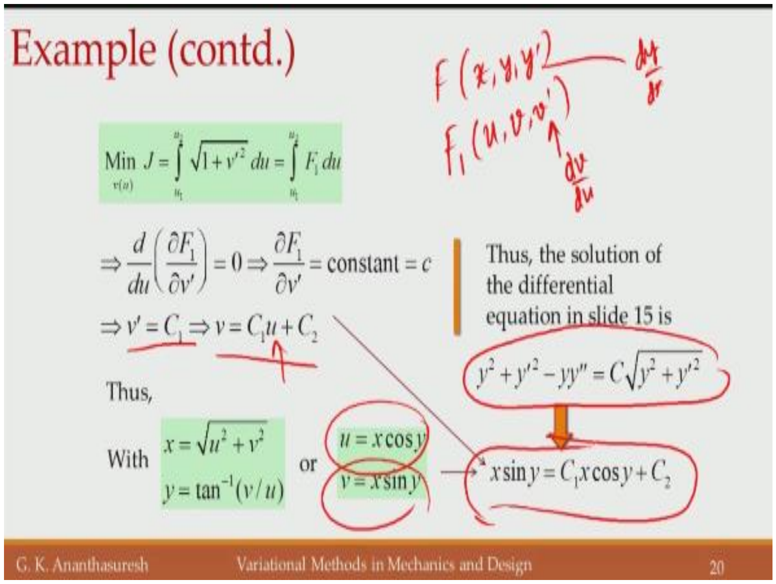

And that is easy because it depends only on V ‘ no X and no u no u ‘ here so no V and no you because whatever we said about integrand being a function of x y y’ now that is equivalent to say f1 I am call it changed integrand you v and when I say u ‘ what I mean is vu dv by this is V ‘ sorry this is V ‘ is V ‘ that is dv dt u okay that is just like y ‘ was dy/dx when we do that this thing turned out to be a very simple thing it just happen to be V ‘ equal to some constant.

So V equal to that and we got the solution if you substitute that over here whatever that looks like a formidable equation to solve its solution now we have this now from V you go back and say whatever thing is right which is x and y and on the right hand side we have this C 1 u. u is also here so you got the final solution okay so now we have used to things we have used the change of variables and the first integral so we can use them whenever we have in need.

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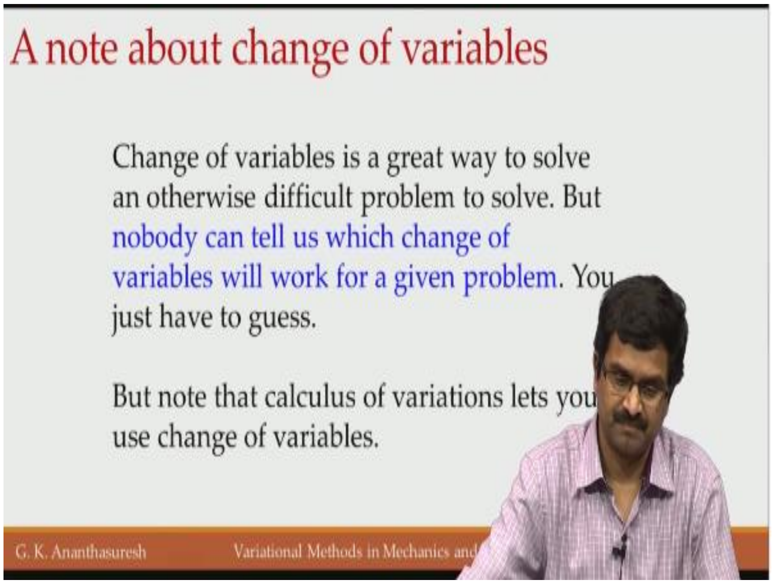

But we must note that given a differential equation that comes from a calculation problem nobody will tell us what change of variables to use it comes from your own intuition and experience to see when you should change using the variables and how to do the change of variables okay there are no there are some rules but not general rules that will work all situations okay.

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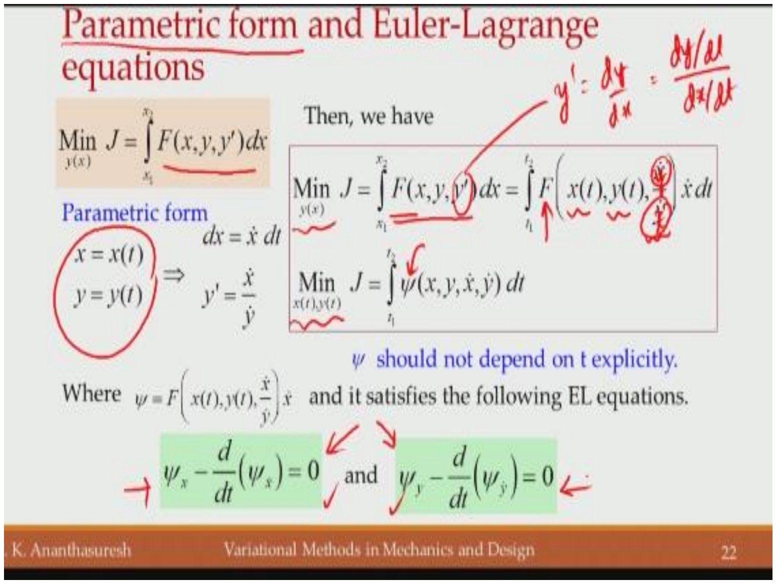

Another thing we do is to go for parametric form ok that is instead of using only change of variables let us say we try all of that or it does not strike us what we will change the variables to use then you go for parametric form by that what we mean is that we have a problem which has x YN right earlier we changed both x and y interms of U and V and we just had different set of symbols you V and V ‘.

Now in parametric form what we do is to express x and y in terms of a third variable called time or T, that is a time but that is there okay now in the parametric form when you do it whatever integrand you had in terms of x y and y ‘ it will become X(t), y(t) and x and y. what you want is d y ‘ right so y ‘ is dy/dx what you want to divide multiply by tdy /dt divided by dx / dt okay that is what we have in fact it should be other way around this should be y dot xx dot because that is dy /dx is just doing your time.

So just make that correction over there now instead of that being F now it will be different function because when you substitute expressions do change you have a different expression side and on that we say that the ayah grant equations are satisfied this can be proven or checked against not a major result say that whatever integrand f that you have now converted to sigh that can solve these two equations okay.

When with these equations all you get the solution for that x(t), y(t) sine note that you have two unknown functions opposed to one and one the change parametric form instead of y(x) being unknown you have X(T) and Y(T) as unknowns accordingly you get to differential equations okay.

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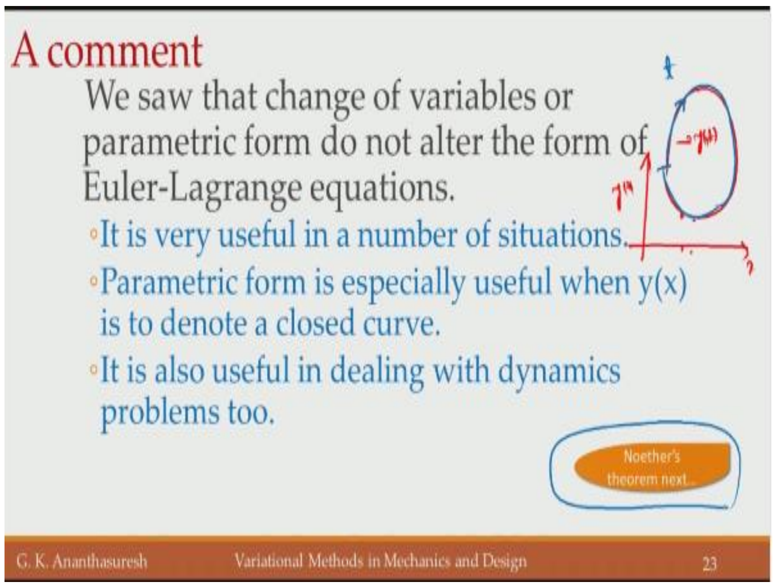

So here is a comment that this parametric form is usually quite useful in many situations especially when y(x) is to denote a closed curve right for example if I have a problem like this where I have to find y(x) we always think that you know y(x) in open curves at one point there would not be two things but optimal design our geometry problems also we have closed curves it is not a function really if this is X and this is y (x) for the same value of x there are two values of Y okay in those cases parametric form helps so that now you can go around you can go around the curve with a parameter T which will go to T equal to zero to all the way L okay, having discussed this we will go to a more impart concept of notice theorem in the next half of the lecture, thank you.