Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Okay so we were solving this problem of making the stiffest beam for given amount of material we got our design equation a joint equation and complementarity condition and the concern the equilibrium equation and we said that when we looked at the adjoint equation.

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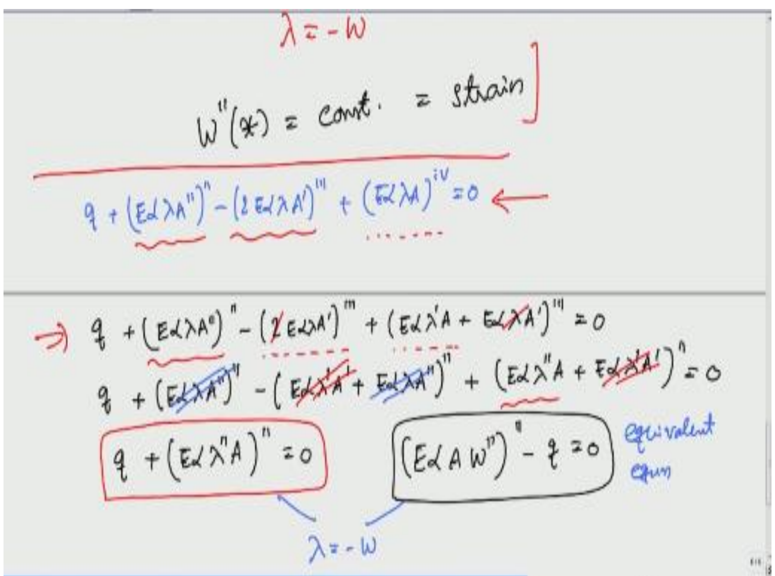

So let us look at the equation when we took the variation Lagrange respect to W we got an equation that looks like this okay and then we said that if you simplify you will get to this which leads us to the conclusion that λ equal to minus W that leads to a very interesting optimality criterion that strain is uniform and hence stress is uniform and then strain energy density is uniform the simplification which I have done here.

Which I hope you did after the previous half of the lecture so this is what we need to simplify which if we do step by step you do not have to expand everything ahead of time because get cancelled if you see the first step implies the first line implies this line where I am leaving this as it is and I am also leaving this as it is and expanded only this last one okay then this 2 and this get canceled and then next time I am still not expanding this.

But I expanded this thing and this particular one also I expand it so everything now except q has a double derivative again I can see this and this so this and this get cancelled and this and this get cancelled leaving out only this term which is this if we compare that with the equilibrium equation that we had which was $(E \alpha A w'')'' - q = 0$ that was the equilibrium equation from above which we had also derived in the last lecture equilibrium equation.

If you compare these two we immediately conclude that λ equal to minus W which winds up in the design equation gave us the condition.

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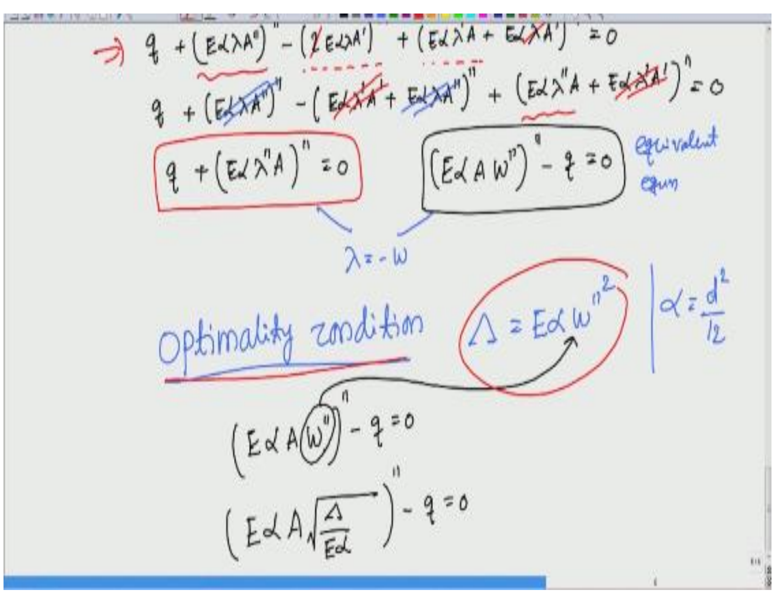

That λ is equal to (E α Aw “)” ^{2} so the optimality condition here the optimality condition here is λ equal to (E α Aw “)” ^{2} right again you see how it came about we had this as the design equation and then we substitute for λ double prime minus w double prime that gave us this okay again let us say man remind ourselves that α is d square by 12 because in this problem we have assumed that D is given its part of data.

So this is the condition we got by writing or manipulating the design equation but there is no design variable here a this is exactly what we had for the bar as well but it does help us compute area o cross section because we know the equilibrium equation which is (E α Aw “)” – Q = 0 now from here we can substitute for w double prime okay so if we do that I get (E α A√ Δ/E α) right that that is w prime we substituted and then double prime minus q equal to 0.

So even though there is no A explicitly over here in the optimality condition it does come about when you put that into the equilibrium equation.

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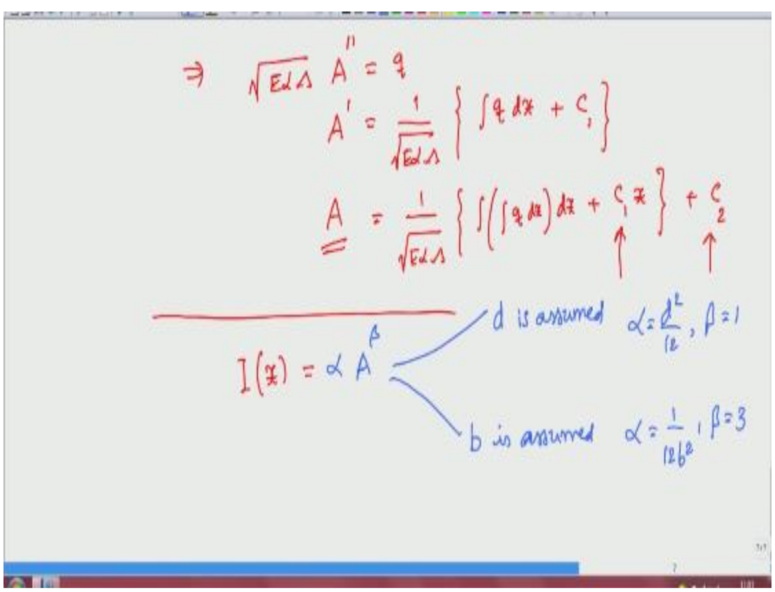

So once we have here we know that these are all constants in fact I can rewrite this as square root of e α λ all are constant except that we do not know λ times a double prime will apply only to a because others are constants right so that is equal to Q which is given to us so if you want to get the area of cross section I just need to integrate twice that e α λ that I have okay times I will have integral of Q of X first time let us like step by step a prime.

I will get QD x q is a function given to us we can integrate it and then there will be a constant okay then one more time if I do I will get 1over square root of E α λ then we will have this one more integral (q dx)dx + C1x and then we will have c2 okay now we have condition the boundary conditions which we are not writing but I encourage you to do this because it is a an application of calculus of variations.

We had written all the general boundary conditions you write them which will enable you solve for this constant c1 and c2 if you know Q you can integrate twice and then c1 c2 the boundary conditions then you get a of X the problem is solved okay analytically that we do not have to resort to numerical things here okay if Q is a load that is not integral then you need to do numerical integration only for Q twice.

And then you get the answer okay this can be done what happens if we go for the moment of inertia f sorry second moment of area I not inertia this I we had written it as a raise to β and we had two possibilities d is assumed that is what we did until now to solve this and the other we had said is B is assumed right, so this gave us α equal to d^2 /12 and β equal to 1 which was easy and now if B is assumed α because one over 12 B^2 and b equal to three okay.

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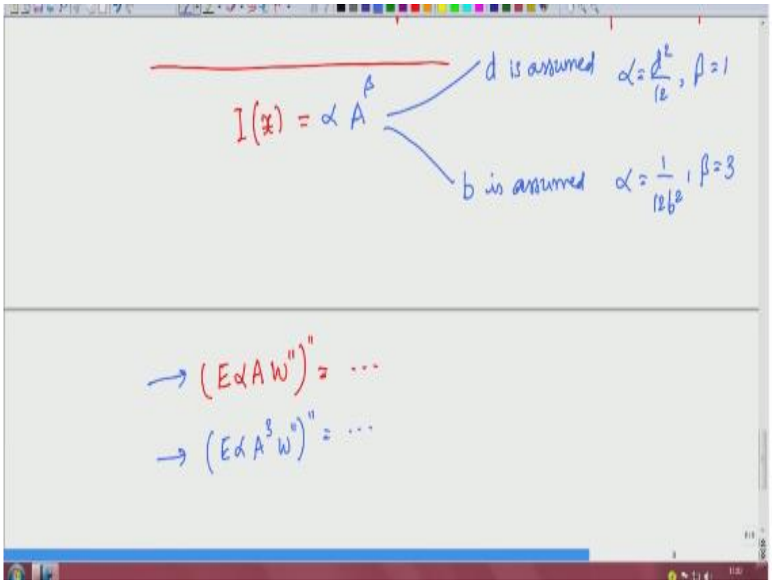

Imagine that we had to expand this term that is we had (E α Aw “)” which we expanded it that we got something before now that becomes (E α A^3 w’’) so expanding this will be much easier than expanding this and doing the calculation now you see a cube is their β equal to three it will become lot more terms but that can be done if you go to Mathematical you can do it by hand you can do it by hand as well.

If you use Mathematical are a symbolic manipulation software we will do all the expansions for you and then you can write down or Lagrange equations just simply taking partial derivatives.

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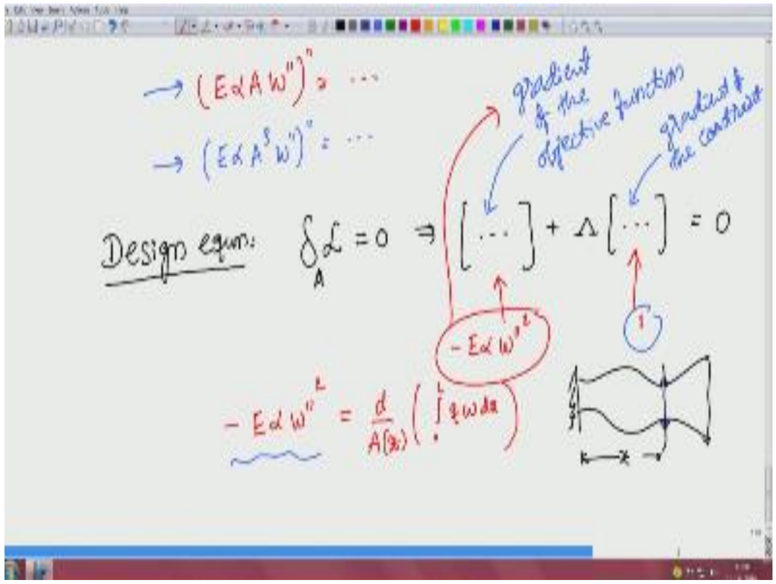

As we have seen for the design equation and that joint equation so what we have here you can do all of this using Mathematical our maple and our by hand which is doable finally what you would get when you do all this that is something that we should remember we will be getting the design you equation will be getting the design equation because of taking Lagrangian varies respect to design variable equal to zero okay.

And this is a design equation we have particular form there will be some expression x λ x some expression equal to 0 that is form of the design equation we had already said that this particular thing will be gradient it will be an expression meaning at every point x gradient of the objective function variant of the objective function okay that means that if I take objective function and take its derivative with respect to A.

In the sense of calculus of variations whatever you get will be the expression for every value of x if you evaluate that you be the precisely the gradient and this particular thing is going to be gradient of the constraint when I say constrained not the governing equation that is just a state variable controlling equation that is really not a constraint the real constraint here is a volume constraint that is what.

Because the corresponding λ is there for that constraint that will become the gradient of that if we go back and look at the case when D was assumed our design equation they go all the way up here a design equation look like this λ minus e α w $w'^2=0$ if I now plug that inhere for that problem so this is e α w w'^2 square with a minus sign and this is one right that means that this is the gradient of the objective function objective function here is the mean compliance integral of Q W is that integral if you take derivative at a particular a of X okay FX is a function let us say a particular value of x if you take a of X or that cross section.

If you change only their how much at what rate as the objective function change is given by this expression so let me reiterate that saying that in this problem as it will happen any problem if you look at what is in the parentheses ye α w w'^2 is the derivative of a of X a particular a of X any value of x of the objective function which is the integral 0 to l q w DX and this has to be understood really well because you have an expression to say what happens I have a beam okay let us draw a beam profile.

Let us say that is optimum let us say these optimal somebody has told you it is an optimum now or even otherwise some profile is given let us say at some value of x let us say this is our X at some value of x if I change the cross section little bit little bit okay let me use some color some material I want to add then obviously under the loading q if I compute w by solving equation I will have a different mean compliance right compared to the one without this little additional blue right.

So what is at what rate that is change is given by this okay and likewise in this case gradient is one because the volume constraint we have integral e DX the respect to a if you take derivative get one and that is what we got okay design equation will always have this form and a joint equation will always have the form similar to equilibrium equation numerically if you have a finite element routine to solve the equilibrium equation for W you can also solve λ because what will be different is just the loading.

In this particular case loading turn out to be minus Q let us look at that equation again we got if this is the equilibrium equation we got this as the ad joint equation for solving the ad joint variable λ here if you see the loading real loading is minus Q here it under B plus Q meaning that loading is minus Q okay so this is a consequence of the objective function which we have Q integral WQ q w DX.

So what we have there is essentially the ad joint load which is simply gradient objective gained of the objective function with respect to the state variable with a minus sign in this case okay so if you follow this procedure you get in a joint equation that look exactly like the equilibrium equation if you have an adequate routine to solve for W you can also solve for λ which is very easy straightforward once you know q + λ you have the expression for the finer the derivative objective function in the design equation if it can be analytically solve like we did in the previous example of assuming d to be constant are given we had analytical solution. But now if you have other situations let us say other objective functions and more constraints you may not be able to solve it analytically.

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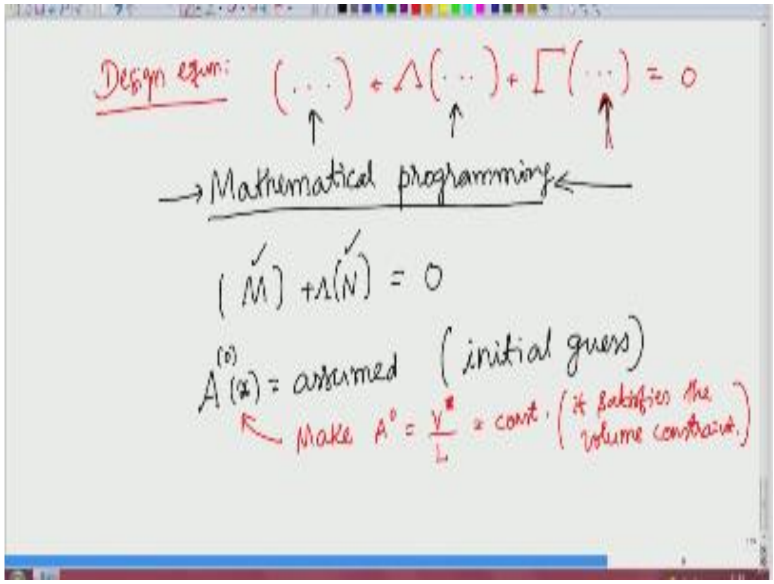

So let us say that we have in addition to volume constraint we have some constraint and stress or buckling or frequency whatever then design equation design equation which is obtained by taking the variations Lagrange in with respect to the design variable we get something and that something again is the gradient of the objective function plus λ into gradient of the corresponding constrain there the volume constraint let us say I have another constraint whose let us take it as a functional type of constraint let us say the gram multiplier transponder is right.

But that also will have something here and that will be equal to 0 and this will be the gradient of that constraint okay so you have expressions for gradients coming out of this analysis so if you do not want to solve analytically or you cannot do analytically for complicated problems this procedure this analytic pressure is still applicable because you get expressions for the gradients once you have that you can go and use any of the nonlinear or what we call mathematical programming techniques which are plenty these days readily implemented in let us say mat lab mat lab toolbox has FM in con consume minimization routine would solve using sequential quadratic programming and trust region methods.

And there are many more and any of those mathematical programming techniques are constrained minimization algorithms can be used to solve because now you are able to supply the analytical gradients that is the beauty of this analytical procedure we have the gradients when you have gradients algorithms will be happy because they do not have to compute the musing finite difference which will be time consuming because we have to perturb each variable at a time and do repeated finite element analysis if you have let us say 100 design variables you have to perturb hundred times that λ a 1 λ a 2 λ a 3 and so for that will be time consuming whereas now we have analytical gradients okay.

That is if you want to use this package programs are blind box blind or black box programs where you just supply your problem and the gradients they will give you the solution but if you want to write your own programs the design equation is still useful let us say I have a design equation such as this let me just for the sake of writing I will just say this is m + λ times n okay let there be another constraint but we'll set that aside let us I get something like this m1n will be some expressions.

Which we can compute as soon as we saw the governing equation that governing equation we can solve whenever we have a so we assume let us say iteration is 0 of X we assume that in any optimization problem we need to have an initial guess you have to start somewhere initial yes you take that so that it satisfies the volume constraint okay make sure that it satisfies volume constraint that is easy because you make it uniform okay then you will make it let us say we have a beam of length L you are given some beast our volume you may cross section everywhere beast r by l if integrate you will get V* right.

So if you do that make a 0 equal to constant I would say in fact v* if we have v*/ L constant because both are given which star is given L is given so it satisfies volume constraint it satisfies the volume constraint okay.

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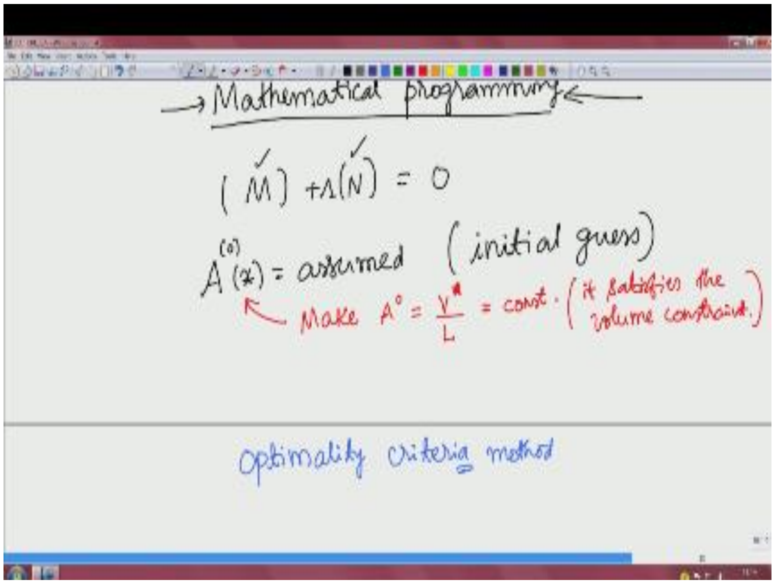

Now taking this equation we can use that optimality criteria method optimality we are using the optimality criterion here that is what is called optimality criteria method we use plural even though we have one criteria but remember that we had a joint equation other criteria they are all combined into this what we do is that M +λ n which should be 0 we add it to get a 1 so a 1 of function the first iteration.

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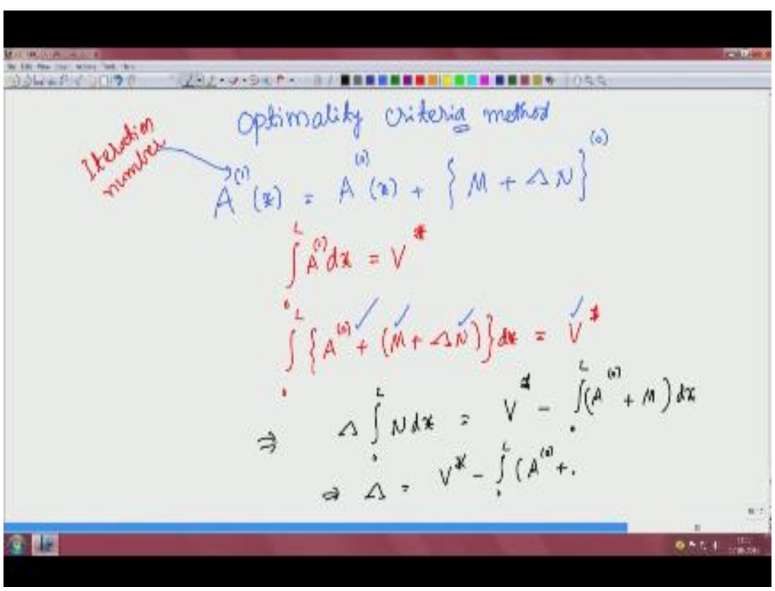

The first iteration so this refers to the iteration number this refers to iteration number okay so this we say we add it to the first iteration function we had Plus this M plus λ times n which is all calculated at zero th iteration okay m we will have some expression the case of the D assumed we got this as minus e alpha w, w ' square and n was one so we will always be able to compute that once you solve the governing equation where state variable is computed and hence its derivatives can be computed there is this λ that you do not know but for that we have the equilibrium the volume constrain which is active by the way because we got λ equal to e I w, wm square.

Which is the design equation if λ equal to 0ww ' equal to 0 w, w ' equal to 0if put in the equation then that will lead to a conflict that something 0plus q equal to 0 minus q equal to 0right that why we said λ cannot be0 that means constraint is active constraint is active less than equal to sign goes away then we have 0 to L a DX is equal to v star now not less than or equal to any more equal to because λ is not 0 λ is 0 w, w ' is0 w double ' is 0 equilibrium will have a conflict okay.

So once you have it we substitute the new area a 1 that is here I put a 1 that means that I have to do 0 to L a 0 plus M plus λ n all of this integrated with respect to DX should be equal to v star if you look at that what are the things that are known this we know this we know this we know what is not known and also v star is known what is not known as λ it is only one equation right we can actually calculate that we can calculate this now by taking all the things that we have to the other side so we start okay first I think not really okay let us just do this I will write the λ part of it λ into 0 to L n DX n is equal to one here but let us do it right.

In general you may have an something else so n DX equal to v star minus 0 to l a 0plus M DX okay I have taken this to the other side and I have this now so what I get is λ is we start minus integral0 to L a 0 plus M DX divided by0 to L.

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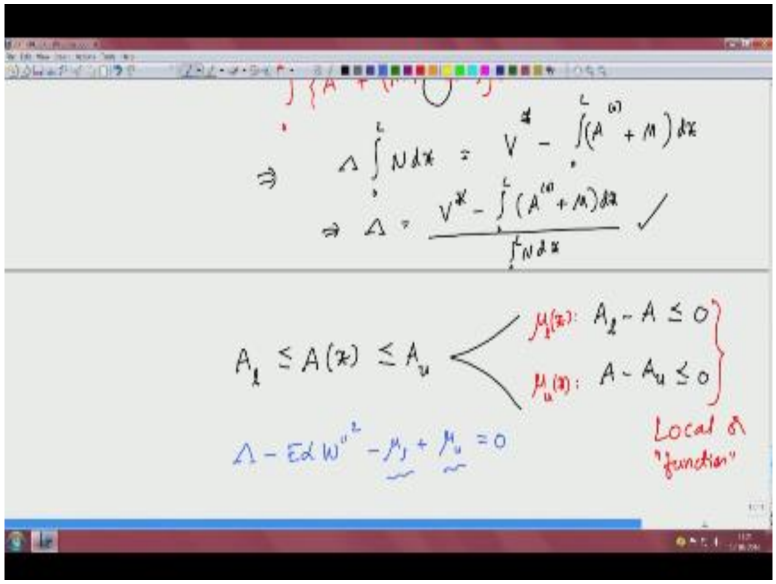

And D X okay so basically have n equal to 1 it will just become L but in general you can get λ that is also analytical because our constraint here is linear in area of cross section if it is nonlinear we are do that also numerically okay but in whatever case AC single variable λ is our variable now we can find that using an equation once you have that λ you can update we have one once you get a 1 you get a 2in a similar way that is wherever one is we have to replace that with 2 this becomes 1 this becomes 1 then you have to get λ for that iteration using the same way.

And you keep on repeating it until this becomes zero meaning that a k plus 1 is very close to AK or to our satisfaction very close you have to define when you want to stop how accurate you want optimal structure to be optimal profile to be you stop meaning that then this will be 0converge that is exactly what was our condition that M plus λ n is equal to 0 that is what we do ok that is how the optimality criteria method works in numerical sense whenever you can all solve the problem analytically we had done this for the bar we will do this for the beam in the next lecture.

So you can see that before that before you go to the numerical implementation in the next lecture we should also consider the fact that you might have upper and lower bounds on area of cross section if you do this sometimes area of cross section may already be negative or maybe 0 which you do not want tolerate or become so large in some places that you may not want it so I can put in additional constraint that area of cross section at any value of x has a lower bound and an upper bound right in fact in this there are two constraints.

One is since we always write everything in the form of less than equal to L minus a less than equal to 0 a minus a u is less than or equal to 0 for these are local constraints right because these are function type constraints local constraints these are local or function type constraints because they are everywhere for all values of x so I will use lowercase letter M ooh we can call it mu L of X and then mu u of x two additional multipliers okay so accordingly when you write down the design equation this will have minus mu L will come here plus mu.

You will come the design equation so for this problem if you are too right we had this capital λ minus e alpha w double ‘ square and then we will get minus mu L plus mu u equal to 0 right now we do not know this mu and mu u when do they come because we also have complementarily conditions that come from these constraints right that is.

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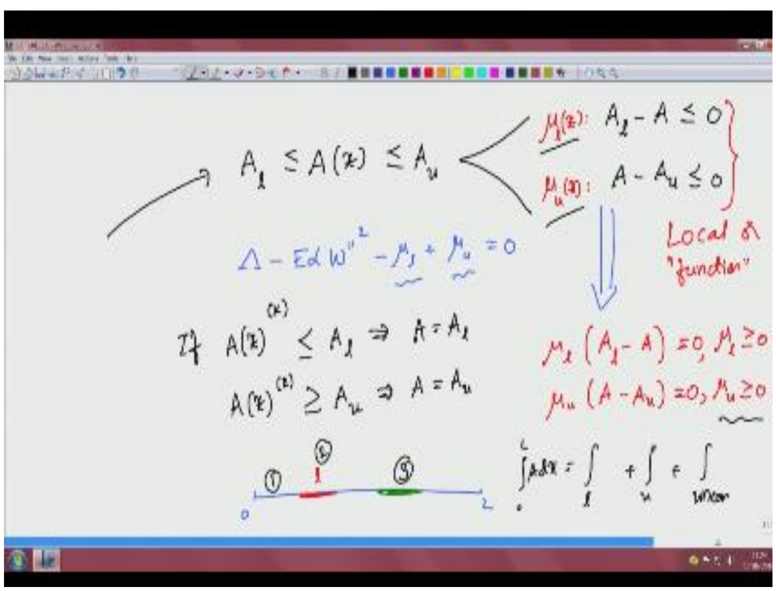

We will have mu l x a l minus a equal to 0since they are less than equal to we want mu L to be greater than or equal to0 and then mu u times a minus a u equal to 0 and hence we get mu u you get an equal to 0 this these two functions that we have here can only be non-negative they cannot be negative that is what these things illustrate so you have to find them actually do not have to find them because whenever it reaches that when you are updating your area profile in this manner when you are updating in this manner we as you find the uppercase λ you check after you do this whether area cross section anywhere is going to disobey these bounds when the disobey you push them to that.

If something is exceeding au you bring it down here you and fix it at that when something is going below air you just make it stay at all in that case what happens is that area classic is already known because if you make anywhere ax of Kate iteration numerically are doing if it is less than a el less than equal to we can say then you simply say a equal to a el for that point for that value of x okay similarly if a of X K iteration is greater than at you, you basically make that a at that point you so if you have the domain okay 0 to L but different values of x there could be some domain where it is reaching the lower bound let us use a for the lower bound there could be somewhere that could be reaching upper bound okay.

Rest of the blue is in between which is controlled by your expression according to the design equation so when you are trying to find the λ using this equation you do not have to worry because that 0 to L is now split into three different things we have the first one no constraint second one lower bound constraint third region upper bound constraint you accordingly split your 0to L a DX into three regions lower bound region upper bound region and then unconstrained region.

Okay and you do similarly this calculation you will always be able to get λ and you proceed okay so we will solve a problem where we will have lower and upper bounds on the area of cross section and numerically illustrate to you in MATLAB with a code that you yourself can also run after we go through a few examples both for a bar we had done earlier now we will do for the beam so that we understand this clearly in the next lecture thank you.