**Indian Institute of Science**

# Variational Methods in Mechanics and Design

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In the last lecture we discussed how energy methods can be used to formulate the governing equations for a beam we started with the definition of strain energy and work potential constructed potential energy took its variation but also saw how minimum pressure energy can give to the weak form of the differential equation or the principle of virtual work and that in turn leads to the boundary conditions on the differential equation which is actually the force balance.

The three ways of looking at static equilibrium we saw it in the case of a beam and we also said that that can be seen for the dynamic case as well; today what we will do is we go to structural optimization of a beam.

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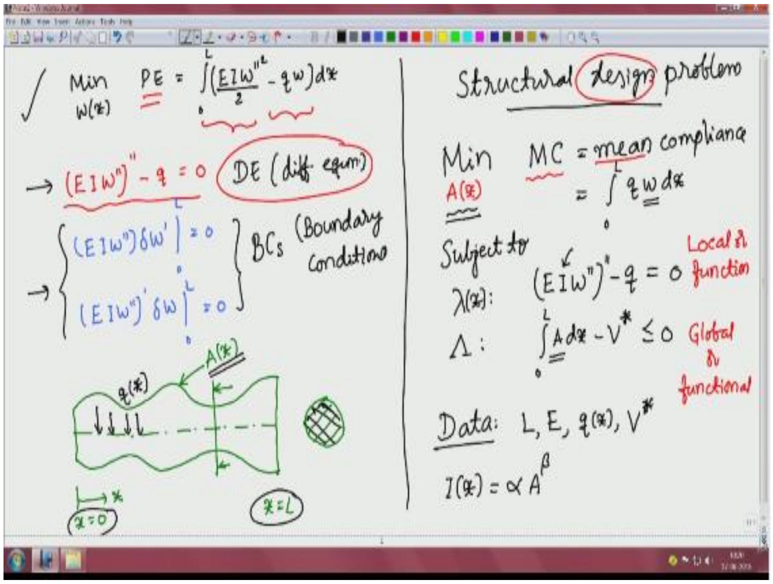


So let us recap what we discussed in the last lecture where we wrote potential energy here where we have this strain energy and then the work potential what potential once again is the negative or the work done by external forces, now when we take this potential energy and take the first variation and equate it to 0 and do the integration by parts to get the differential equation so this is the differential equation we write DE for it differential equation.

And we also have the boundary conditions as we write as B C's boundary conditions this is for a beam which has variable cross section that is a is a function of X, X goes from x equal to 0 to X equal to L and this Is the cross section are different values of X you have changing cross section and for such I think this is the differential equation and these are the boundary conditions okay, now let us go to the structural design problem.

So structural design problem this is to say that variational methods or calculus of variations methods not only help as do analysis which is this we are analyzing the beam finding its deformation and stresses and so forth we can also do design okay so we can pose a problem which we had already done for bars now we do for beams let us say I take the case of minimizing strain energy of a beam we already have an expression for structure of a beam we have done it for the bars actually.

Let us do that instead of strain energy let us do another measure of stiffness minimizing strain energy is a measure of maximizing stiffness is equivalent to maximizing stiffness there are many other measures for stiffness one of them is called mean compliance will call it MC mean compliance which is given by basically integral 0 to L of Q times w DX okay that is basically saying that the work done by the applied force Q that is acting on the beam right.

So if this is the beam is the neutral plane so there is Force Q of X acting and that multiplied by the transverse displacement w is our objective function so work directional force if that is minimized it is called mean compliance mean in the sense that we can take this and divide by the length that should be the mean but dividing multiplying by constant does not make much difference for an optimization problem so we do not actually put that even though we say it is mean compliance.

It is mean mainly because we are looking at the load and the work done by that load or at every point in the domain zero to L and summing it up or integrating that is why it is called mean I can always divided by L but that does not matter okay we basically work done by the external force that goes by MC main compliance, how do we minimize mean compliance we want to design define a variable which is area of cross section of the beam a of X.

Just like we had done it for the bars okay that is our design variable subject to as we had looked at the general framework for structure optimization problems there are design variables which is in this case a of X but we also have state variables which is this w the state variable should be governed by differential equation for a beam the governing differential equation is right here this is what we have already written that is this one.

So let us write that that is the governing equation $(EIW'')'' - Q = 0$ that is the governing equation for the state variable W when we want to minimize the mean compliance or maximum stiffness if we are given a lot of material infinite amount of material we can of course make any beam as stiff as it can be made that is infinitely stiff by making area of cross-section infinity but that is not allowed in practice dream theory itself will break down.

If we do that but from practical consideration for optimization when you want to minimize mean compliance or maximize stiffness we should do so subject to the volume constraint that is we cannot have any amount of middle that we won we are given only let us say a certain amount of material v star that we will put less than equal to 0 nobody tells us at this time how much material to use even though v star is given to us.

V star maybe given to us but we may use only fifty percent of it if we can find an optimum will be happy that is why you put the less than will do they are okay and as we had said earlier when you have a functional type constraint which is this second constraint we should put a Lagrange multiplier with an uppercase because only a scalar as opposed to lambda X for a function type constraint again let us recall.

This is what we call global or functional type and strained whereas this one the first one is local or function type constraint okay this inequality this is any inequality that is okay we know how to deal with that using complementary condition for the inequality but let us understand the problem that we have a governing equations in local or a function type constraint we have a resource constraint which is a global or a functional type constraint.

Let us also clearly right what is known to us data so first of our length of the beam is given to us young's modulus of the beam is given to us and the loading q of x is given to us and the volume v star is given to us okay, is this statement of the problem complete actually it is not because we have area of cross section as our design variable but I find area of cross section over here but I also find I is I known to us it is not known.

That is I of X we do not know but we can express it in terms of area of cross section by writing it as some alpha times a of X raised to some parameter beta where alpha beta are numbers how does it come about let us look at that.

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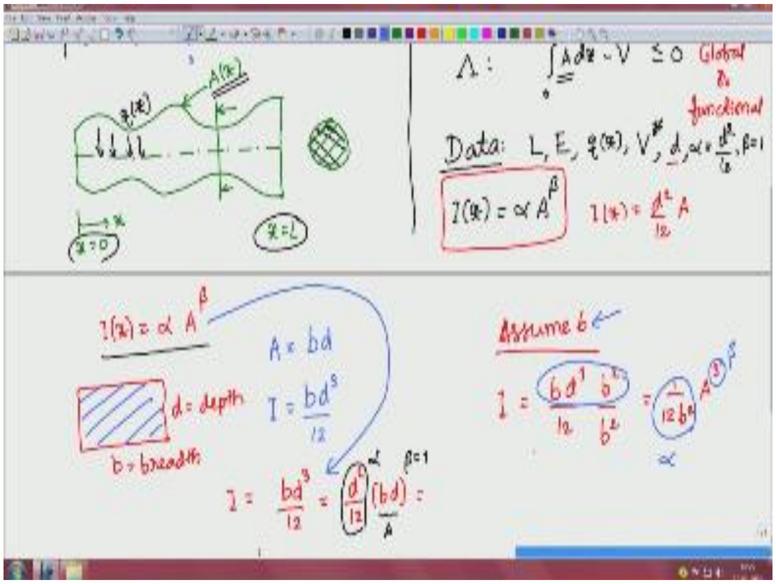

For different cross sections this alpha beta will be different so we are writing here I will rewrite it we have I of X we want to write as alpha times a raise to beta let us take rectangular cross section okay let us say the breath is B and the depth is d okay so this D is depth of the beam which is the dimension in the player of deformation of the beam if there is a beam that is deforming the cross section will be deforming there will be two dimensional across-sectional rectangle.

One will be in the plane of deformation other is perpendicular depth is the dimension the plane of deformation and breadth is the dimension the perpendicular direction for the beam so if my hand is a beam when I am deflecting like this right this thickness is a depth and this width that I see from the top is the breath normally thickness with all those are confusing when you take beams that deform in play in verses out of plane and so forth it is always good to use B for breath d for depth okay.

Again these the dimension in the plane of deformation of the beam okay so if you have this we all know that or many of you would definitely know is that this I which is the second moment of area is given by $bd^3/12$ an area of cross section is given by B times D for this rectangular cross-section okay so here if I want to write in this fashion we can write that is to say in this case I which is $bd^3/12$ I can write as BD okay.

It depends on what we assume has known breath or depth to begin with let us say that we assume that B is the variable whereas d is taken so I can write this I as $bd^3$ /12 right. So if I want to put area I need to multiply by $B^2$ and divided by $B^2$ so I can I can I can do well let us actually take d as the fixed value so I will take out $d^2$ put 12 as it is x BD if I do this essentially what I am saying is that this quantity here is my α okay.

Because D I am assuming I am assume the depth of the beam and vary the width okay and better that we have will be equal to 1because it is just the area BD is area right so we get in the form that we wanted okay let us take that here so when we go back to the problem statement over there let us assume that we know d okay, and we know this α which is as we just did is $d^2$/ 12 and then β = 1 so that when we take I in this form I will have simply $d^2$/ 12 times a so in this particular case I of X is $d^2$/ 12 x a, a is BD so I get $BD^3$ at fault this is by assuming the depth of the beam okay.

On the other hand if we want to make be assume so if we assume b right then I can write I to be we have $BD^3$ / 12 I know to get area so I will multiply by $BS^2$ and divided by $B^2$ then what do I get I get 1 /12 $b^2$ x $a^3$ because this particular thing is now $b^3$ $d^3$ that is $a^3$ in this case this becomes our α this becomes our beta okay obviously this way of doing it will be more difficult than assuming d and proceeding we always when we start take something simpler that is why I am taking d to be given. So that I can make the problem simpler and later on we can consider be to be given and very d okay

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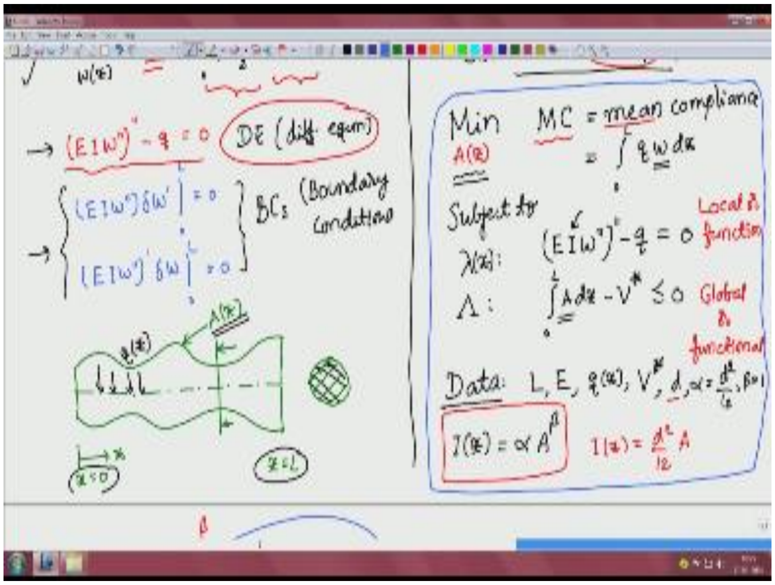

Now taking this problem the problem that we have the statement that we have let us encircle it the problem that we want to solve is simply a calculus a variation problem since we spent quite a bit of time and understanding calculus of variations we should be able to solve any problem that is given to us when it a problem statement is complete meaning that everything is properly written up then any number of constraints.

Whether equality inequality local or global does not really matter we can just mechanically do the variation write the differential equation boundary conditions and formulate the equations which when we solve we solve the problem of structural optimization okay again I really want to remind you that we had done this four bars now we are doing for beams okay.

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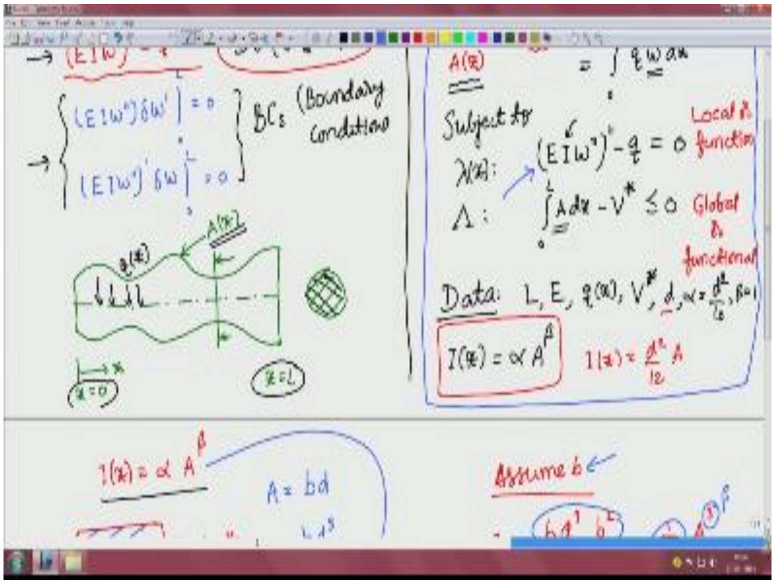

Now the first step that we do when we solve an optimization problem is to write the Lagrangian so the Lagrangian here we have the mean compliance which is the objective function 0 to l q w DX and then we have two constraints one is a local constraint whose Lagrange multiplier is λ and we have to multiply that with the constraint itself which is e I w''- Q okay yet if you want to look back you can see what we have written we have written this constraint and x Lagrange multiplier λ similarly we have to do for the global constraint where the multiplier is capital Δ.

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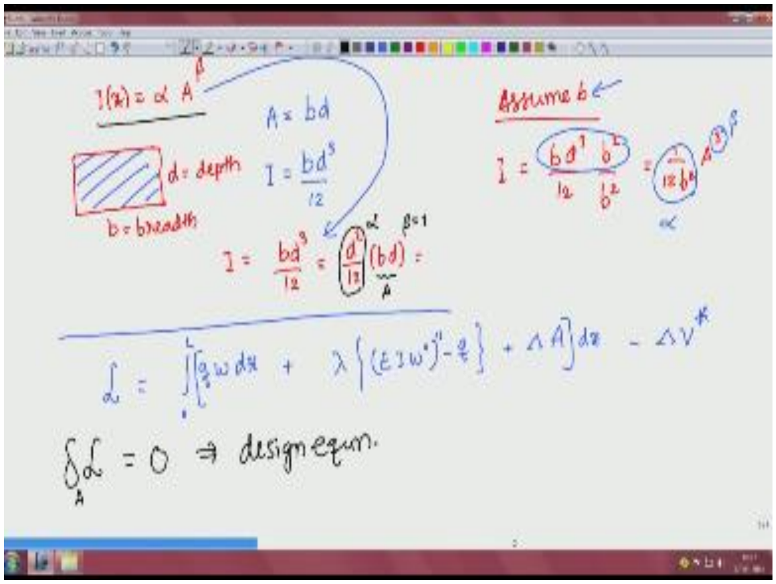


So $\Delta$ x a okay because that is what is under integral sign here ad x so all of this x DX and then we have $\lambda$ v* that just a constant where when you are minimizing a constant such as this does not matter though basically that v*$\lambda$ times V* will be there okay these are Lagrange Lagrangian so what do we do as we had done these steps we have to remember as we are doing simple problems bar and then now the beam so first thing we do is we take the Lagrangian variation with respect to the design variable which is a here okay which we have to equate it to 0 if we do that we get design equation okay.

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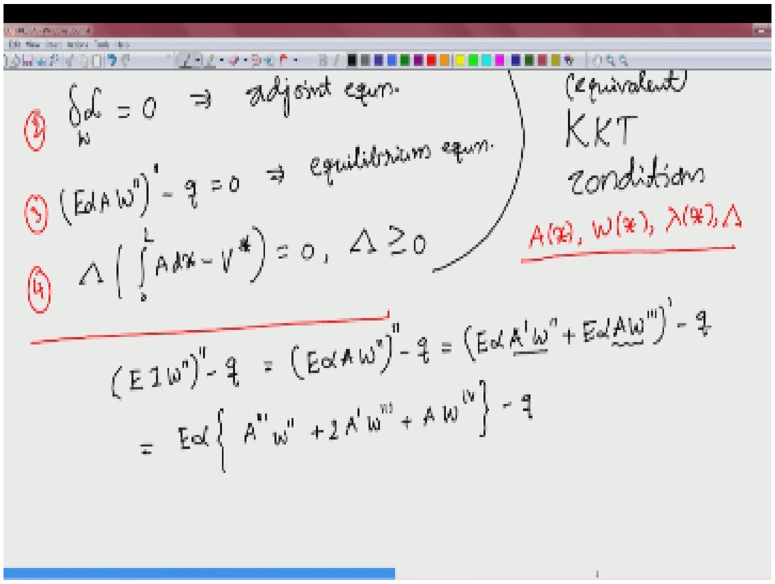


And if we do this for state variable equated to zero we get a joint equation okay and when we take this there are basically two variable state variable and the design variable so we do not need to take variation anymore but we have the governing equation when I am dating necessary conditions now we have the governing equation which is e I w’’ which will simplify because if you see this I here we need to we wrote it as that α into a where α is $d^2/12$ okay.

So that is what we do because I wise we take care evasion here we wonder what I is again that is now λ a so I can also replace this calling it α a -s Q = 0 that is our equilibrium equation equilibrium equation and then we have the complementarily condition for the constraint that we have which is then equality 0 to L a DX –V*r okay is equal to 0 and we have this λ than or equal to 0 basically these are what we have here are basically equivalent of KKT conditions Karsh Kuhn tucker conditions I would I must say these are equivalent they are the same arguments are applicable here those ought to be we have.

So again we should always count to the number of unknowns and equations to see if we have enough equations to solve for those unknowns the unknowns are we do not know a of X that is our design variable we also do not know W of X we do not know λ X and we do not know this upper case λ over here okay so when you have this in fact if you look at the data you have to check you look at all the variables that are in the problem statement and then see what in the data whatever is not in the data we should say that that is our design variable.

So here I am saying is a design variable right that is an unknown so we list it in the unknown part and then Δ we do not know capital λ we do not know undoable you but do not know everything else we know here so that check will always be useful okay now once we have these we can write and see we have written the equations design equation ad joint equation equal in equation and the complementarily condition and the condition that lamp should get than or equal to 0. Once we have these we can so proceed and solve the problem.

So we have four variables and we have four equations if I can number one two three and four now let us look at the equation again the Lagrangian is given here this DX is extra because we already have the DX over there so q w+ all this Lagrangian let us write the design equation before that we need to expand this term okay let it let me do that we are now solving in detail so that e I w''- Q we first substitute for e over I which is α times a w'' let me expand it once e α into a' W''+ e α a into W''' and then one more prime we are done differentiation once q remains as it is one more time it will give us.

I think it is safe to take a alpha outside that will there in all of them we have a'' I am taking derivative of this now a'' W''+ a' W''' that I am going to put 2 hear because when I take derivative of this I get a' W to confirm product rule and then into W fourth derivative minus Q okay having written that.

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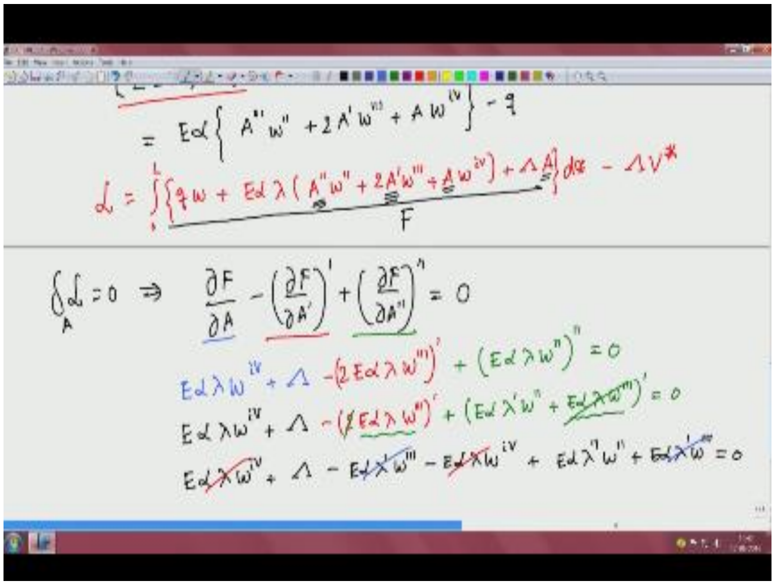

Now let us write the Lagrangian one more time Lagrangian which is 0 to l q w+ e alpha λ because λ was multiplying this thing over there times a double ' W double ' plus 2 a ' W triple ' plus a w fourth derivative and then we also had the upper case λ times all of this x okay let me use a different bracket because we already have that bracket let me use this curly bracket DX minus λ v star okay for the design equation we have to take variation of the Lagrangian with respect to K and equate it to 0.

So we had read Euler- Lagrange equation by taking the variation of the leg a Lagrangian with respect to a so we look for things basically if I call this whole thing as my integrand f then I have to write ∂ f by ∂ a, a is there is here we also have a is also here and we also have a ' here and we have a double ' here so I would write ∂ f by ∂ a times ∂ f by ∂ a ', ' plus ∂ f by ∂ a double ' double ' equal to 0 so it is a good exercise for a Lagrange equations application so we have this right so based on this we can write the terms ∂ f by ∂ a is in two places one is this e alpha λ into W fourth derivative.

That is what is multiplying a and then we also have upper case λ okay that is the first portion so that is that let us use color coding second one okay that is minus a ' there is e alpha λ 2 W triple ' that is ∂ f by ∂ a ' and all of this we should take one derivative okay and then we have this thing is plus a double ‘ that is e alpha λ W double ‘ then two ‘s two derivatives of that equal to0 okay let us simplify it a little bit here so we have this one is nothing to simplify ye alpha λ W fourth derivative plus this λ this red one okay.

It may be a good idea to leave it alone because the fourth term will have you know something coming there so I will just put this as minus 2e alpha λ W triple ‘, ‘ and then let us go to this last one let us take one derivative of this that will give us a alpha λ ‘ w, w ‘ plus e alpha λ into W triple ‘, ‘ equal to zero okay because λ is also function of X now we see that this thing here is also here this is minus this is plus I can cancel up this2 and this term all together okay.

So one of these gets canceled so we have a little bit simplified let us proceed to further simplification 4 plus λ minus 1 will have okay that is e alpha λ triple ‘ let us expand it now that will be e alpha λ ‘ W triple ‘ minus e alpha λ W for derivative we are happy because this is the first term and this get canceled okay and then we have this the last term which will give us a alpha λ double ‘ λ double ‘ W double ‘ plus e alpha λ ‘ W triple ‘ equal to 0.

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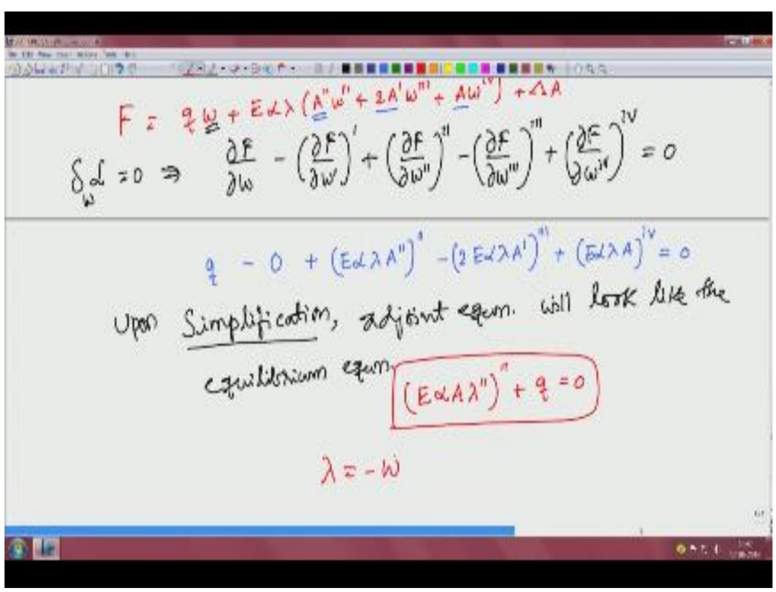

Again it is a happy situation because we see that and that cancel so what we get with this is just this λ then we have plus e alpha λ double ‘ w, w ‘ equal to 0 let us keep it aside for now because we do not know what λ is as yet so then we proceeded to do this variation of the Lagrangian with respect to W okay so that is over there okay for us to write I think I have we have to rewrite this Lagrangian down here so that we can easily take the variation so instead of writing the whole Lagrangian let me just write the integrand of the Lagrangian in that thing.

So I will just write F okay that is Q W plus λ times there was a fourth derivative right so e alpha λ okay q w + e alpha λ into a double ‘ W double ‘ plus two a ‘ W triple ‘ plus a w-4 3 by 2 plus λ a that is integrate so once we have it when we write the variation of the Lagrangian with respect to W equate to 0or Oiler Lagrange equations it will be ∂ f by ∂ w we have that term here minus ∂ f by ∂ w ‘, ‘ that is not there w ‘ is not there plus we have ∂ f by ∂ w double ‘ double ‘.

We had w ‘ here we also have triple ‘ and forth ‘ so it is a very good exercise for calculus of variations problems triple ‘ triple ‘ plus ∂ f by ∂ fourth derivative and we are taken for that we would use there that should be equal to 0okay let us do that even though it may look intimidating I think but we can use as a routine process ∂ f by ∂ w will give us q here minus w ‘ term is not there that is 0 plus w double ‘ is there that is going to give us e alpha λ into a double ‘ and then double ‘ minus triple ‘ we have e alpha λ with a 2 into a ‘ and that goes has triple ‘.

And then we have e alpha λ into a and that goes to fourth derivative that is equal to 0 okay so W there is only QW ‘ is not there w mm we have this term which we have w triple ‘ we have this term w fourth derivative we have that okay Andy alpha λ is there for all of them now we need to simplify it okay that simplification I would urge you to do it on your own simplification it is just algebra like we had done many terms get cancelled in fact you would see that this ad joint equation that is what we get when we take Lagrangian variation with respect to the state variable adjoint equation simplify upon simplification okay.

In this problem adjoint equation will look as always like the equilibrium equation okay this is a lengthy simplification you can do it in a few minutes if you do that it is going to look like this it is going to look as e alpha a into λ double ‘ double ‘ plus q equal to 0 this is how it's going to look like if you compare it with the governing equation which is this alpha a www ' minus Q here it is just the plus sine alpha a w, w ' so comparing these two we can conclude that λ equal to minus W okay there is a minus Q here is a plus Q once you know λ equal to minus W we go back to this equation.

And then find something if I do this it will become alpha w double 'square equal to zero because λ equal to minus W so long double ' equal to minus W double ' so we get this now what we get here again is that λ is equal to e alpha w, wm square what does it mean λ chi alpha are constants w ' should be constant that means that it is not a function of X so w, w ' as.

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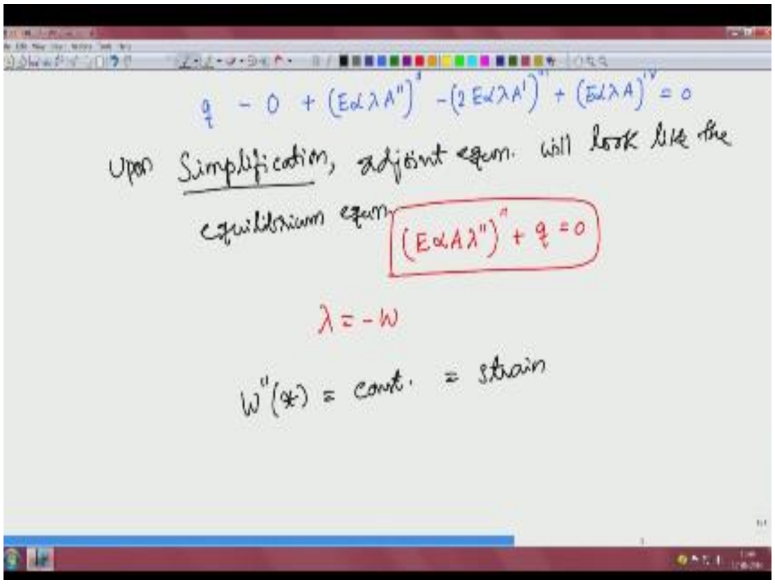


We know is strain it is a function but here it is a constant and that is strain and hence x e that will get stress that is constant so this exactly the concrete we had arrived at in the case of bars that if you want to make the structure the stiffest for given amount of material then a strainer stress or uniform throughout the be a m meaning they are not functions of X and once you know this you have enough equations to solve just like in the bar which will consider in the next lecture.