Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello we were discussing a special feature of calculus of variations which is called general variation normally we have made functions to be unknown but over a fixed domain x1 to x2 if it is one variable case. But now we are relaxing that we let the domain of the function that we want to find is also variable because that leads to an interest rate of problems that we discussed partly last time we will discuss more today. So looking at the problem we have this concept of general variation.

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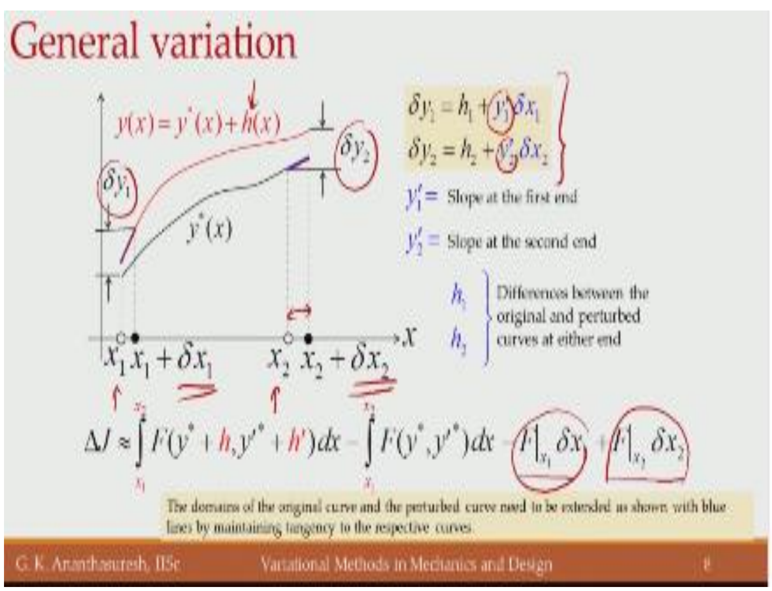

Where we as we can see here so we are looking at a function y* which we think is a solution that minimizes a functional J which is which has an integrand f which depends on y and its derivatives, then we say that our domain x1 to x2 is not given to us that is variable so we let that have also variation as it is shown here so we have the domain itself from x1to x2 is going to be undergoing the variation just like why undergoes variation which we have here Δy1 and Δy2.

We also let the domain very and here we differentiate between the h that we are using and Δy that we have here so which is the relationship that we have derived in the last lecture it is not really a big derivation all we are doing is extending the curve from here to here there to there because x1 to x1+Δx1 does not exist but x2 to x2+Δx to exist so we need to interpolate this way extrapolate this way and that way that gives us.

Basically looking at the slope here that is why we have y1″and then y″ over here and then y2″ over here that is how we got the Δy and Δy2 and what we did here if you look back is a very simple manipulation when we have this integral to be done from x1+Δx1to x2+Δx2 we divided into x1 to x2 and then added and subtracted these portions that we needed to do because this additional thing we need to do that is nothing but F at that point because Δx2 is so small we get this term and then we have to subtract because we are not doing really from x1 to x2, x1 2x1+Δx1 that we are subtracting okay,

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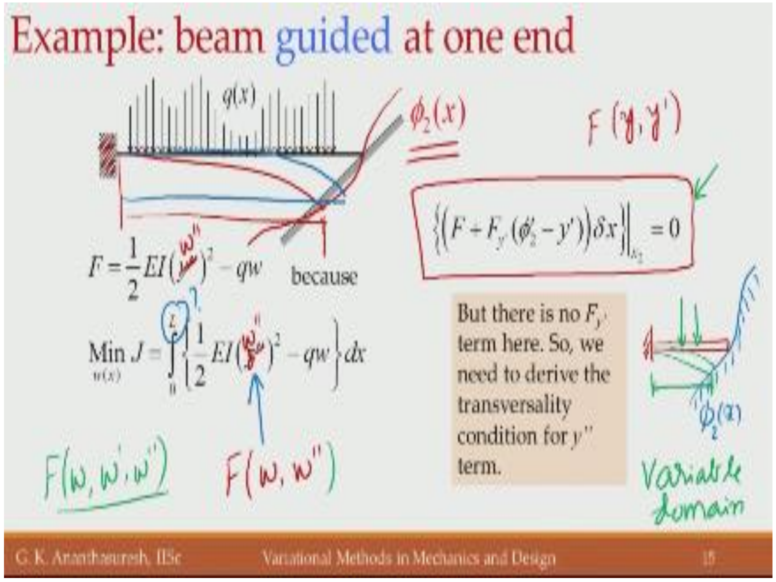

With this we got the boundary condition of which looks like this right, this is what we had when our integrand was a function f y and y′ only then we said at the end of the last class that if we have a functional such as this here this integrand right, is a function of W and W″ it is actually not why it is w w''^2 because that is what will be there in the beam right, so we are talking about a case where we have a beam that is allowed to slide on one end this is what we are discussing last time.

So let me draw it here again let us say I have a beam this is the beam and if this is not fixed at the right end but is allowed to move on a curve here I have shown it as a straight line but now we can say it is a $\phi(x)$ that curve is given it is on a slope like this now if we apply force on this if you apply some force on this beam how does it deform is what we are asking again there are lot of possibilities but now our integral is not 0 to l.

Because that l is not fixed because when the beam bends let us say like this it is going from here to here not the complete length so what should be the boundary condition for this now because the domain is now variable that is why we need this general variation, okay so for that we do not have a condition like that yet and that is what we will do now because our integrand now has not just Wand W′ in this case of course w′ is missing. But a general case is that our integrand depends on y or w, w′ and w″ okay.

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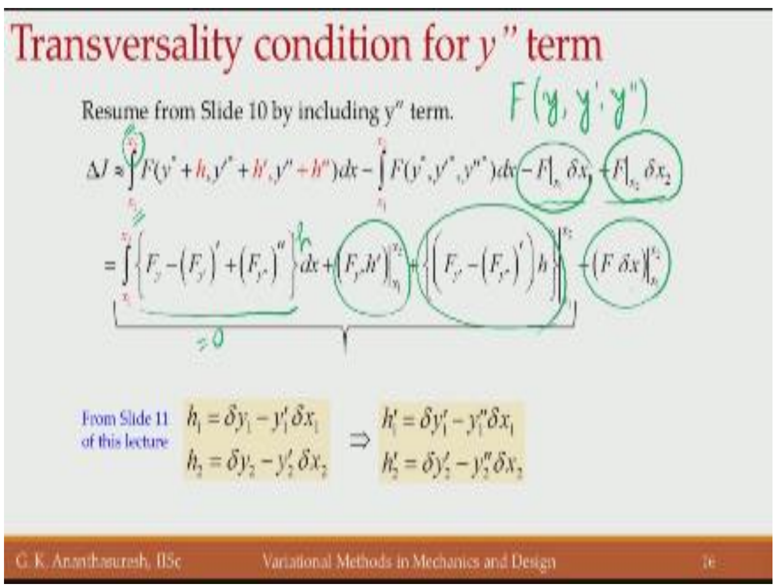

When you have that we can do the same thing we take a general one now as if our integrand has y, y′ and y′′ if you have it again we do the same thing in the sense that we do this changed functional from x1 to x2 and then add this part and subtract this part because the new functional was supposed to go from x1+Δx1 to x2+Δx2 but we are doing from x1we do not want from x1to Δx1that gives rise to this term and then we are stopping at x2 here.

So we need to add the extra part or Δx2whichgives us this term and again if you do the usual integration by parts and all that we get the boundary condition we get the differential equation part where fundamental lemma will give us actually they should be a Δ or h should be there let me just erase that so this is yeah, there should be an h here and then we have h′ and all that we apply fundamental lemma and make this thing equal to 0 that gives a differential equation we get the usual boundary condition and then we get this additional condition here okay, and then this h the usual two boundary condition that we get and then FΔx.

Now when you have h we said that h is different from Δy now because h is at some point again if you go back to this at any point.

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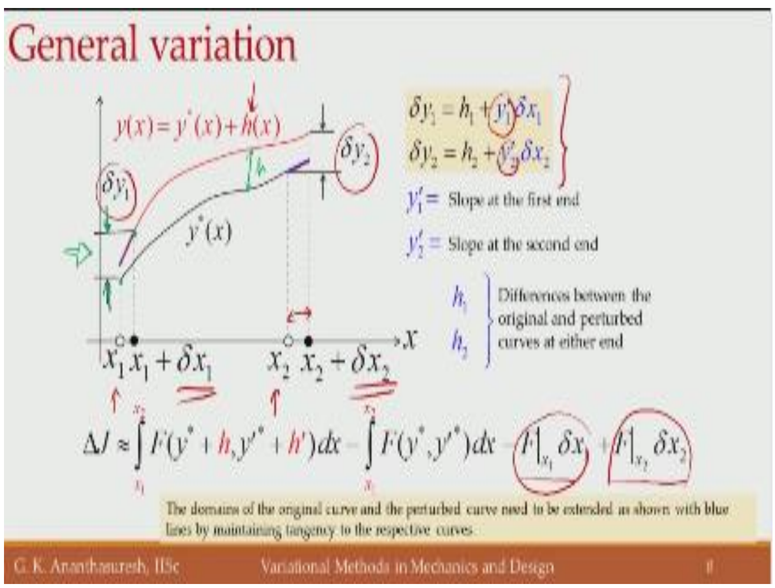

The distance between these two curves is actually h as our usual perturbation the function is variation the function but what we call Δy is a corresponding points x1 was here it has moved to this point x1 y1 was here is move x2 y2 here so those to be compared that gives us Δ y1 okay, so this is now Δy1 okay.

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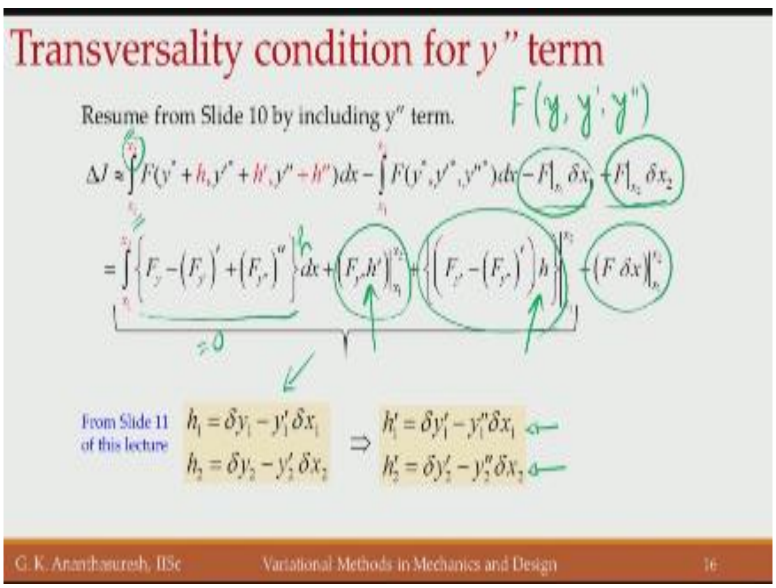

So the same difference we have to make here saying that h1 and Δ1 is related in this manner which we had already used now we just take a derivative of this so h1′ becomes Δ y′-y1″ times Δx1 likewise Δh2′ h2′ will become Δy2′-y″ Δx2so we can substitute for this h′ here and h here at x1x2 meaning h1 h2 h1′ h2′ if you do that we get the boundary conditions that look like this.

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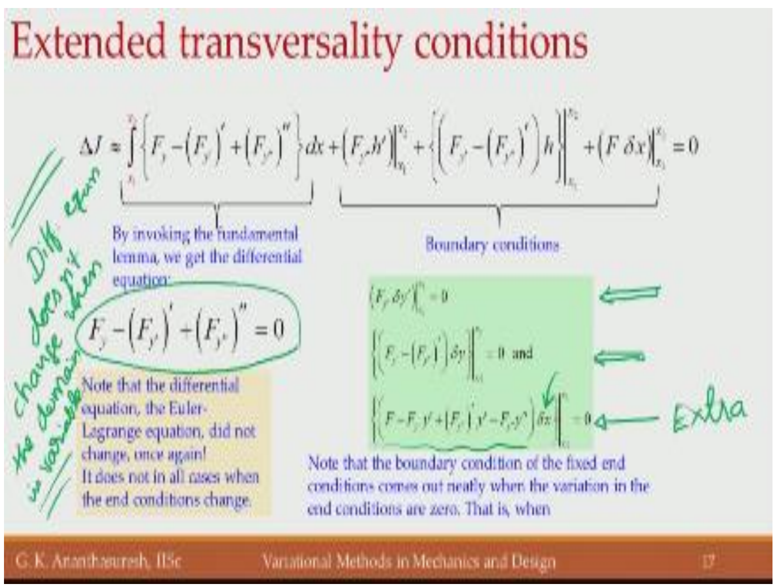

Now we had three boundary conditions there is the first one that says $F_{y''}\Delta y' x_1 x_2 = 0$ that is the usual one that we already had and similarly we have this condition $(F_{y'} - F_{y''})' \, x \Delta y$ that is also known to us already but we had these two we already know but this third one is a new one because of variation in the domain whenever domain is variable we have to satisfy this because Δx in that case is not 0 so that must be 0 a long thing that we have here okay, that is the condition.

Now it back and differential equation of course remains the same that does not change so differential equation does not change this something important to remember does not change when the domain is variable when the domain is variable yeah so that is something that we must remember okay, well the boundary condition we have this extra boundary condition that involves y and y' and y'' in some sense okay.

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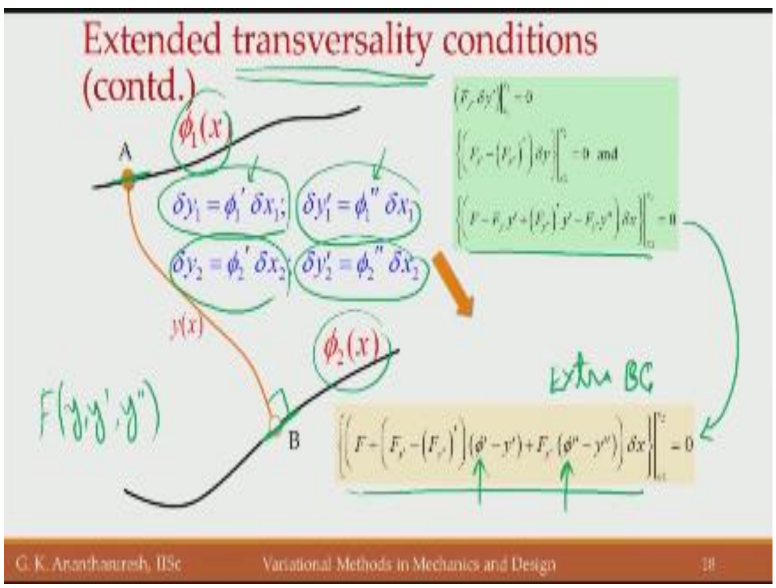

Now this is basically capturing what we have when we have an integrand that depends on y y' and y'' we would have the boundary condition which this is expanded version this is the extra boundary condition extra BC, now if you see this and what we have here are not the same because at that point we still had y and y' but now we have let that point move along a function p1x v2x earlier Δx was domain was variable $\Delta x1$, $\Delta x2$ but now the $\Delta x2$ and $\Delta y2$ are related because they have to lie on this line $\Delta x1$, y1relatethat they lie on that line so we have this relationship Δ y1, Δx1 are related and hence when you take derivative these two are related here it is v1′ here it is p1″and then we have this relation and this relation.

If we now substitute this Δ x1 or Δy1that we have in terms of the other we get an equation that is this long one that we get by adding the new thing appear here as this ϕ′ and ϕ″ that come about the boundary curves where the boundary hast o go okay, this is more general version of the so called transversality conditions, transversality because these only when you have a particular form of the functional which is f(y)$\sqrt{1+y'^2}$ f(x) not f(x) $\sqrt{1+ y'^2}$ if you have functional then these curves being normal like what is shown here the orthogonal.

I transversality holds for all others we still call them transversality conditions even though they are just very general ones okay,

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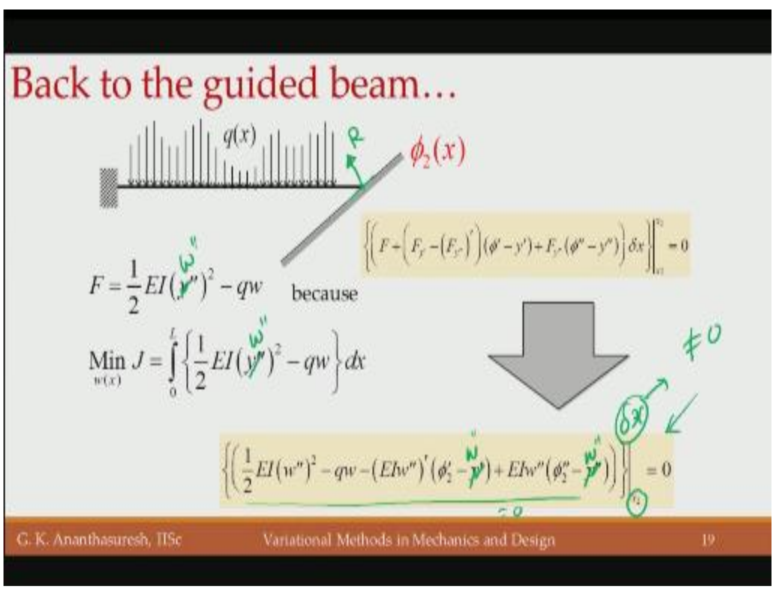

Noting this in this case v2 is given to us we go back to this equation that we have and look at our functional okay, so we look at our functional that depends on again this is not y″ this is w″ this is w w′ and when you substitute all the relevant things over here okay, ϕ2 that is given as a straight land all that we get this condition okay, this multiplied by Δx at x2 because that is variable but then ∆x2 is not 0 because it is variable so what multiplies this thing is equal to 0 that is what we have put, okay because this is not equal to 0 okay.

What does it physically mean if you were to compute this ½ EIW ´ ^{2} - qw - EIW ´´ V 2 ´ - y´ + this actually not y´ again they should be w ´ EIW w´ ww-p 2´´ - w ´ okay this is just a type right now if you do that at x2 write that other end that is L that you have what it should mean it means that when it is allowed to slide here okay there would not be any force there is no reaction force whatever reaction force that will be at this end will be in that direction okay that is what it means physically the tangential component of the reaction here along the curve is going to be zero because free to move in that direction okay

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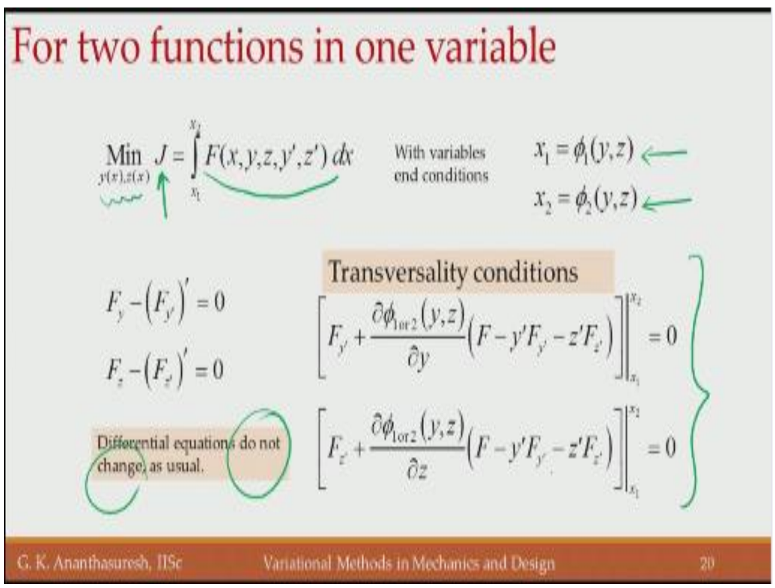

Similarly if we have two functions in one variable then what happens that is we are minimizing a functional J with respect to two functions y (x) z(x) simultaneously where our integrand depends on xy zy ´ + z ´ okay then we get we know the differential equations again differential equations do not change we're emphasizing that is only that we say the boundary conditions are different we are saying that x1 andx2 the boundaries are controlled by two functions v1 and v2 okay there is are lationship between y and z at one end and other end two functions are there they are connected to each other right.

In that case we can stab transversality conditions that are shown here those of you are interested can derive these in the same manner that we used that we followed when we get the transversality conditions we can do that note that now we have two functions y of x and z of x and they happen to be satisfying relationship at either end at x1 and x2 but using that when we do it we get transverse condition of this form because again the domain will be variable in this case x1 x2 when they have to satisfy you are not able to freely move because between y and z there is a relationship.

You cannot have x whatever you want and that is when you get these conditions okay for many other problems also you can derive transversality conditions because again let us recall the kind of generalizations that we have used in extending calculus of variations of one function in one derivative extended to many derivatives and then several functions and then we also extended to from one independent variable x to y x and y 2D x y z 3d for those also one can write transversality conditions because they are very useful in many problems pertaining to engineering physics.

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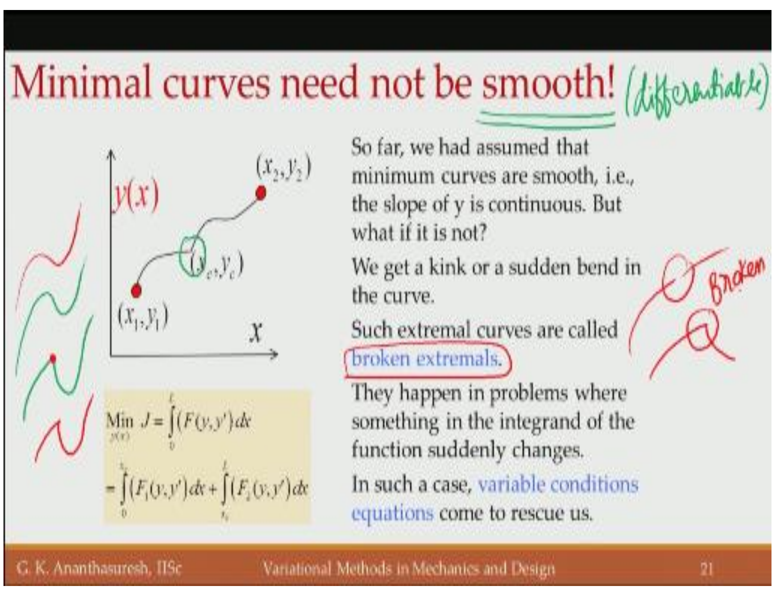

And in geometry as well and there is this notion that we can now very quickly use to what we talked about smooth functions so far whatever examples were considered the solutions I will always been smooth meaning they are differentiable what that means is that we can take derivative of the functions but what about points like this right so we have this is a smooth function this is a non-smooth function if you look at this point which I will put a big red dot there that point is not smooth life not differentiable because there is a no unique gradient at the point it is going there and then suddenly changing it is direction abruptly.

As opposed to a curve that smoothly goes like that right so for such things if a solution is of that kind will soon cite an example where that happens a very simple problem but that has a non-smooth solution how do you do that how do you deal with those in fact such extremal curves are called broken extremal broken in the sense not that a line is there and it is made into two pieces not like broken like that normally if you take a wire and that wire bends like this you say the wire is broken that it is not made it to pieces but it is just that at that point it looks like somebody broke right it is broken okay.

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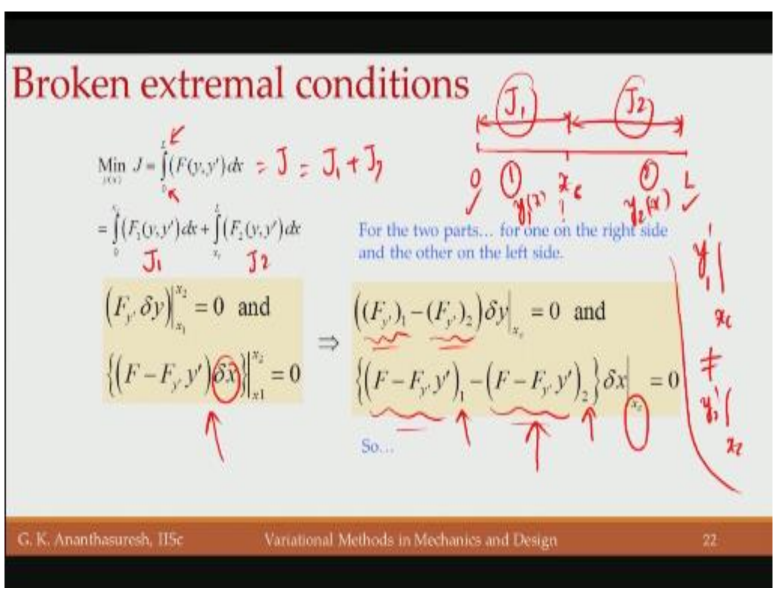

So we can use this variable n conditions to talk about this broken extremals okay or non smooth things so we look at that middle point in a interesting way so if I have a problem where I have to go from 0 to L okay I will draw the domain if I have to go from 0 to L if I suspect that at some point in between there might be a broken point meaning that a non deficient will thing let us call that xe or original J the functional we can split into J1 and J2that is from here to here J1 and from here to here we call J2 to then we write down our boundary conditions that we have discussed in the last class and today that we write these conditions.

Now what we say is that whether you take x1or x2 because what multiplies this δ x for either case for this x see this is the end point for this domain okay that is the first domain if I take y 1 curve this is beginning point is known this is not known and here beginning point is not known endpoint is known right at this interface that is x see what we say that what we say is that this thing and this thing should be equal and so it should be this one and that one so f y´ and F - Fy´ y ´ will be equal that interface they ´ y 1 ´

If I say curve on this one is y1 of x on this other domain if this is y 2 of x we are not saying that y 1 ´ is equal to y 2 ´ at x C because that is the broken extremal so we are not saying y 1 ´ evaluated at x C is not equal to y 2´ evaluated x see they are not equal because the thing is not differentiable right but these quantities are the same they are the continuity conditions ∂ f/ ∂ y ´ evaluator for y1 at xc at4y to a taxi they are the same and also this F - Fy ´ x y ´ is continuous across the interface.

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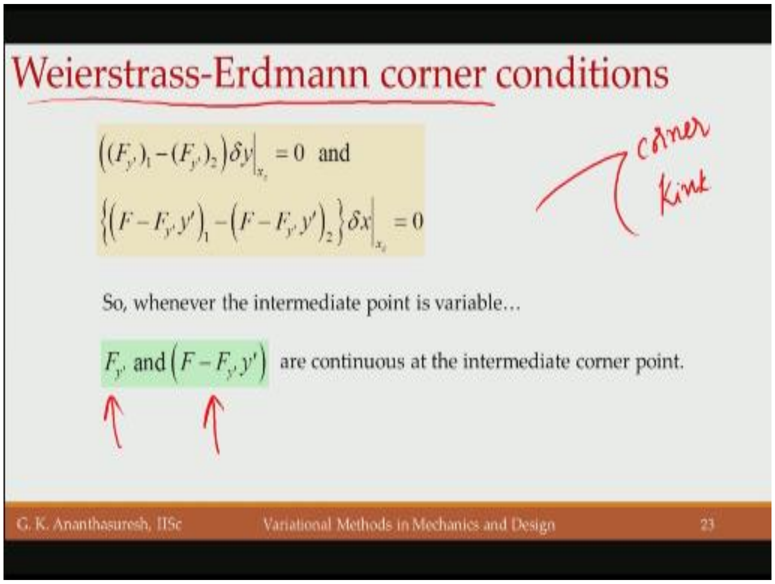

So you can see variable and condition how it comes about if you just play the domain into one and two and evaluate J as summation of two functional s okay so if you do that we get these conditions they are known as Weierstraas Cardinal conditions corner meaning again this is a corner okay like a non differentiable point okay a kink is a corner they called corner conditions okay and what is continuous of the colonel condition is what we just said Fy ´ and F- Fy ´ times y ´ okay.

Though are the continuous and they are known as worst as Weierstraas Cardinal Conditions.

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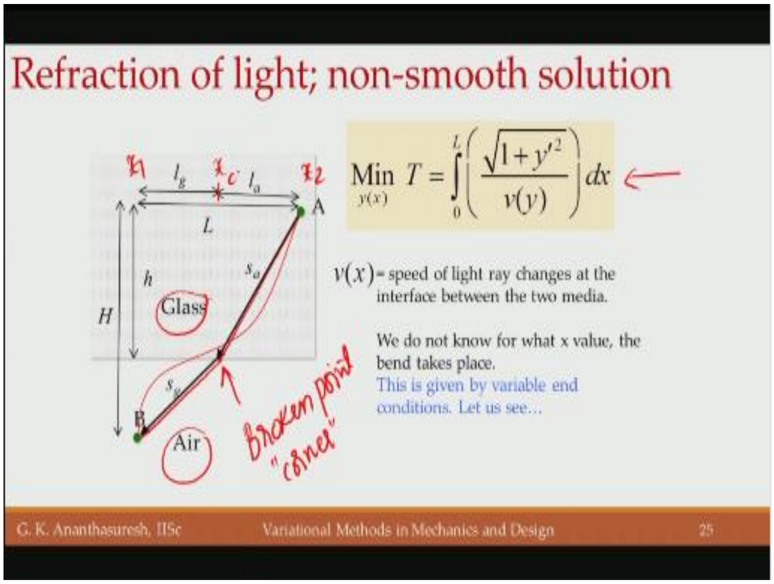

Now let us take an example an example that we had taken in the very beginning to talk about fermat conjecture about refraction of light the light he said does not take the least the distance path it takes the least time path and we had said like a funny problem if there is a dog and there is a rubber duck in the water the dog does not go like this because it cannot swim at the same speed as it can run on the swimming pool floor so what it would do is to take a path that is broken like this again if you look at refraction sort of a dog running on swimming pool floor and swimming in the swimming pool we say light rays when they are in glass vs. air their speeds are different so it would go like this and bend over to the other point right.

So this is our broken point or corner this is our broken point R corner so we can apply this corner condition so here we had minimize the time the functional remains the same differential equation means the same and has a solution being a straight line here and here remains the same except disc on how do you find this point if I say this is x1 and this is x 2 right over here we have x see the interface at that point we have this condition.

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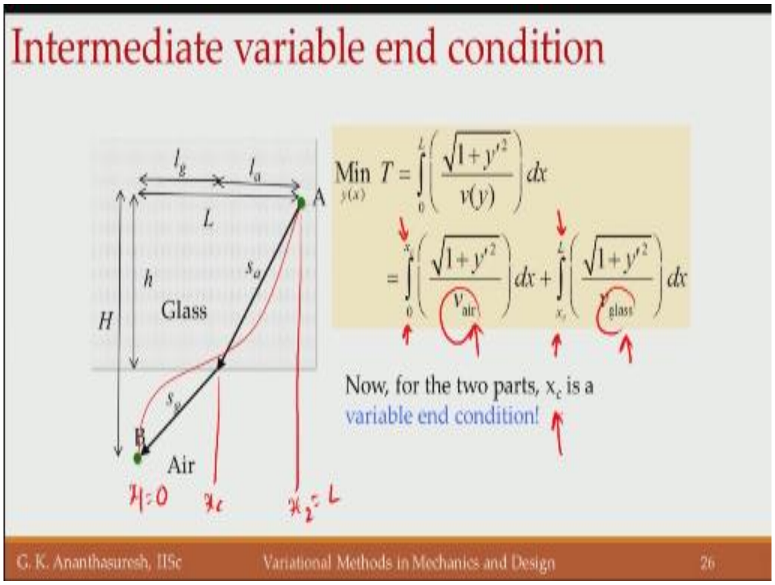

That is 0 to x 0 to x C and then x c to L so we have two things here this is in air and this is a glass so we are going from x 1 to x C in air and then XC 2 x 2 are equal to L this is x 1is equal to 0 0 to x C air okay and then XC 2 L in class the speed is different now yet minimize this sum of two functionals that is a variable end condition is this XC.

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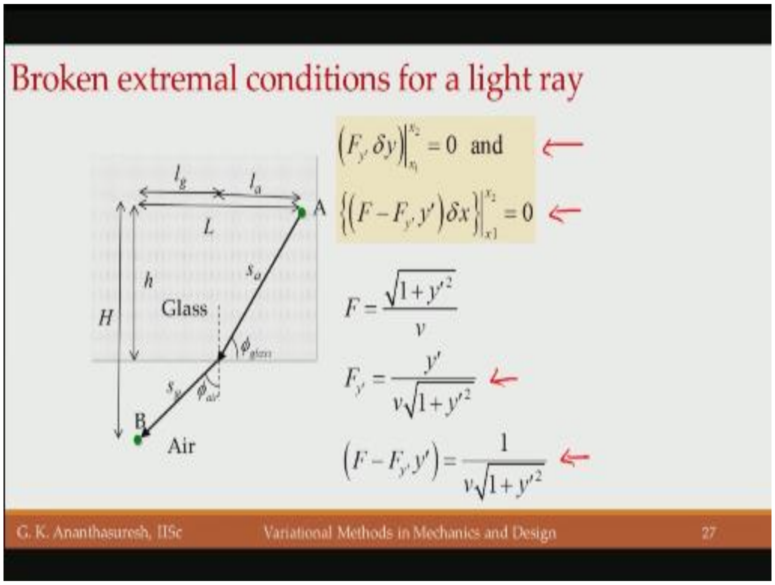

There we know what is continuous which is a kernel condition these two right so here F is 1+ $y'^{2/}$ V that is what we had in our thing over here that is a F for our problem integrand that depends on y and y´ and Fy ´ is given by this and F- Fy ´ times y ´ is given by this these two should be the same.

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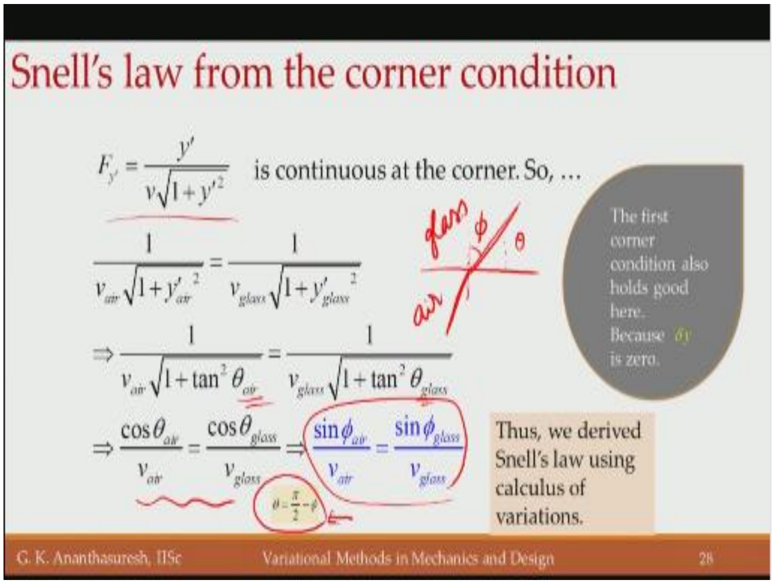

That is what we do whether the interface whether the discontinuity of this one will actually give us the what we call the Snell's law because y ´ we have VI air and this we have put and we also see from trigonometry that y primary thing about tan θ in the air tan θ in the glass right if you do this we get the relationship that sine Ø/ v air again we have converted the tan into Ø because of the way normally in refraction when you state Snell's law you measure the angle from the normal at the interface.

If we have these the interface the light ray comes like this and goes right we draw the normal and we measure the angle here whereas when you say a curve we measure the slope here so this is θ this is π that is the relationship here theta is π - Ø that is why from θ that we had we converted to Ø here we get Snell's law so the very first problem in calculus of variations involves an advanced concept of general variation to arrive at it earlier we had done this problem using one variable normal not calculus of variation normal thing where we say XC is unknown and we had done that we assume that these two are straight lines.

Now we are differentially that tells you a straight line we say variable and condition here there is a corner the light ray has a kink at the point where it is going from air to glass and that can be solved using this way stars Edmond corner conditions.

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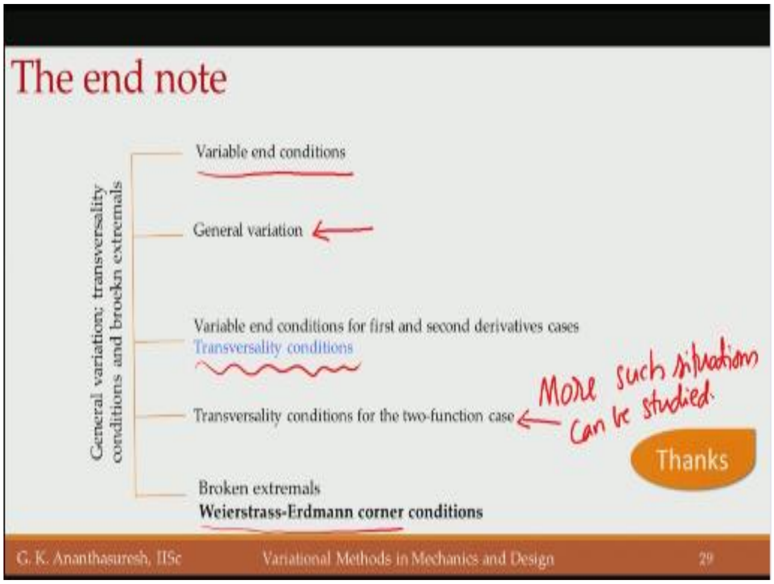

So what we discussed here is how to handle variable end conditions that dependent on this concept of general variation where the domain itself is perturbed and that led us to what are called transversality conditions and later we said that in between there could be a broken point which is a kink a non differentiable point or a corner so we have judgment corner conditions we also quickly looked at a two function case in fact more such cases can be studied.

That is just you know one thing that we have y 1y of x and z of x and y and z could be related to each other by either n such more such situations can study it okay we just considered one example as an appetizer okay so this is one important thing calculus of variations where the domain itself can be variable and we can consider that as well okay moving on will return to mechanics a little bit more later so that we can apply all that we have learnt to mechanics and design the next few lectures thank you.