Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello this is lecture 2 on the course variational methods in mechanics and design.

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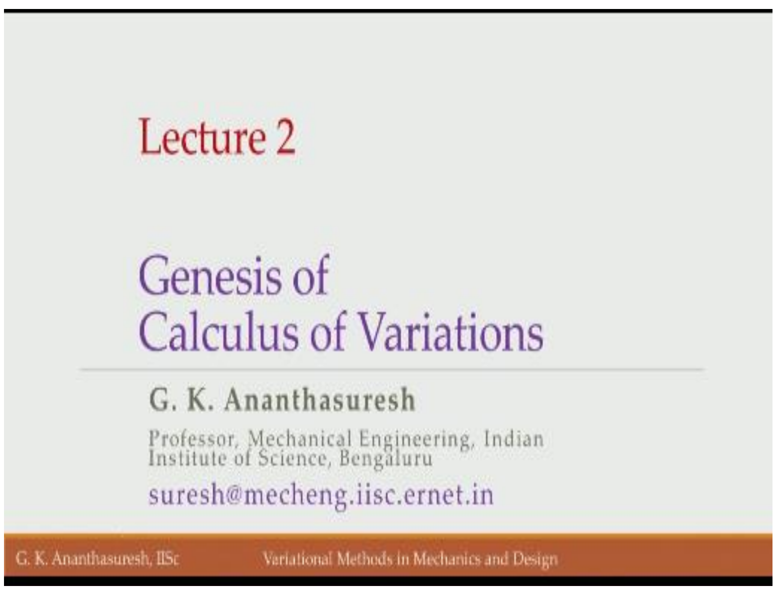

In the last lecture we discussed what variational methods can do for mechanics and design or structural optimization in particular when you say design structural design are structural optimization. Now we look at the main topic which is calculus of variations which is the basis for the variational methods. In calculus of variations there is a long history that we need to discuss that is what we will do we will talk about the genesis of calculus of variations rather how this field this mathematical feed came into existence.

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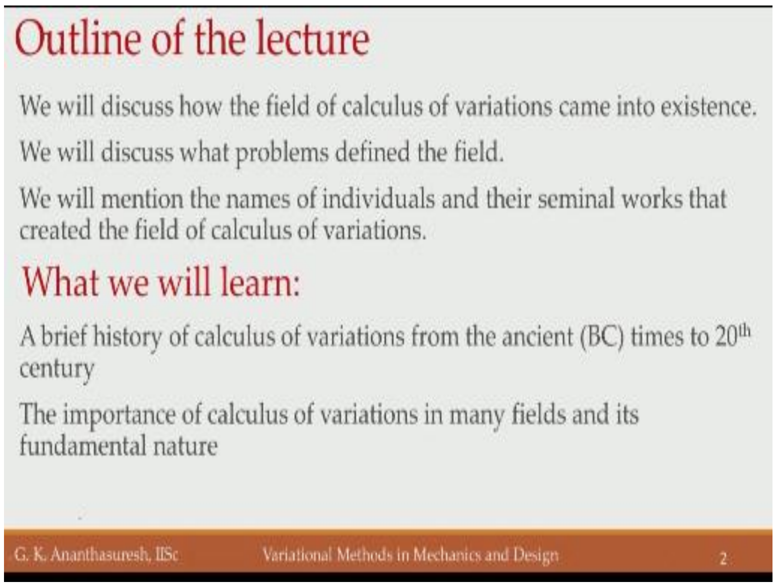

So here is the outline of today's lecture first we will discuss how this field evolved, and we will also discuss what problems define this field because usually interesting problems call for a new mathematical technique and will discuss those problems that define the field of calculus of variations, and then we will mention all the people who did important contributions to calculus of variations.

So we get a sense of history of this field what do we learn in this lecture we will learn two things one is we learn about brief history of calculus a variation starting from ancient times to 20th century and we will also look at important character variation in many fields will be discussing only in the field of mechanics and structural designer special optimization, but will mention what other fields also have applications of calculus of variations because it is a very general tool okay.

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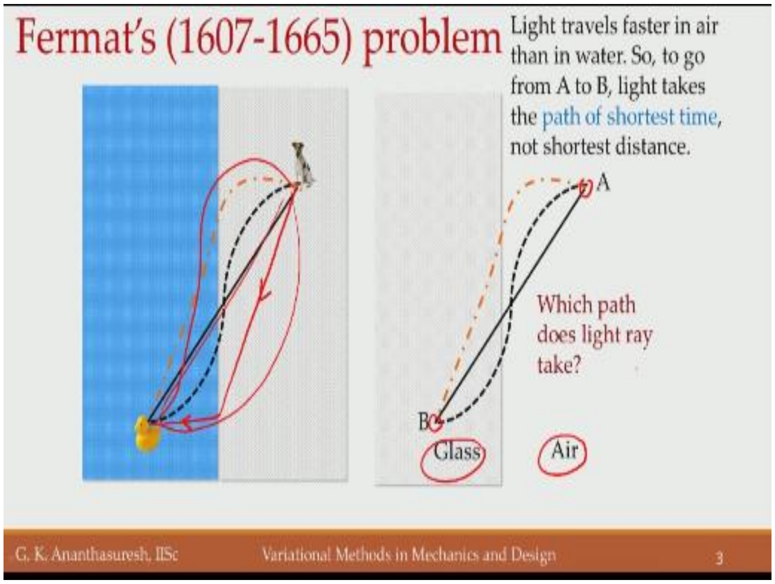

Let us consider this very old problem goes back to the time of Pierre De Fermat who lived in the17th century 16-071665 and his problem is put in a rather you know daily life like thing here let us say you are at a swimming pool and there is a dog let us say this is your dog there and then there is a rubber duck that you are thrown into the swimming pool and you ask your dog to fetch it okay.

Dog has certain speed of running on the floor the swimming pool and duck in order to fetch the duck it has to also swim in the swimming pool water there is a swimming speed and these speeds are different. Let us assume that dog can run faster on the swimming pool floor as opposed to swimming in the pool, so which path does a dog take to go from where it is now to where the duck is.

Three different paths are shown the solid line here refers to the shortest distance path right because that is the line joining the two points there is a shortage distance, but it is that the shortest time path is a question right, because the dog has to go fastest to the duck if he takes this straight line path what happens is that it will be running some distance on the swimming pool floor and some distance in water swimming.

So instead let us say it takes a path that looks like this that is dog starts from there goes here and then goes here this way the distance in the pool swimming distance is the least, but it has to run longer distance on the flow right. So that may not be the fastest again even though swimming speed which is lower and hence a distance is minimized but overall distance increases, so what is the best path okay.

And there are different curves are shown and we can draw many more curves right, so we can draw a number of curves one of them is going to be the least time path okay. This problem although it is put in the daily life situation, but actually that is what light rays also do and that is what was the observation or hypothesis of Pierre De Fermat that is what is called a Fermat problem, Fermat has lots of problems after his name this is one of them.

So light rays when they go from point A to point B which path do they take let us say here we have air and here we have glass medium so A is in air and B is in glass for it to go it has certain speed in air and certain speed in glass so which path does it take same problem as what we discussed here with a dog and a rubber duck okay. So here light ray Fermat conjecture that it takes the least time path and not least distance path okay that is a difference. Least distance path is straight lines we know it but what is the least time path okay.

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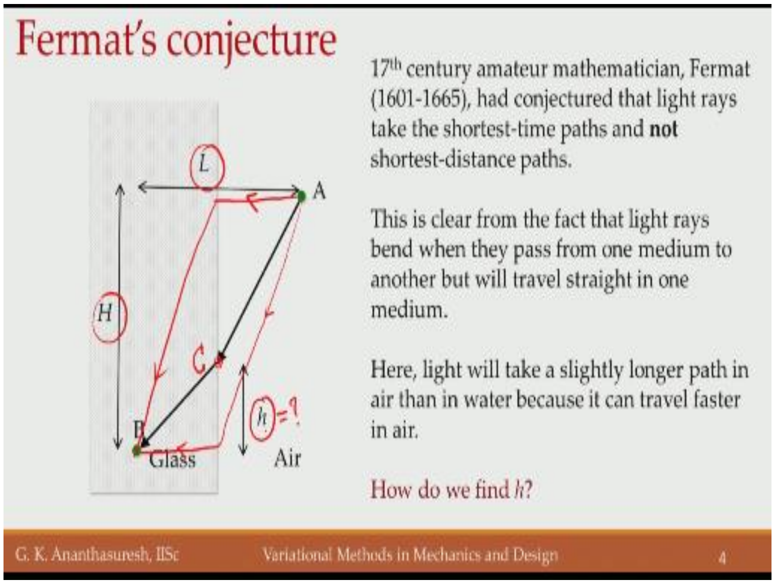

Let us pose this as an optimization problem so let us say that point A and point B are separated by the distance L horizontally and distance H vertically and let us say that it will hit the glass and air interface at this point okay, because we said that no going from here till here and then going that way that is from A all the way here and then going this way is not going to be the fastest path because overall distance has increased and definitely this is not the fastest path because if you go back to the simple analogy the swimming speed is less here compared to air then this will not be also right.

Somewhere in between, somewhere in between let us make that variable H that is what we do not know. Once we know that H this point gets located then we know A to let us say this intermediate point C and then C to B and that will be very clear. So we have to know this H how do you find that H okay, you have to find H we can pose it as an optimization problem to minimize the time taken from A to C plus time taken to go from C to B the total time will minimize with respect to H and we get the answer okay.

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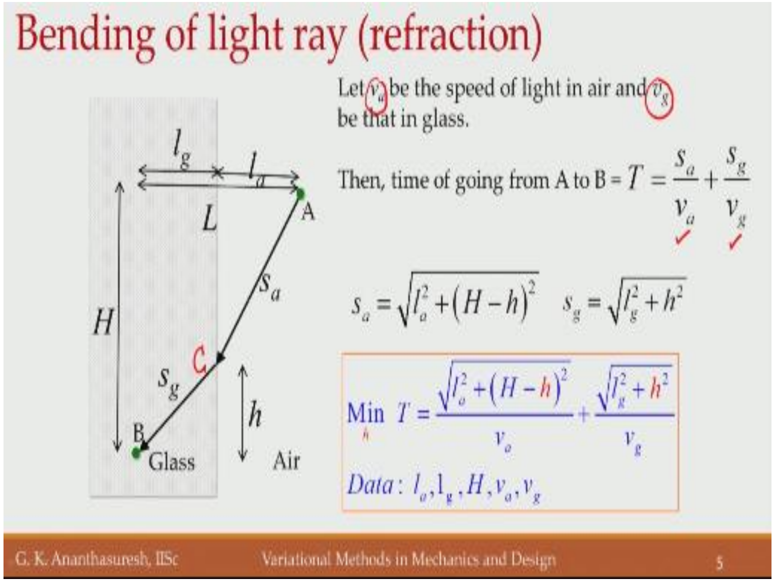

So let us pose it as a finite variable optimization problem so there is distance SA that the light ray would travel from A to this point which we called C and then C to B distances is SG. So if you look at that first let us assume that in air it has a speed of VA and in glass it has a speed of Vg okay, then the time of going for this point from A to B will be time to go from A to C which is that that Sa distance divided by the speed which is Va and then from C to B is this term Sg/ Vg and what is Sa you simply use Pythagoras theorem where we have this La square plus h the total h square that is Sa and then SG is $Lg^2 + H^2$ okay.

Because how much is La, how much the Lg is also known to us so we know how to get Sa and Sg but they do depend on this case. So in order to minimize that time T which will be Sa /Va Sa we have here / VA Sg/Vg we have this total expression or variable is H which is indicated in red that is what we are minimizing with respect to and all these are known quantities data okay.

So we have a function of H which we have to minimize with respect to H in order to find the least distance path which is segmented path A to C then C to B.

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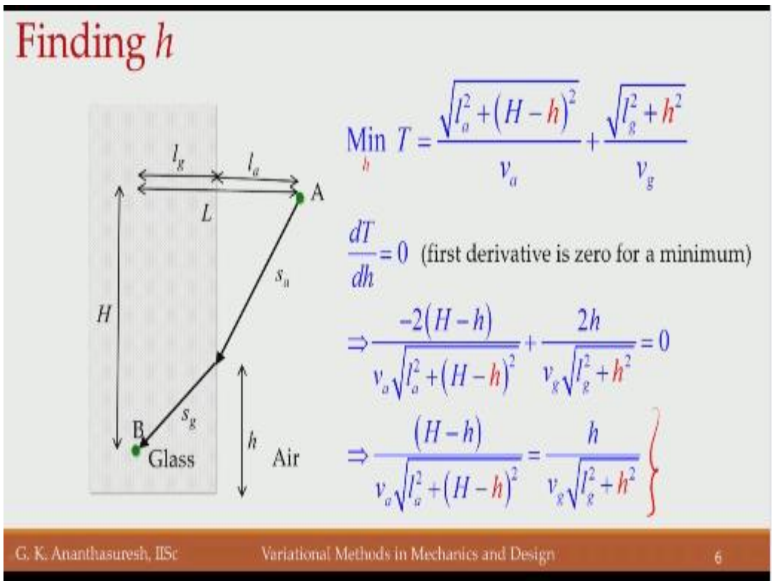

So minimize this in order to minimize the condition is that the slope or rather the first derivative DT/DH should be equal to 0 that is a condition for a minimum and a maximum but now we are taking that this is a minimization problem either case the first derivative should be equal to 0. So you take derivative of T where the expression is known take differentiate this respect to H which is done here for this first term and the second term which is shown here.

Se simplify a little bit and mood terms around so we get a an equation like this h-h / Va times square root of La^2 + h-h^2 = h/Vg square root of G^2 + H^2 okay. We still do not if you want we can solve for H here our aim is to find H we can solve for we have an equation in H now which we got by equal derivative of T with respect to H equal to zero we have the equation we can solve for H.

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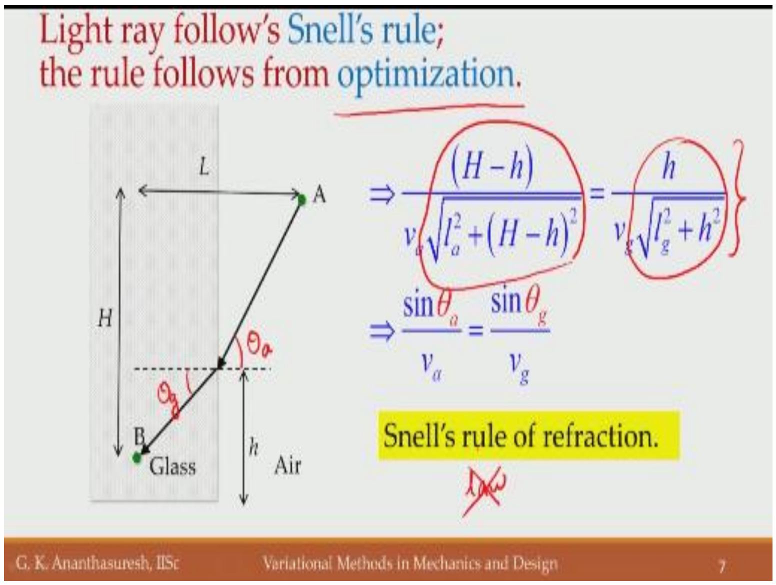

But instead let us look at what is in here which is this is the condition that we got in the last slide that is dt/dh=0 gave us this equation now we recognize that this part the one that I am circling now there is nothing but sinθa okay where sinθa if I indicate that this is the angle so sinθa would be with the normal let us say this is θa okay that would be that H - H which is this one divided by this quantity that we have this hypotenuse which we have now okay.

So if you look at that, that is sinθa and this will be θG okay sinθg this other one this part is sinθg okay. Now what we have is what we call Snell's rule of refraction sometimes people call it a law and that is not correct because we say something is a law which can if it want it can only be verified by experiment but here it is not true we have derived it by following thermos conjecture that says light rays follow the least time path.

So light ray follows the Snell's rule if you say the rule itself follows from optimization okay. So in this problem we have derived what is normally called Snell's rule of refraction by assuming that light rays take the least time path okay.

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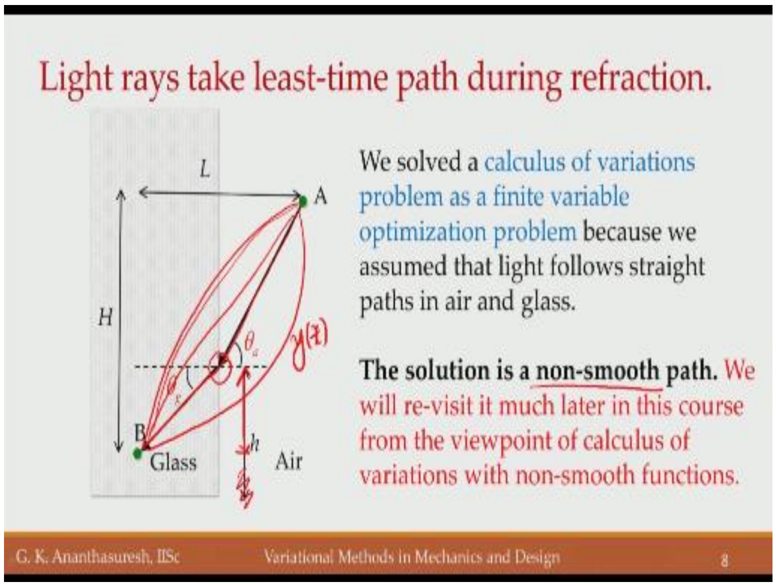

Now if you look at this problem we originally said that there can be lot of different curves right so we have a number of curves we wanted to get the best curve that minimize the time but we have used it as a one variable problem and define this H which is from here to here okay not from here to here from here to here and made it a one variable optimization problem but a solution and recognize that as Snell's rule,

But later on much later in this course we will revisit this problem to interpret it as if it is a continuous curve how that leads to this non differentiable point or what we call non-smooth path. So the very first problem in calculus of variations has a non smooth curve as a solution this is a curve only thing is it is two straight lines that are piece wise continuous at this point there is a non differentiable point that is derivative is not uniquely defined at that point there is a kick in the path.

And that can be handled in the framework of calculus of variations where we say there is an unknown function if I call this Y(x) that unknown function itself can be computed.

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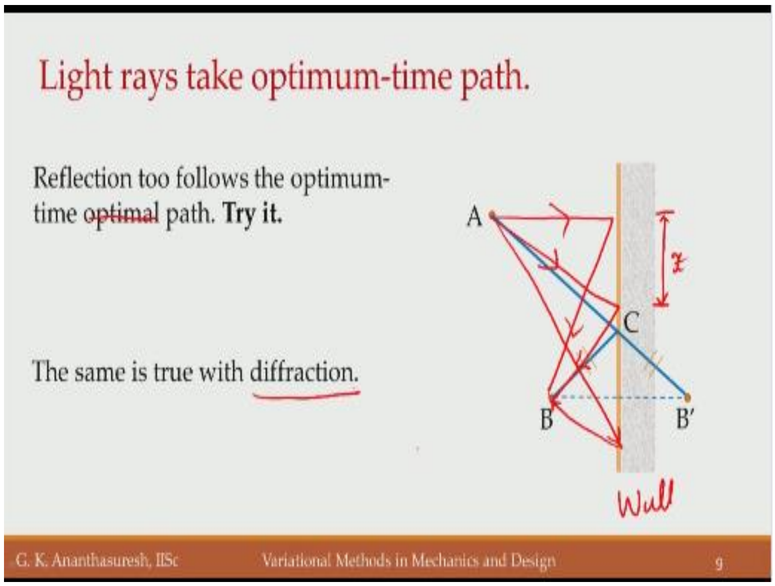

If while talking about light rays let us also talk about reflection let us say we have a point A and point B and there is a wall here and we need to go from A to B but in between we must touch the wall once okay, that is again there are several paths I can go like this at our I can go like this and several other ways that I can if I want I can go here and then go this way and so forth there are several paths.

Which one do you take which path do you take the same question as what we asked in the case of refraction we can talk about that in the case of reflection also. If you actually work out this problem you will find that the light rays even in the case of reflection they try to find the optimum time path okay optimum time I do not is optimal again optimum time path you can try this problem by assuming again let us say this distance now where it hits if I call that some x okay.

Then it will go from here to there because this distance is A and B are given with respect to the wall so if we say this is X you can solve the problem and get the answer okay. But this is true with diffraction also not this reflection refraction, but also with diffraction okay if you work out the problem minimize time in this case of reflection time and distance are the same because you are in the same medium where the speed is the same right.

And the optical theory of diffraction or geometric theatre diffraction also uses this optimum time path to get solutions okay, because light rays try to minimize their term they are always in a hurry even though they are the fastest things that can let move but they try to minimize the time to go from one place to other place okay it actually fits with the nature of light rays right because they are the one that can go as fast as or faster than anything else okay.

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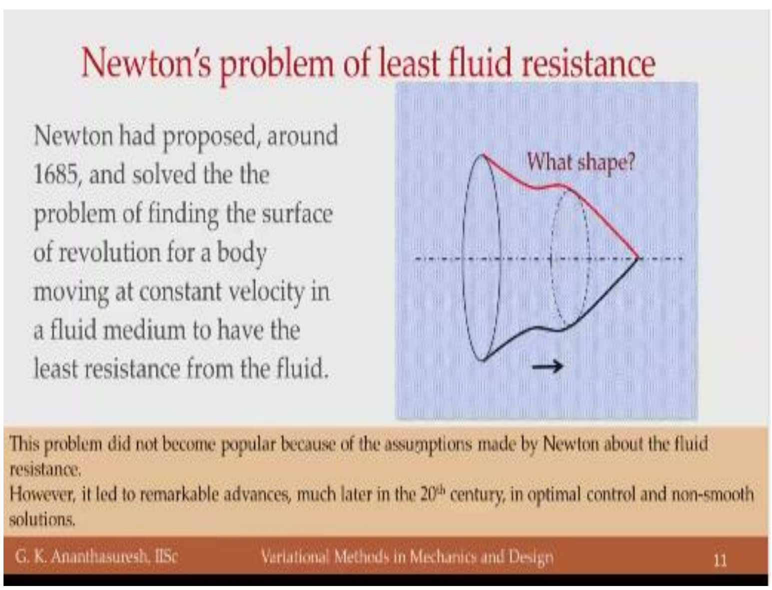

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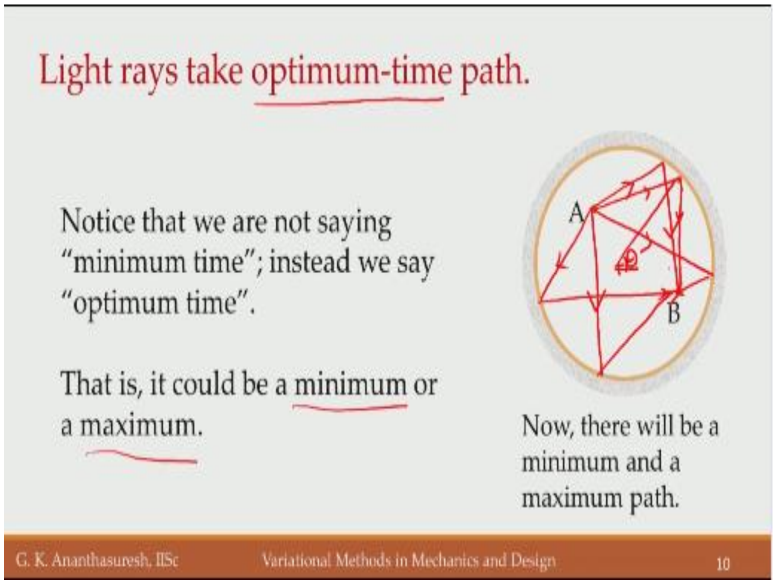

All right okay, but then notice that I am saying now optimum time rather than saying minimum time okay, the reason is that when you say optimum time it can be minimum time or maximum time that can be a minimum or a maximum locally right, and in the vicinity of those can be minimum or a maximum.

Sometimes let us say we take this example of a circular mirror right there are two points A and B now we say the same thing I want to go from A to B, but I have to touch the mirror wall okay I basically support reflection to go from there to here there are several paths right I can go from here touch the mirror here and go there or go like that and go this way or go like this and come back to B there are several paths of all of these which one will you take it turns out if you solve this problem that is you take let us say in order to locate a point you take this as angle θ and then minimize the time taken with respect to θ as θ varies that is where the light will fall there and come here right.

So if you vary that angle θ and get the time to go from here to here to here then you minimize it you will find that not only minimize there is also maximum, so light rays actually take optimum time path so that was the correction made to Karma's conjecture they do not take the least time path always they take the optimum time pass sometimes it is maximum, sometime thesis minimum that is something to bear in mind in this problem okay.

So historically this is one of the problems where you have a non-smooth solution but is a calculus of variations problem.

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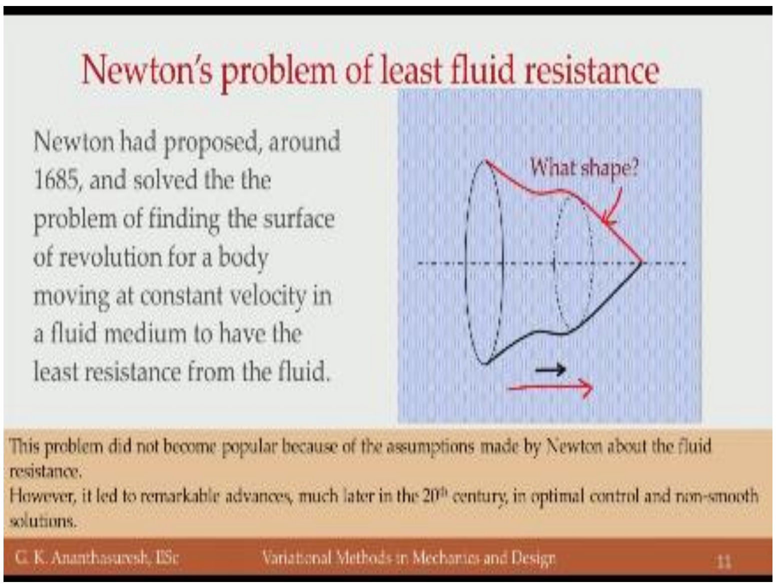

Newton who lived after firma he had a problem of his own that fits the calculus of variations. He wanted to find the optimum shape what is shown here this shape okay, which is surface of revolution which is for a body that is moving in a fluid what is what shape will minimize the drag on it, minimize the resistance of fluid on it as it is moving in the direction that is shown here if it is moving like this what particular shape will have the least drag okay.

But this problem has not become very popular in the history of calculus of variations because Newton apparently had made some mistakes or some assumptions which were not valid later on but he had thought of this many other people might have thought of problems like this where the unknown is a function that is what we had discussed last time. The unknown itself is a function okay.

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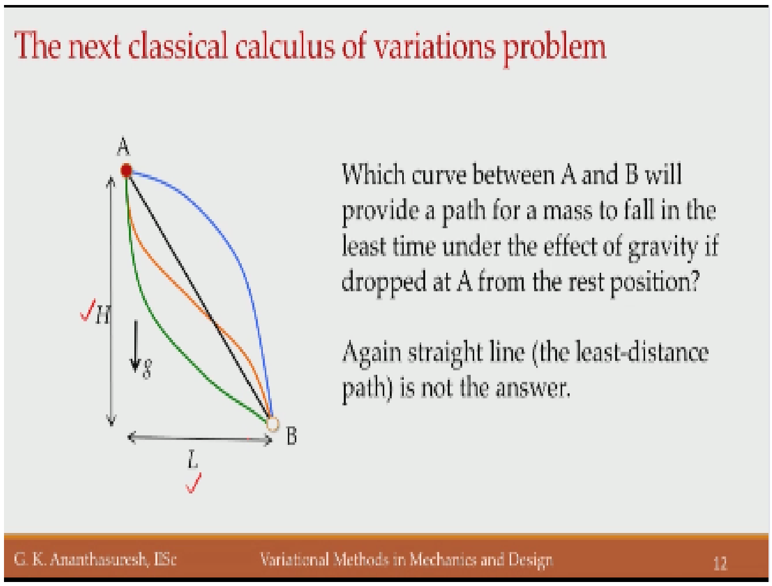

So this problem is a classic problem in calculus of variations which is again points A and B go to from A and B different conditions here under the influence of gravity different curves are shown there is a straight line joining A and B and the blue curve and red curve another black curve or a green curve where all these curves and let us say a bead is starting there and then due to gravity it falls down, right in various ways right, among all these curves which one will make this bead fall from A to be separated vertically by distance H separate horizontally by distance L again the least time, okay.

Which path among these if you say straight line like inclined plane that is a least distance path but that is not least time path okay, this is a classical problem that started calculus of variations historically, okay.

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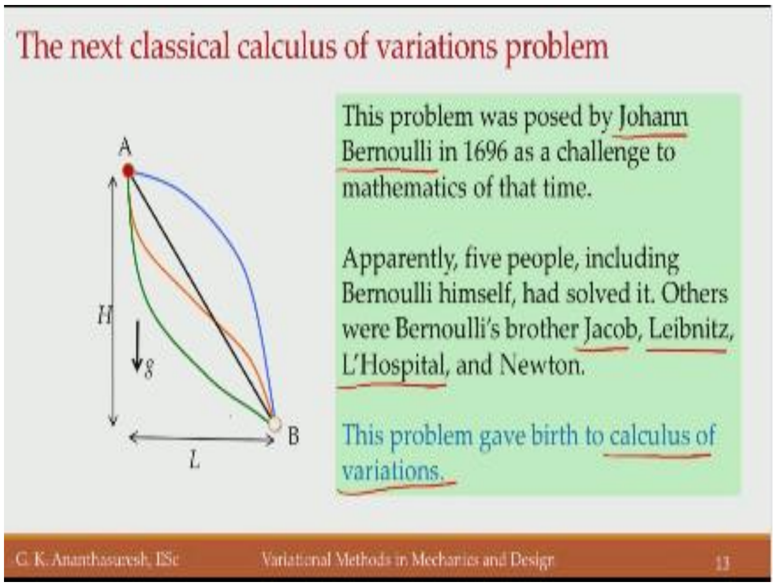

So here we have to note some historical details one is that this problem was posed by Johann Bernoulli, there are four different Bernoulli's is one of them. In 1696 he poses problem in a mathematics journal and gave time till Christmas for everybody to respond with answers and at the time he had solved it himself and he also had a brother named Bernoulli Jacob elderly he also had solved it and Leibniz who was his close associate she had also solved it and L'Hospital with all whose name you must be familiar in the case of limits finding limits these including himself Jacob Bernoulli, Leibnitz, L'Hospital four people had solved.

And then he was actually waiting Johann Bernoulli was waiting for the solution of Newton but Newton had not responded we send that problem again to him without writing a personal letter because there were rivals in mathematics and mechanics and other things. So Newton saw that he got a little irritated apparently he reached home at 2'o clock in the night and his knees gave him this problem and he solved it then only went to bed and then he sent it to Johann Bernoulli the solution without even signing the name, right.

Then Johann Bernoulli he saw that solution it was obviously correct, another solution this is a fifth solution to the problem and he immediately that as Newton's because of the elegance of the solution that Newton had sent so the famous code that Johann Bernoulli made even though Newton why his rival is to say that the strike of a lion can be told by looking at its paw the imprint of the paw on that, so that means that Newton stamp was there in the solution. Anyway five people solving a problem gave birth to the field of calculus of variations, okay.

Where the unknown curve in this case the straight line and a king straight line is not a solution a continuous curve is a solution that is why this problem has become a classical problem in calculus of variations.

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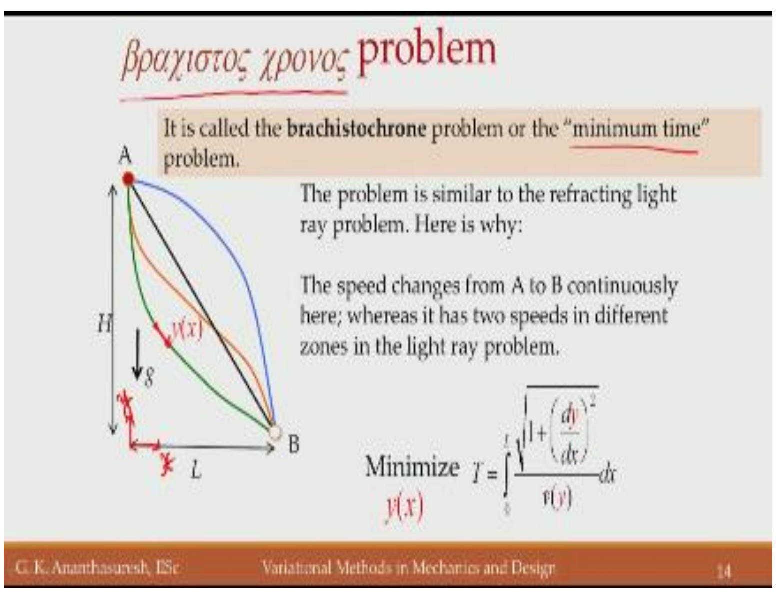

And that goes by this name if we can read it okay, that is basically Greek which says brackish to Kronos that in English is brachistochrome problem chrone means time brachisto is a root for least or minimum, minimum time problem. So Farman problem also was been minimum type problem, this minimum type problem this is more famous because the solution of these a continuous curve people read determine what that curve is all these five people that we talked about to the previous slide had actually solved it.

This problem with point A to B the speed changes continuously unlike in the refraction problem refraction problem there are only two speeds in air and glass our swimming pool floor or in the pool itself right, whereas here it is a falling body their acceleration only the vertical direction nothing in the horizontal direction right, is only guided by this wire or a path. So here the speed of the bead at different points is continually changing so you have to derive an expression for it and post the problem unlike the refraction problem that we looked at okay.

There it looks like this so when you actually get the time it is an integral now you take a small piece here and call it dx and then say how much will be the fall here and because there will be some speed and you can get the speed of that at that moment and then see how much distance you get the time we will get a small time to go from this point to this point and then integrate it from 0 to l if you take your access here like this, this is x and this is y then y(x) is our function in terms of y(x) it turns out to be this integral 0 to l $\sqrt{1+(dy/dx)^2}/ v(y)$ which we have to derive from conservation of energy.

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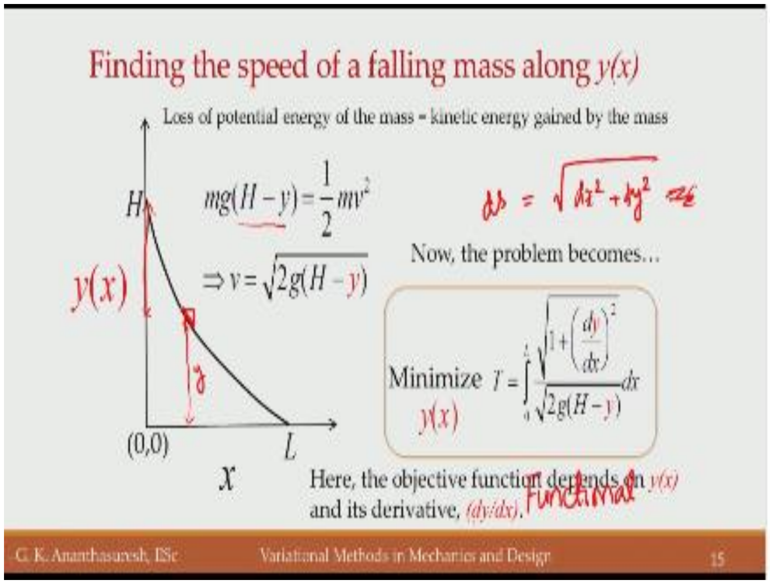

So loss of potential energy for the mass equal to kinetic energy gained by the mass based on that you can get the speed so mg(H-y) let us say it is at this point that will be the y here okay, that is y H-y is the height through which the bead has fallen that is from here to here right, that you be at H-y mg that the potential energy lost this kinetic energy gain peaking at the speed v, okay. So we get the speed v at that point now the problem becomes where the v was there we substitute this √2g(H-y).

So this one basically comes because the small dx ds that we take there that using Pythagoras theorem this is dx and then dy that is ds will be $\sqrt{dx^2+dy^2}$ that is our ds okay, that isour ds, okay and that ds is put here now what we have done is where we have taken dx out okay, of this that becomes $1+dy/dx^2dx$ that is why we get this, so this integral later we will know what this is called, it is called a functional it is not a function it is a functional, okay.

If you minimize this with respect to y(x) in the refraction problem we had only one variable called H now we have a function y(x) itself as an unknown that is a nature of calculation problem this is the classical brachistochrome problem, okay. And it depends as you can see depends on the function y(x) and it's derivative dy by dx.

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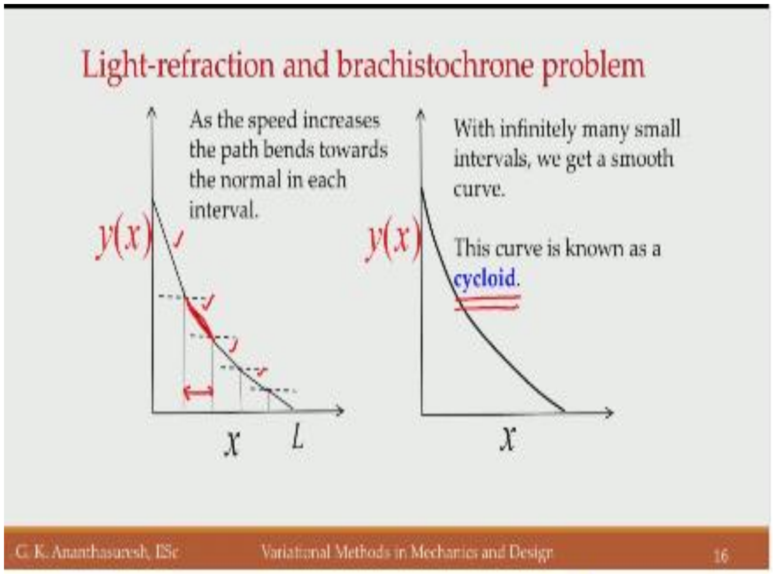

Now if you want we can actually solve the light refraction problem brachistochrone problem in a similar way right, if you think of this discretization here as the speed increases let us say we take this interval very small thinking that in that small time short time this path is very small so we can assume that the speed is constant there we can get the speed changing speed there, there, there, there and so forth we can actually pose it as a what we did light refraction problem and sum it up, right .

So if you do all that minimizes respect to all these different heights you can solve it using finite variable optimization problem and that is how Euler ended up solving as we will discuss later. But anyway the solution for this problem that we post which is to minimize that integral with respect to y(x) the solution is a well-known curve well known at the time also called cycloid.

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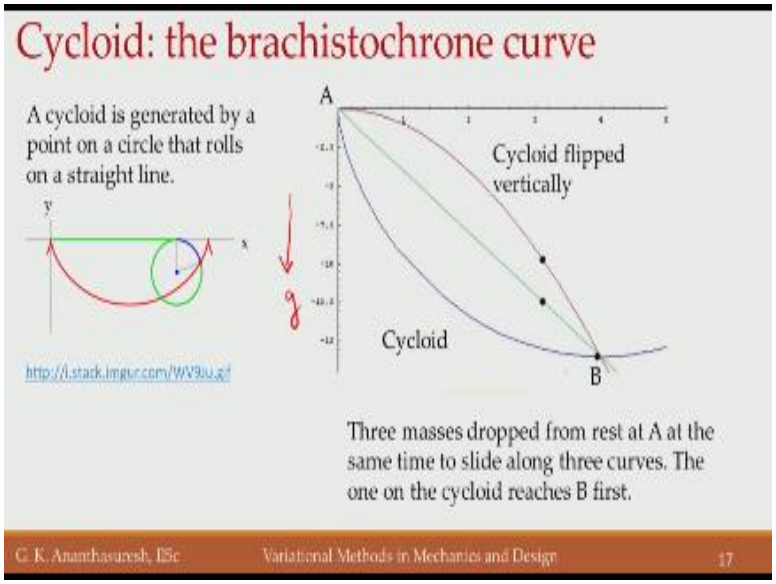

So cycloid is a special curve which as we can see the animation when you have a disk rolling a point on the disk will trace a path which is what we are seeing here a point on the disk as it is rolling it traces this red curve that is a cycloid. If you take part of that if this is our point A, okay and this is point this curve is the cycloid okay, so the cycloid we have here the other cycloid which is vertically flipped and there is this green line which is line joining points A and B.

Now if all these three points that is this, okay so we have this point and this point and this point all three points let us say start at the same time at A and there is gravity here so let me get the pen, let us say there is gravity this way for point A to point B, then B the B that followed the cycloid curve is already there already close to a point B where are these two are lagging behind, right.

In this particular instance these two are the same you know horizontal distance need not be this takes different time this takes different time, but the question is that cycloid are answer is that cycloid use the least time path from A to B okay, so that is the brachistochrone problem for the solution is a cycloid.

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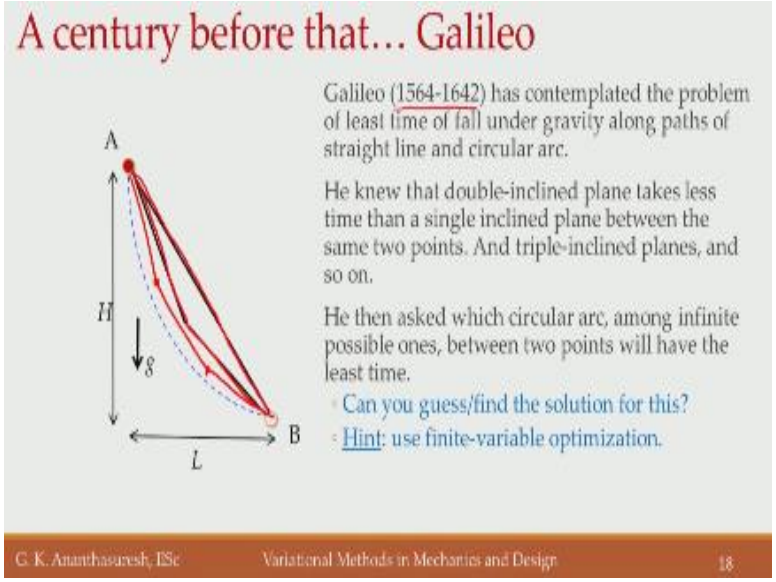

Essentially before that Galileo also had thought of problems of this kind all of you might have heard that Galileo was doing lot of experiments with inclined planes because realize that gravity is trying to pull the things down and he knew that there is a force you are trying to figure out what is effect of the force. So he was trying to do this drop masses from leaning tower of Pisa and other things and inclined planes you realize that if he has one inclined plane such as this one from A to B when it reaches be it will reach with certain speed which you can verify if it goes horizontally how far it goes, right.

Instead he realized that if he puts two inclined planes like that it takes less time that is what is experiment showed and then you realize that if he takes three inclined planes, okay there is a segment here segment here, right then it takes even less time. Then he thought how about a circular arc, if I if you put a circular arc from A to B that also reduce the time so basically he saw the circular arc has a limit of all these several segmented multi inclined plates, okay.

Then he started asking the question, A to B if we have points A and B between A and B infinite number of circulars can be drawn, among which one of them is going to have the least time for the ball to reach from A to B following that path that is a problem he posed. Some people say that Galileo was mistaken because he thought circular arc is the brachistochrome curve that is the least the curve that minimize a time.

But actually it is not true if you read Galileo's book to new Sciences where he actually probes booth circular arc among all possible circulars are that pass through two points he gives a solution as one particular circular arc, he was not thinking about any other curve okay, which is what Bernoulli later realized you can see when Galileo was right. He was in the16th century almost essentially before Newton and in fact you allude before Farman also okay, or at least they were contemporaries for some time.

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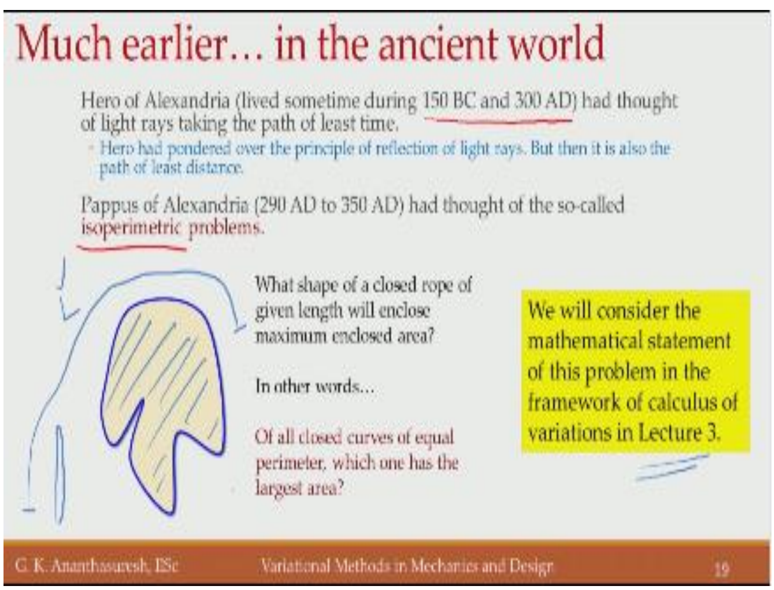

Much earlier now let us we are going back to the ancient world hero Alexandria who lived in BC1500 BC to 300 AD we were uncertain exact when he lived right, he had thought of this lightness problem so it was not something that Farman was the first one to think about he also had thought about what like trees do, do they take the least distance path at least path but then he was not able to you know do experiments are concluded one way or the other.

But other people who thought about calculus variation of a problem one of them is called the isoperimetric problem, okay. Let us say that you are given a piece of string okay, let us say I will take so let us say that I am given a piece of string like that okay, and I need to arrange that into a closed loop such as the one that is shown here the length of this from here to here let us say the l is given to you.

Now we need to arrange it in a closed loop fashion so that the area enclosed by it is the largest okay, the smallest is easy basically you can take this and loop it around this so it will be they are very close to get 0 area right. Now you have to include such a way that to enclose the maximum area that is called papas of Alexandria a problem or isoperimetric problem or queen ido problem there are different stories for the same thing every you know ancient culture there may be story around this type of problem where you are given a certain rope or a string of certain length and you need to put it to a particular ship to enclose the maximum area, okay.

We will consider a mathematical shape into this problem later on actually next lecture but these also calculate variation problem there you have to find this function that define the shape of this closed curve so that it gives you the maximum enclosed area.