Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello we continue with our discussion of functionals involving two variables that is independent variables like x and y meaning that we are we are looking at problems where the domain of the functions that we are trying to find is actually two dimensions we had discussed at length about single variable functions such as y of x are several functions y1 y2 yn now we are discussing functions that are the depend on two variables x and y so make it z as a function of x and y so let us continue with that and solved couple of problems so we appreciate how to deal with two dimensions and also an example for three dimensions which we had discussed in the last lecture.

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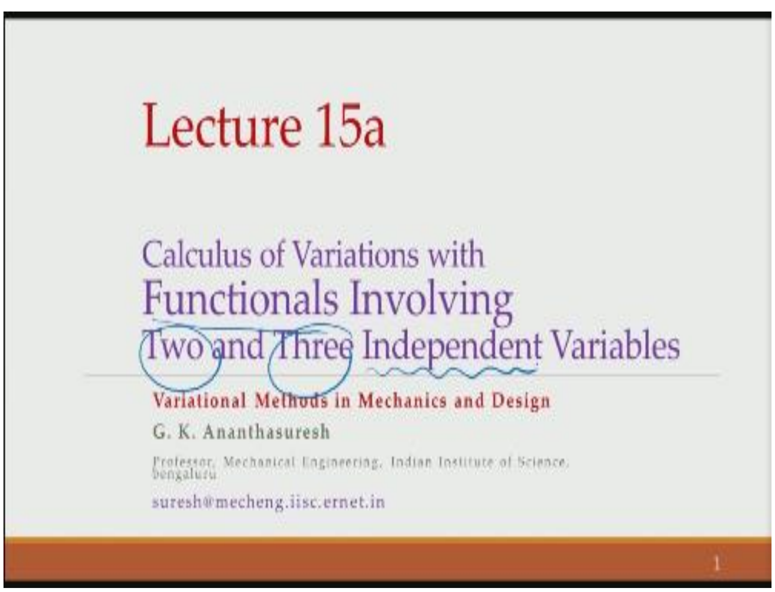

So looking at this we are looking at functions or functional involving two and three independent variable that is what we call them like xyz which are the ones that form the domain of the functions we are trying to find.

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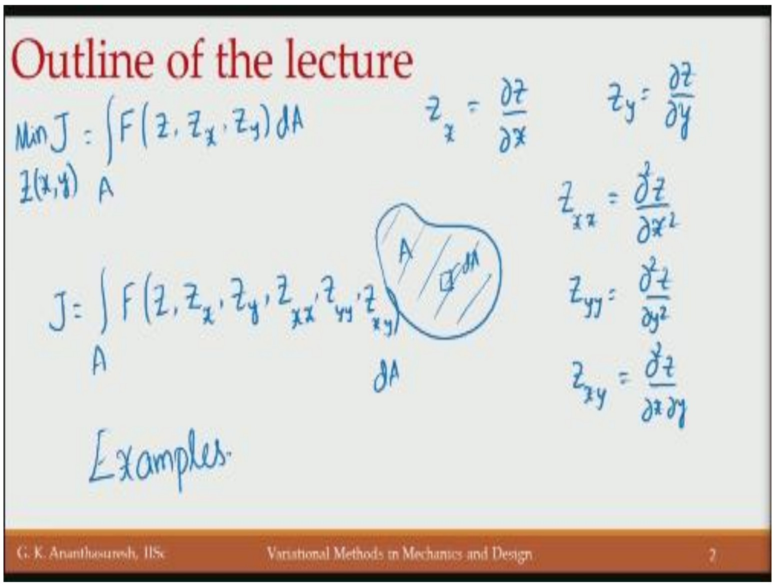

So what we will discuss in this lecture is what we had already discussed that integrand is a function of z and zx and z y in our notation zx if were call is ∂z /∂x and then zy is ∂z /∂y so we have a functional that depends on x and y which we write like a dA that is if I take an arbitrary domain like this somewhere I take a small area which I am denoting as dA and then I integrate over the entire area that is what this means that is our J the functional will you minimize with respect to if right we are minimizing with respect to z x and 1 okay.

That is our problem we discussed that and the boundary conditions we discussed it in the last lecture we will do a problem that uses that and will also consider a case where our functional again another J which depends not only on z x and z y but also depends on z_{xx} based on the same notation you can imagine what z_{xx} stands for it stands for second derivative of z with respect to x twice so we can also write z_{yy} that is $\partial^2 z/\partial y^2$ and then we can also do a mixed derivative z_{xy} that is$\partial^2 z/\partial x$ and ∂y or $\partial y/\partial x$ right.

So we also look at this after that anything else you should be able to do it on your own okay and in fact we do not encounter problems that involve more than second derivative in engineering and normally physics applications okay so this is what we will do and will do examples of both of these cases in geometry and mechanics and that also translates to design or structural optimization okay.

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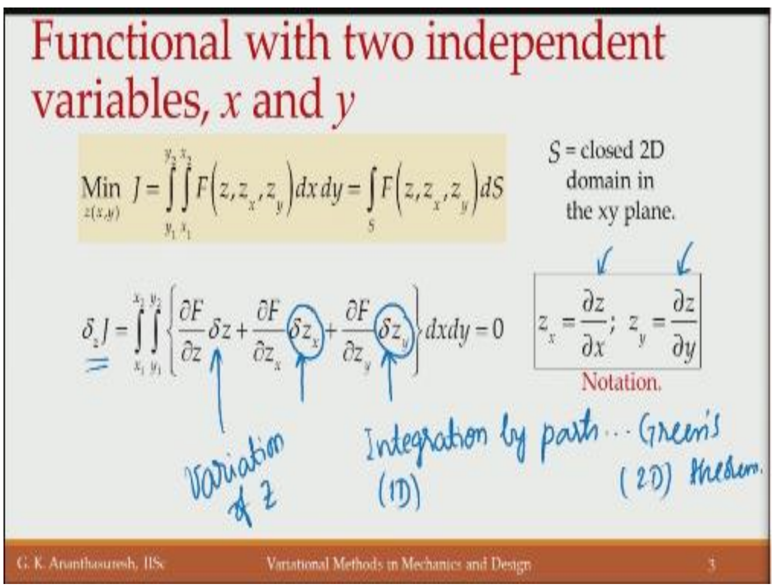

So just to recall when we have the functional that depends on z x and z y again the notation is given here z sub x sets of why we take the variation I am just reminding so that you can see how we do when there are higher derivatives at xx zy yz xy we take the go to variation like this equate to 0 and then we try to eliminate not try to in fact we eliminate δ z x δ z why because δz is the variation right so this is what we called as variation of z small perturbation on that but here we have partial derivative of various respect to x partial derivative of various respect to y we need to do integration by parts we do integration by parts anyway that is in two dimensions that is in 1d in 2d it basically means applying greens theorem okay.

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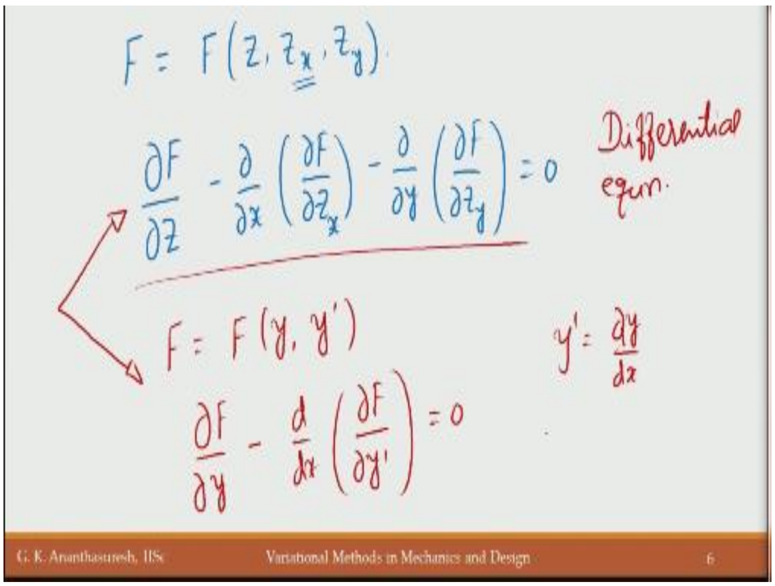

Then that gives us the differential equation as well as the boundary conditions so how does the differential equation look ay like again let us remind ourselves that integrand here is a function of z z sub x z sub y okay so this gives us a differential equation that is $\partial f / \partial z$ - $\partial / \partial x$ ($\partial f / \partial z$) sub x that is wherever that exists only that partial derivative you take and then take partial root of that with respect to x $\partial / \partial y$ of $\partial f / \partial z$ y equal to 0 so this is the differential equation this is the differential equation okay if you actually compare this with when the functional we had only one variable like why and we had y´ right.

When we had that the differential equation what we call Euler- Lagrange equation was like this row f $/\partial$ y´´ equal to 0 again if you recall y´ is∂ y 1 just go by ∂ x or dy/dx that that was y ´and you can see the similarity between these two okay but we know we also now to derive them so differential equation is easy to write or what we call Euler-Lagrange equation the boundary condition requires little work.

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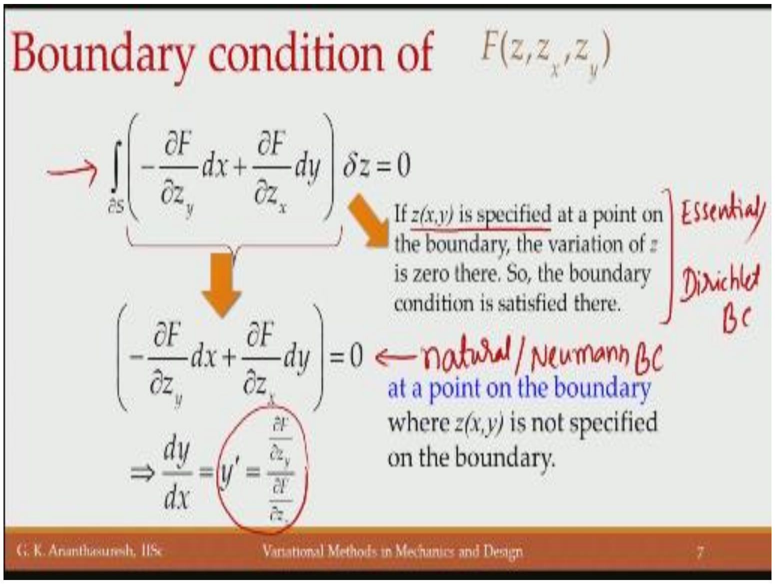

Which we did last time and through integration by part which is greens theorem applying that this is a little trick to eliminate the δz sub xδ z sub y we got a boundary condition which is shown over here right so whenever δz is 0 that is that is specified then boundary condition is satisfied when it is not specified what we call this one is the Dirichlet boundary condition or what we call essential boundary condition meaning that Dirichlet are essential directly or essential boundary condition where z is specified at a point on the boundary or a portion of the boundary then there that does add a zero the condition is satisfied.

In other place where it is not specified we get this is called the natural boundary condition are no I'm and boundary condition this is boundary condition BC boundary condition and that here means this okay so that is what we had derived last time.

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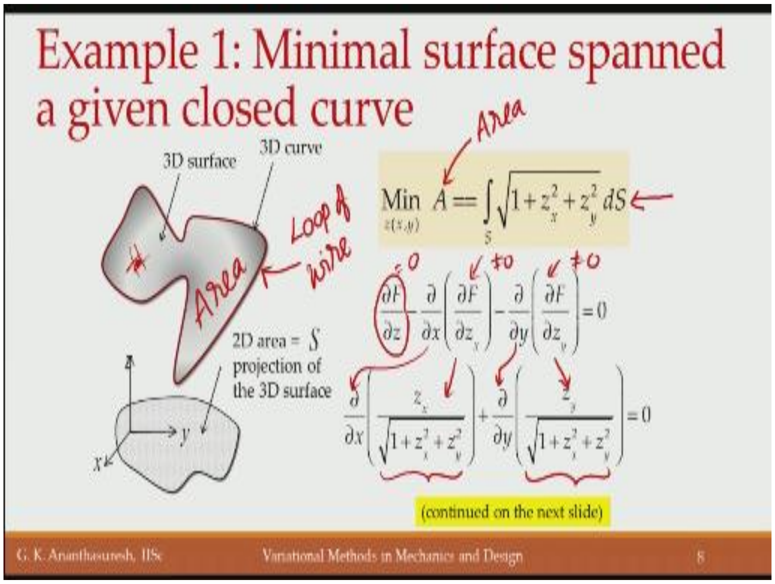

Now let us actually take an example that illustrates what we have just done okay where z should be integrand should be a function of z z sub x that sub y and this is the problem that soap films solve naturally right if you take a loop of wire such as the one that is here okay that is just a loop of wire you should try this experiment at home and all you need for that is basically a wire that is easily bent.

So you take it and make it a loop and in three dimensions you do not does need to be two dimensions make it into 3d dip it into soap water and take it out then you will see a surface such as what we see here a surface forms right that surface that the soap film forms is a minimal surface meaning that you will be minimizing the area here this is the area of the surface whatever area that we see he is he here that area will be minimized and that area is given by the expression that is shown here square root of $1 + z x^2 + zy^2$ essentially.

This is to say that we have if I take a little patch here little patch and we are to integrate over the entire surface the little patch is given/this one + del x^2 + zy^2 okay so when we have this expression of minimizing area you can see that integrand now depends on z sub x z sub x okay easy does not appear here but does not matter we can still apply it is just that in this problem this is equal to 0 whereas this and this are not equal to 0 right those things in fact how much it is we have here ∂ /∂ x of this part and ∂ /∂ y of this part so this is ∂ /∂ x over there and over∂ y what is inside is right there okay.

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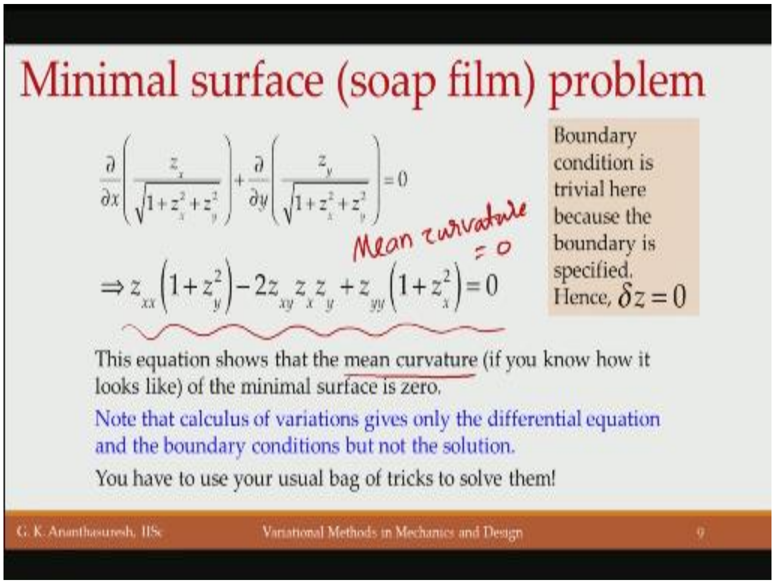

That is what it is if you continue the differentiation you get a an expression that looks like this okay this expression basically defines that minimal surface other surface found by soap films any loop of wire you take there will be minimal surface you will get that and so films instantly solve this differential equation okay how do they do it they do not actually know calculus of variations as we know it but what they try to do is minimize the surface area to minimize the surface energy that they have so films always minimize surface energy and hence the surface area.

So this geometric leaf interpret means that the mean curvature is zero so what we have here is simply says that at every point on that minimal surface you have mean curvature equal to 0 that is something we need to remember and again a calculus of variations only gives you differential equation boundary conditions but it will not tell you how to solve the differential equation for that we have to use our usual techniques.

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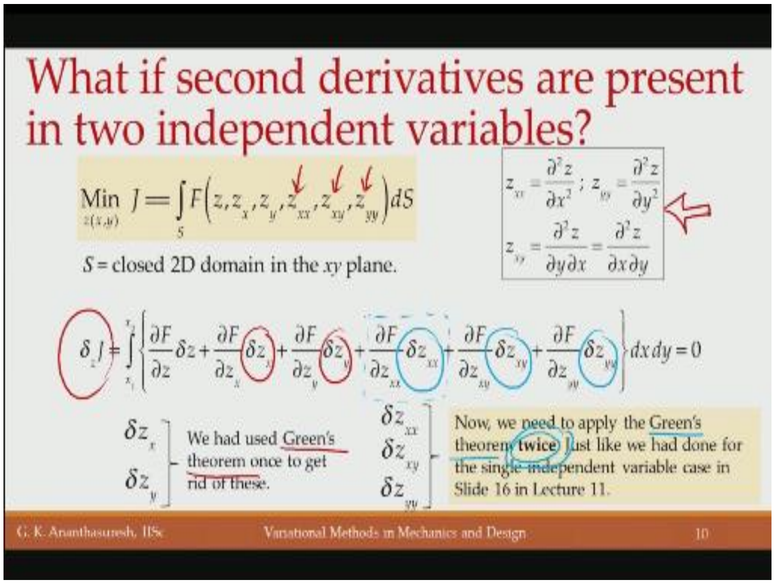

Whether it is one famous problem for this now before we take up a mechanic's example where the integrand is depend onto independence such as xy in the domain let us actually consider a case where we have now these second derivatives also z sub xx z sub xy z sub y by which we had already defined earlier today okay now in order to arrive at the differential equation boundary conditions again take the variation respect to z we get everything and to get rid of these two we apply greens theorem and get to δ z now if you want to get rid of these other terms okay so that is $\delta\, z_{xx}\ \delta\ z_{xy}\ \delta\, z_{yy}$ you have to apply greens theorem twice okay.

That is all there is to it systematically and we have to use the same trick where this whole quantity that we have we represent as a term and subtracting another term from it as we have here.

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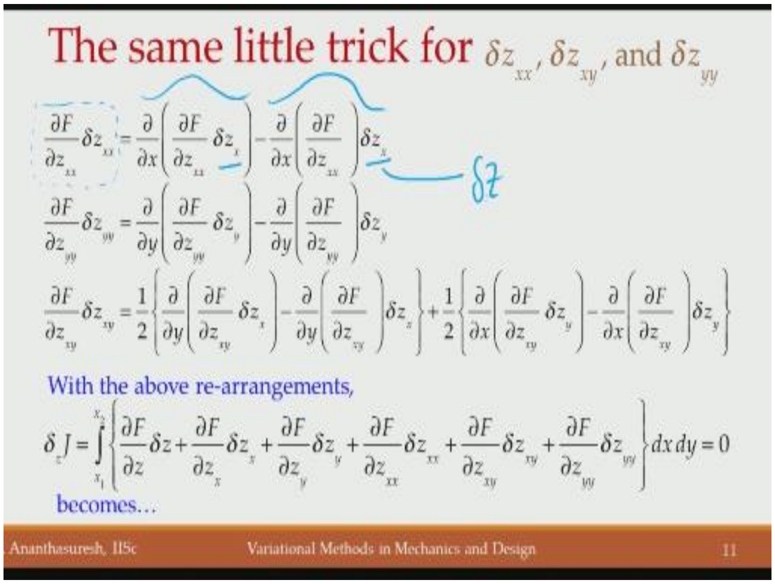

So what I had encircled in the last slide was this that is split into this term and then this term okay because then we will be left with not δ z sub xx but δ z sub x δ z sub x after that you have to

That is split into this term and then this term okay because then we will be left with naught δZ sub xx but δZ sub X letter that is of X after that you have to do one more time to reduce δzx as y2 that is just δZ okay so that is what we have here a lot of manipulation since the slides are going to be there accompanying these lectures you can look through and then go through the terms okay so you have to do some rearrangements and you know group them together.

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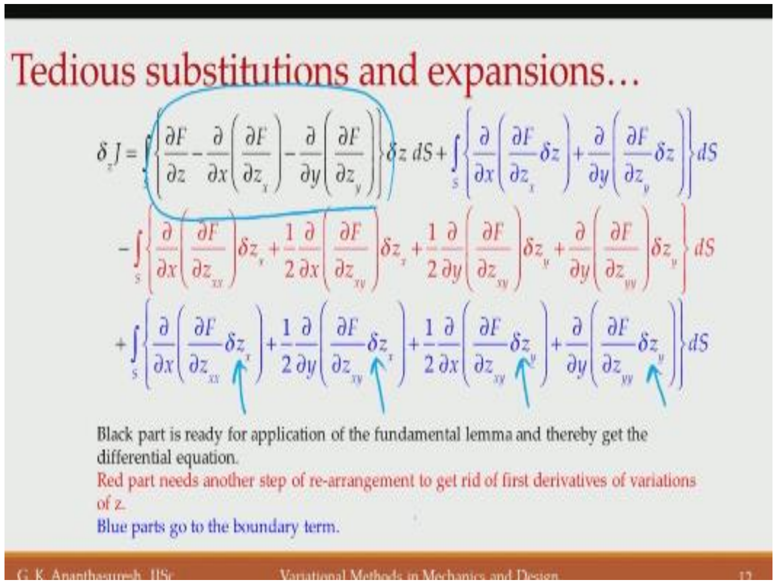

So forth when you do all that you get both the differential equation which is there now this is the earlier one that we had right this is a differential equation part now we also have additional terms that will come up with these are the boundary terms right so we have these we will also get the additional terms for this if you do one more integration by parts if you see we have gotten rid of by doing these theorem once we have gotten second derivative to first derivative.

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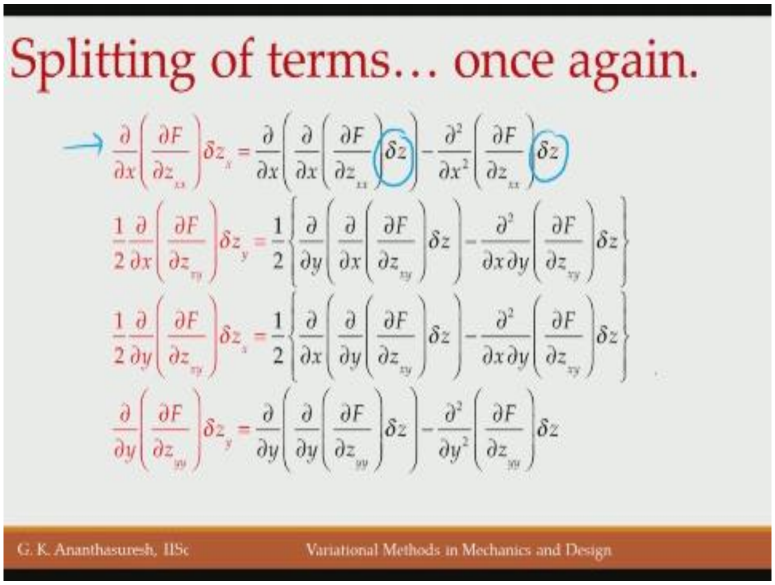

If you do one more time okay the same thing if we do one more time where you replace one term with two terms you notice that we have only δZ are not that as a sub X or δZ sub y okay if you do the same thing again what we get.

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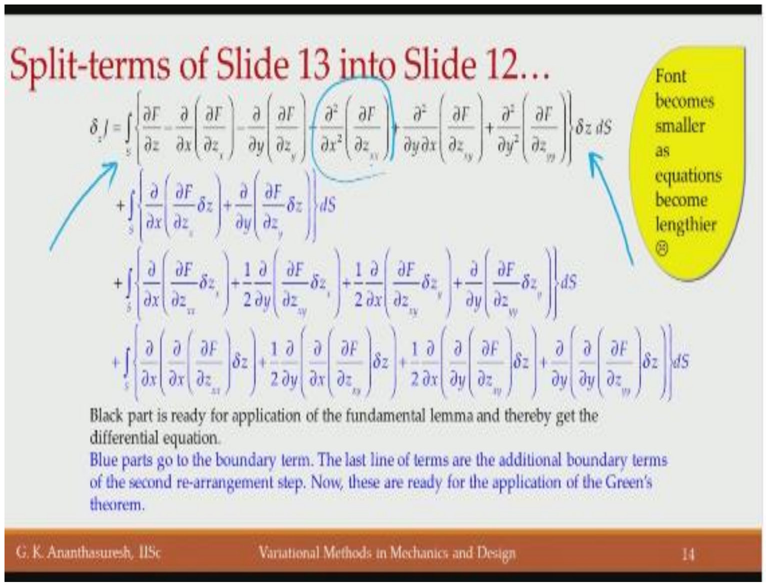

Will be all the terms the differential equation if you see the first row here we can observe what are the term that came differential equation ∂ f / ∂ Z minus ∂ / ∂ X of ∂ / ∂ Z X all of that this and that we also have a second order term ∂ square by ∂ X square of ∂ f /∂ z XX and same thing is mixed derivative and respect to Y also you just need to work out and accumulate more terms that will go into the boundary.

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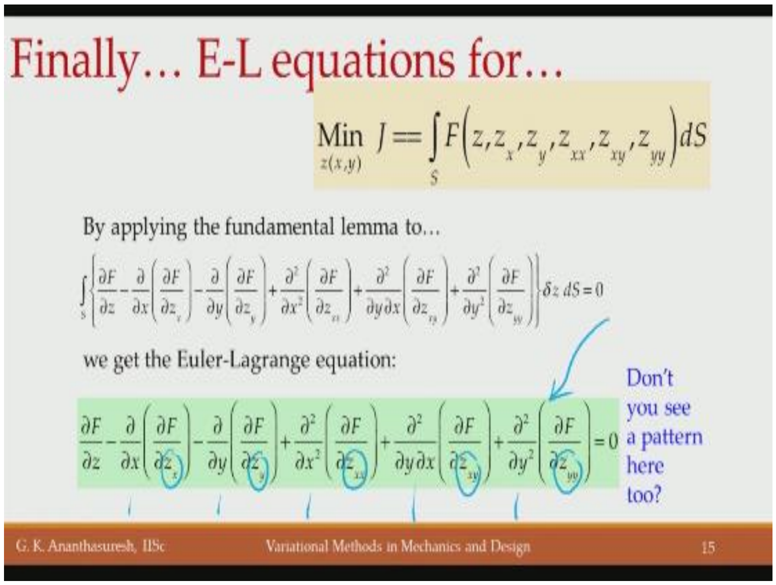

And if you do all that you will get Euler Lagrange equation that looks like that okay and you can see a pattern here now just like we try to saw try to see a pattern when one variable case here also ∂ /∂ X ∂/ ∂ Y ∂ square /2 x square ∂ square / ∂ / ∂ X ∂ square/ ∂ Y square and the same thing is done on set this is by ∂ X ∂ a / ∂ / ∂ square is then by ∂ X Square + and so forth okay. So we can actually write them down without actually having to do anything any calculation.

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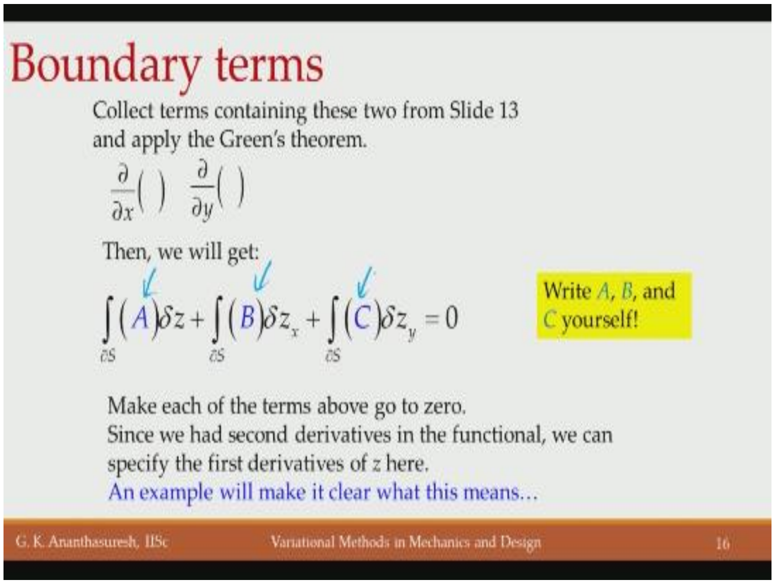

So only boundary terms tend to be longer so these things what I have not shown here ABC they are pretty long you can always take them and write them yourself okay just the real made rid terms and looking at it.

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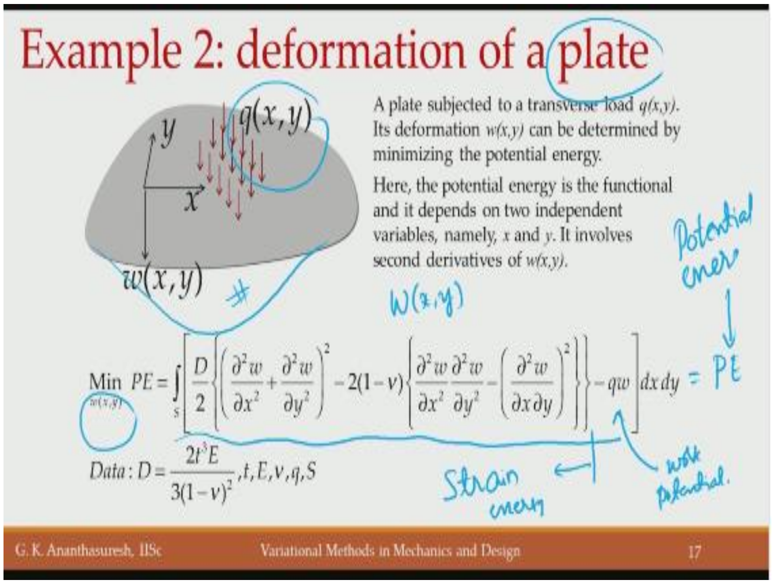

So an example that is related to this is a plate a plate is true dimensional things such as the one we have shown or a dinner plate for example you can constrain it on the boundaries or put a knife edge and a boundary and so forth you can cash in various ways but what this plate does when there is load acting on it load is given by this Q X Y is that it will deform so this plate will deform in some manner okay.

The deformation that we denoted by W here as a function of x and y then you get a differential equation for that what unit is we have to minimize the potential energy with respect to that function W X Y and in this case for a plate from mechanics and geometry you can compute the expression for potential energy of a plate okay that has this is the strain energy that you have up to this point it is strain energy and this is as usual work potential okay.

Q is a transverse low w is the work done that should be the work potential overall we get the potential energy the whole thing that we have is potential energy that is what is PE potential energy G okay.

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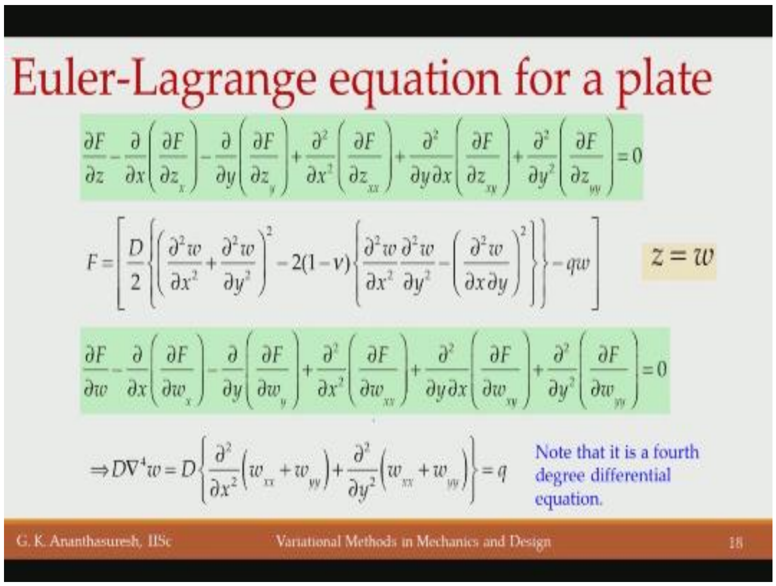

These are the problem were like what we had considered right so this one it depends on Z, Z X that why is that xx and so forth does not depend on all of them there is certainly w right and the second derivatives first derivatives are not there well that is ok just the terms will be 0 if you work this out writing it all the things that are there okay we get an expression by the way here.

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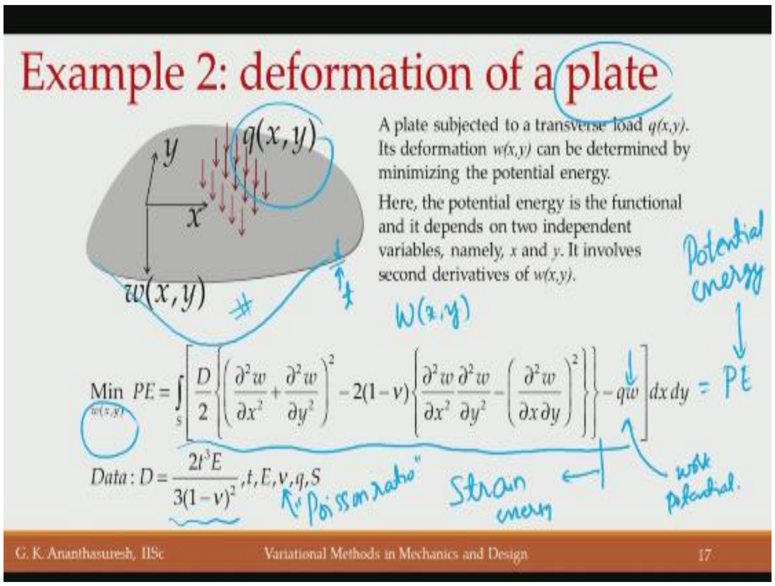

D is a plate modulus which is given by this T is the thickness of the plate okay so if I somewhere we say some thickness that is thickness of the plate e is the Young's modulus and ν another material property is what we call Poisson ratio okay with all of those we have an integrand that depends on w and its second derivatives if you write down the Euler-Lagrange equation then you get what we call plate differential equation which is given by this sometimes for short form we call it dealt to the fourth like here, okay.

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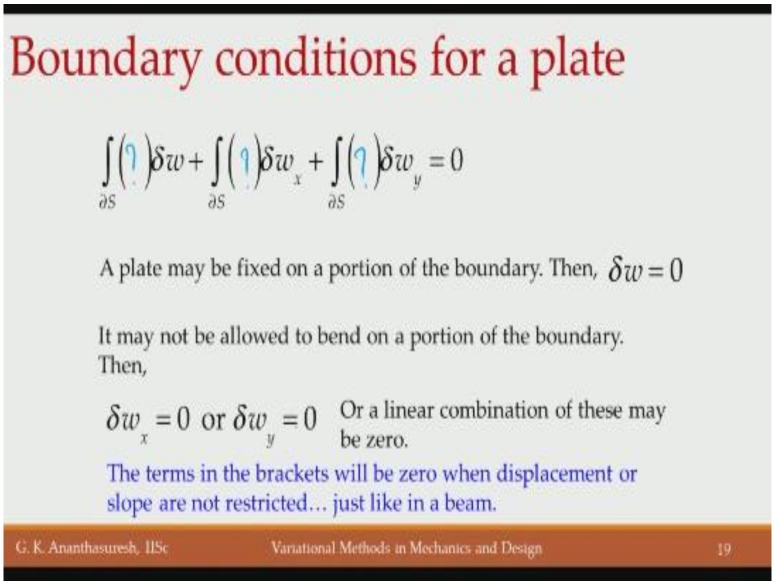

The boundary conditions we have to work out quite a bit in order to see what goes in there what goes in there what goes in there but all the steps are there you just need to work it out for your functional and you get these things and then it will be useful to solve any kind of great problem that is a set of conditions a plate will have their either delish lay our Nyman we can solve those things and based on whether w specified WX is specified w-why is specified we can change the boundary conditions and solve the problems okay.

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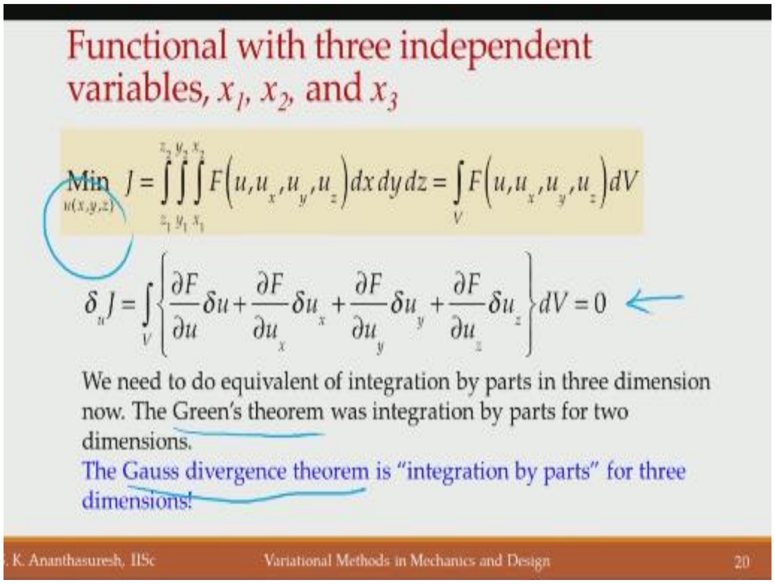

That is about two variables now we had touched upon the three variables and we essentially say that whatever method we have used for one variable x 10 to two variables by replacing integration by parts with integrals theorem in the case of three variables we replace greens theorem with Gauss divergence theorem so here the function U is a function of XY and Z okay and that is the unknown and integrand here will be function of up to first derivatives you use of X use of x and u sub Z.

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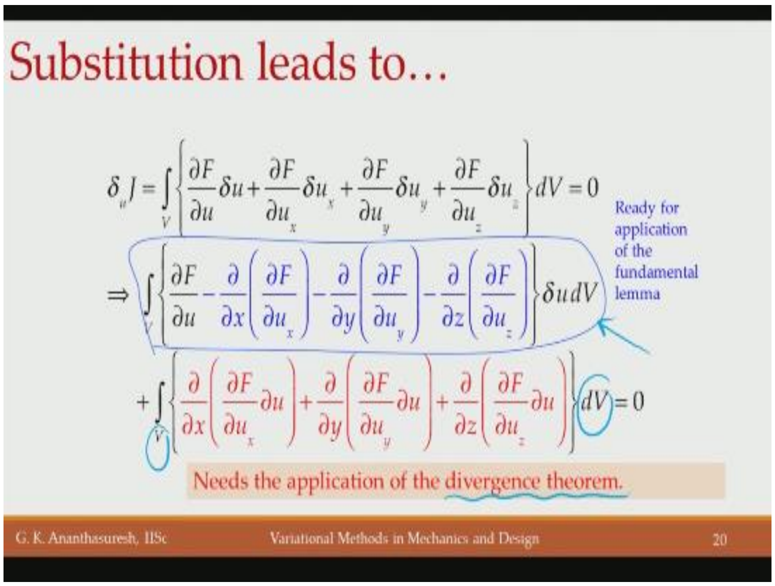

And when we do all this again taking the variation and equating zero replying greens theorem which now becomes Gauss divergence theorem okay so if we do that then we get the differential equation of course the trick that we used still remains the idea is to get rid of this and put into form of just δyou everywhere except that there is Del ∂ /∂ X here but we are trying to avoid trouble okay.

So either you specified or UX is specified the other one will be 0 and then we can go on to write the plate boundary conditions so if you do that we get the differential equation part δarbitrary and then we get this and this one is in the volume integral which we convert to surface integral by using the divergence theorem okay.

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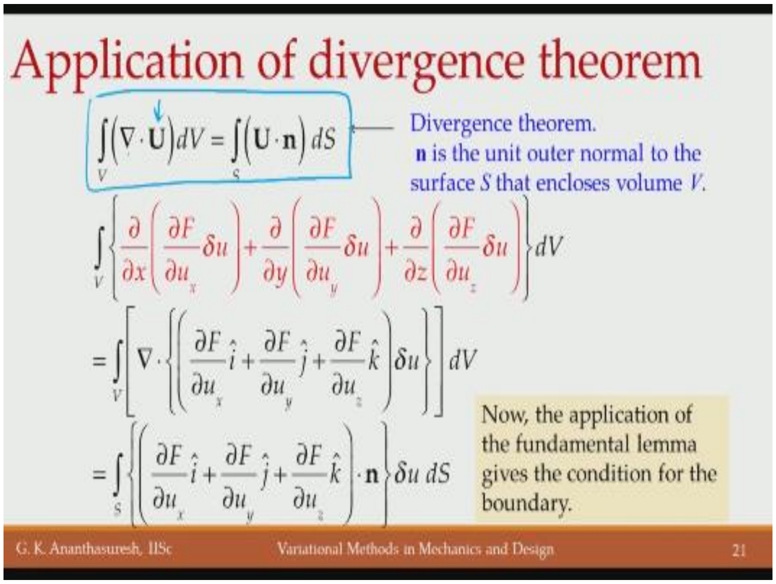

Divergence theorem is this so if you take divergence of a vector in this case u del dot u is called the divergence okay divergence if you integrate inside the volume surprisingly it is equal to the flux that is going out are coming in u dot n TS which is done over the surface of the volume that is enclosed okay that is what we are doing we have if I say divergence it is basically ∂ /∂ u /∂ x +∂ u/ ∂ y ∂ / ∂ Z and that is what we have here.

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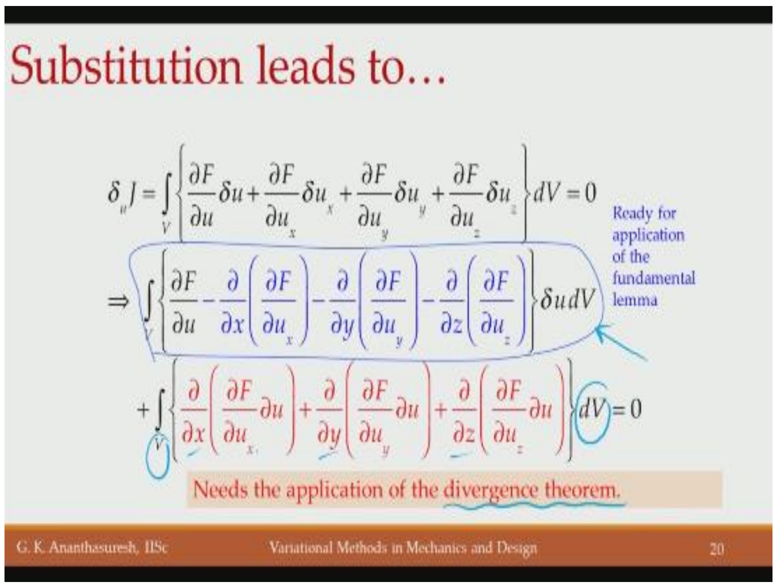

∂ /∂ X, ∂ / ∂ y, ∂ /∂ Z all of these things.

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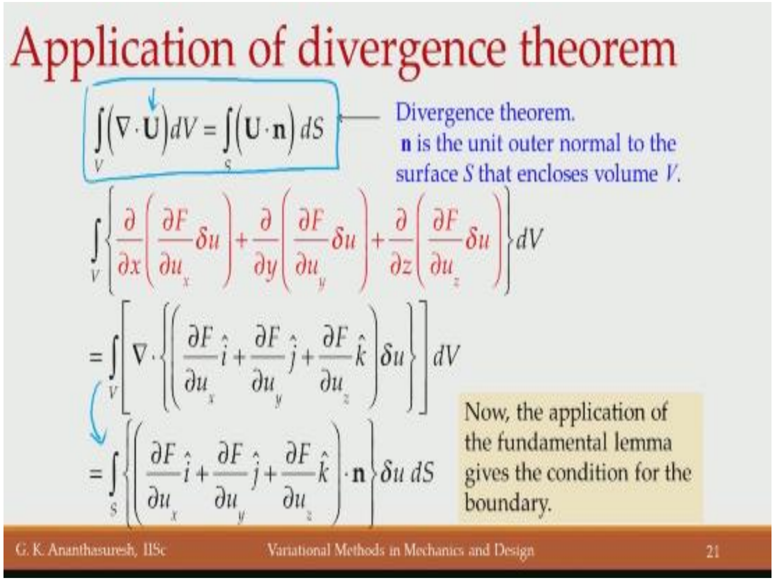

And if we write this the Gauss theorem if you apply from here we get here then we can see what is the individual things are okay that has to be true on the boundary.

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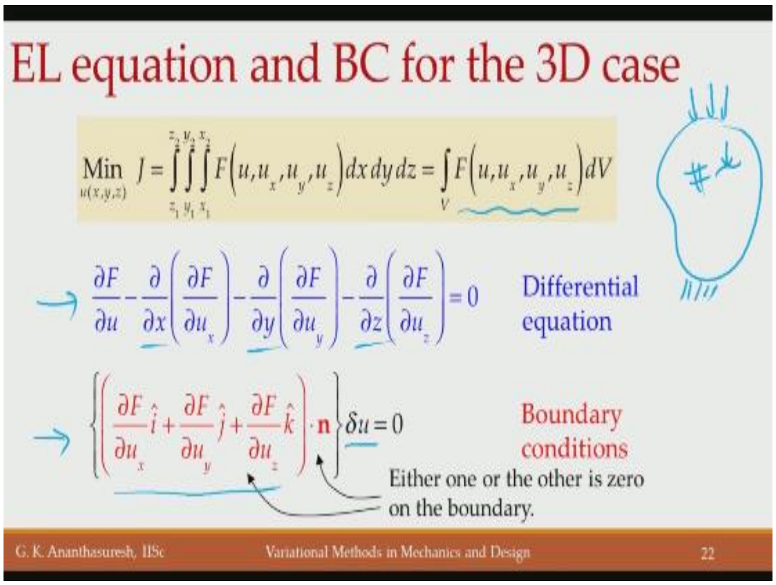

So now if we take this problem for the three-dimensional elasticity case let us say I have a solid ok three-dimensional solid it may be fixed somewhere it may have some forces on the boundary there could be some forces in the integral so like weight and centrifugal force okay in that case we can minimize this it will depend on the first derivative you take the linear strain use of X use of y u sub Z and you itself is a function of XYZ and hence use of X use of x u sub Z.

But also functions of u then you get the differential equation okay that is a pattern there is a dou/∂ X ∂ / ∂ debaters you can close your eyes and right boundary conditions also have pattern but now it is just written in three dimensions because this should be true at every point on the boundary either that is zero or this issue when is this zero whenever you is specified anywhere okay and that is how it looks like.

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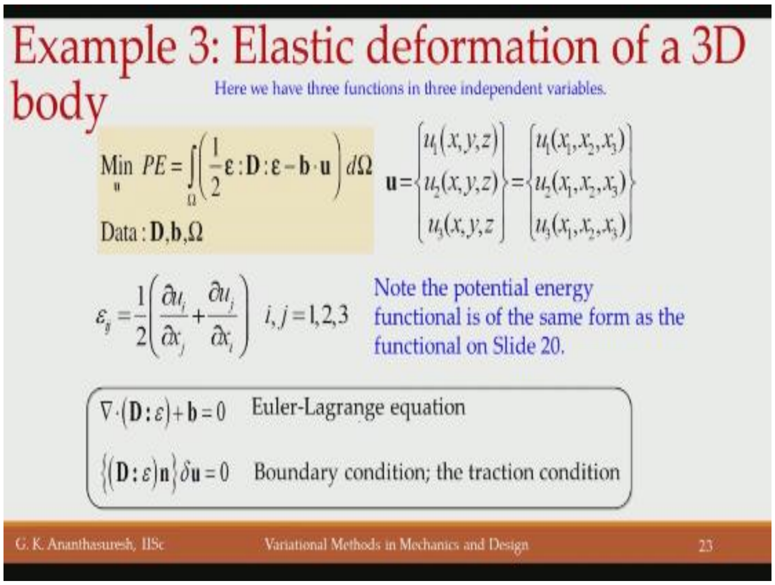

For this if you want an example the best thing is a solid that we I just sketched over here right.

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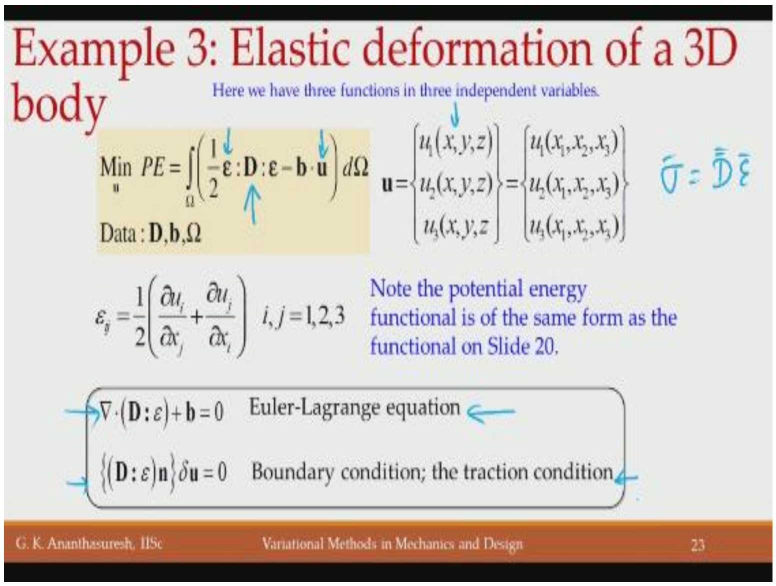

So we can write down potential energy because that is what you do minimized and that is written in terms of stress strain relationship basically d here is stress is equal to D times ε all are we can say it is a matrix form these are in vector form or all can be in terms of form and you have that okay ε here is a strain so and u is the three-dimensional displacement were defining what you is actually and we can write all a Grange equation.

When you write it you get this familiar thing that we use in mechanics of solids when you look at continuum and the boundary condition would look like this okay so that is how we do the three dimensional problems also and as I said more than three dimensions not necessary in three dimensions having double derivatives also is rare but you can still follow the procedure work it out you have to apply Gauss's divergence theorem twice okay when you have more than first derivative in your integrand okay.

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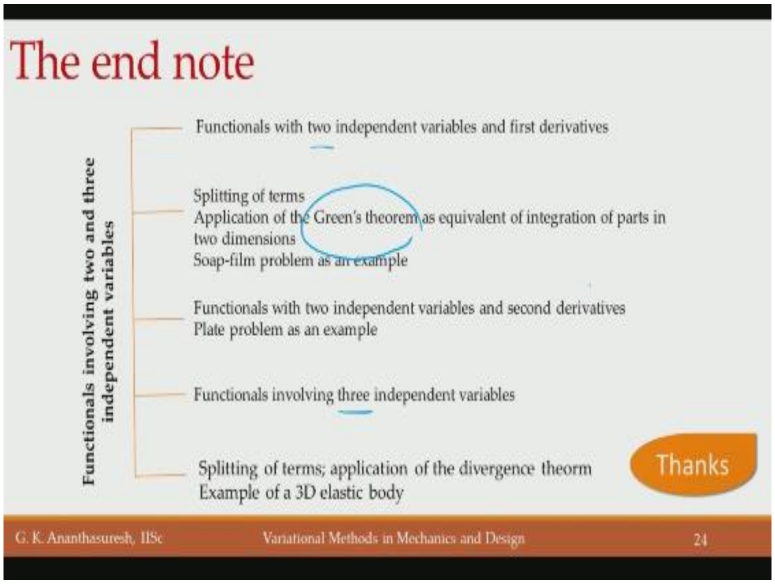

Just to summarize this we have discussed two and three independent variables too and then we have three independent variables and we looked at greens theorem as an extension digression by parts in two dimensions and there is Gauss theorem to help us in the three dimensions okay and we have solved examples of a soap film and in mechanics we did the problem of plate and the three-dimensional art is solid where the boundary condition we take a quick look see the boundary condition.

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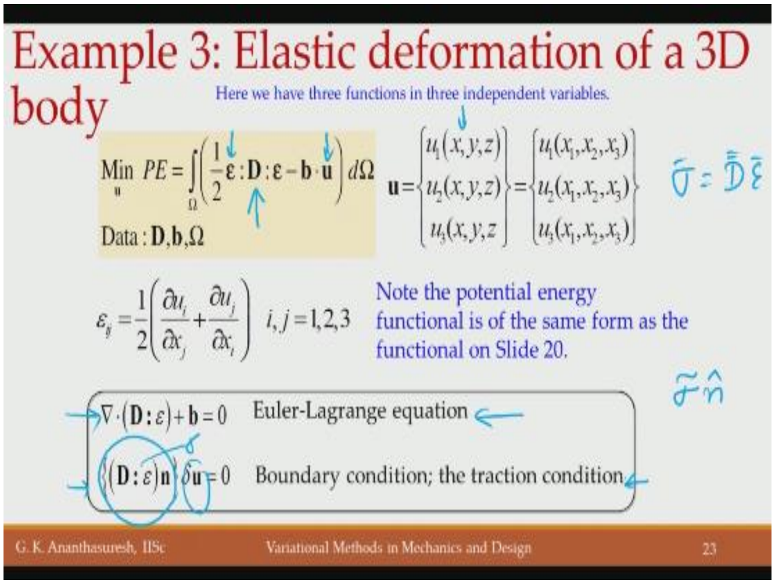

That comes from the divergence theorem which says that d ε the tensor notation basically this is Σ, Σ n if you write this let us say the tensor sometimes we write like that right and this is the unit vector if you do that that basically says that the traction on the boundary is 0 whenever δU is not even is δu 0 u δU is zero when you is specified like fixing it okay when that is not specified the corresponding stress should be zero.

Whenever you have not specified then that portion if there is no force the force can as it is called in solid mechanics literature that will be 0 that comes in the boundary condition so at this point you should see the power of calculus of variations we can solve many problems in geometry and mechanics as we are seeing in this course we can also use calculations to solve structural design problems the next part of the lecture we consider one more concept in calculus of variations and related to mechanics and geometry and design as usual, thank you.